A SMOOTH FUNCTION ON A MANIFOLD WITH GIVEN REEB GRAPH

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Abstract. We show that any finite graph without loops can be realized as the Reeb graph of a smooth function on a closed manifold with finitely many critical values, but possibly with positive dimensional critical point set. We also show that such a function can be chosen as the height function on a surface immersed in 3-space, provided that the graph has no isolated vertices.

1. Introduction

Let $f : M \to \mathbf{R}$ be a smooth function on a smooth manifold. The topological space obtained from M by contracting each connected component of the level sets to a point is often a graph and is called the *Reeb graph* of f. (Historically, it was first defined by Reeb [10]. This is sometimes called the *Kronrod-Reeb graph* of f as well. See [11], for example.) This object is very important in studying the shape of a surface both in Differential Topology and in Computer Graphics (see [4]–[9] and [2, 3], for example).

In [11] Sharko studied the Reeb graphs of smooth functions on closed manifolds of dimension ≥ 2 with only finitely many critical points, and gave several characterizations of those finite graphs which can be the Reeb graph of such a function. It was pointed out that not every finite graph without loops or isolated vertices can be so realized.

In this paper, we consider smooth functions with finitely many critical values, but possibly with positive dimensional critical point set, and show that an arbitrary finite graph without loops can be realized as the Reeb graph of such a smooth function on a closed surface. We also show that such a function can be chosen as the height function on a closed surface immersed in 3-space, provided that the graph has no isolated vertices.

The paper is organized as follows. In §2, we state and prove the realization theorem of a given Reeb graph by a smooth function on a closed surface. In §3, we prove that such a function can in fact be chosen as the height function associated with an immersion of a closed surface into $\mathbf{R}^3$. We also show that if the graph is a tree, then it can be realized as the height function of an embedded 2-sphere in $\mathbf{R}^3$.

Throughout the paper, manifolds and maps between them are differentiable of class C^∞ unless otherwise specified.

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2. Realization by a smooth function

Let M be a compact manifold and $f : M \to \mathbf{R}$ a smooth function. Throughout the paper, we assume $\dim M \geq 2$. For points $p_1, p_2 \in M$, we define $p_1 \sim p_2$ if $f(p_1) = f(p_2)(= a)$ and they are in the same connected component of $f^{-1}(a)$. This clearly defines an equivalence relation on M. The quotient space $M/\sim$ is denoted by $\mathcal{R}(f)$ and is called the *Reeb graph* of f. The quotient map is denoted by $q_f : M \to \mathcal{R}(f)$. It is easy to see that there is a unique continuous function $\bar{f} : \mathcal{R}(f) \to \mathbf{R}$ which makes the following diagram commutative:

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "q_f"'] & & \mathbf{R} \\ & \mathcal{R}(f). \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

It is known that if a smooth function $f : M \to \mathbf{R}$ on a closed manifold has only finitely many critical points, then its Reeb graph $\mathcal{R}(f)$ is a finite graph without loops or isolated vertices (see, for example, [11]). Moreover, each vertex corresponds to the image by q_f of a critical point of f. Furthermore, Sharko [11] showed that a finite graph without loops or isolated vertices can be realized by a Reeb graph of a smooth function on a closed manifold with finitely many critical points if and only if it admits a good orientation. Here, a *good orientation* of a graph G is an orientation of the edges of G such that at each vertex v of degree ≥ 2, there are at least one edge incident to v which flows out of v, and also at least one edge incident to v which flows into v, and that G contains no oriented cycle. In fact, Sharko gave an explicit example of a finite graph without loops or isolated vertices that can never be realized as the Reeb graph of a smooth function on a closed manifold with finitely many critical points.

Our first result of this paper is the following.

Theorem 2.1. *Let G be a finite graph without loops. Then, there exists a smooth function $f : M \to \mathbf{R}$ on a closed surface M with finitely many critical values such that its Reeb graph $\mathcal{R}(f)$ is homeomorphic to G.*

Proof. If G contains an isolated vertex, then such a point can be realized as the Reeb graph of any constant function on a closed connected surface. Therefore, we may assume that G contains no isolated vertices.

Let $f_1 : G \to \mathbf{R}$ be a continuous map which is an embedding on each edge. Such a function can easily be constructed by first considering any injective map $V(G) \to \mathbf{R}$ and then by extending it to the edges so that it is "linear" on each edge, where $V(G)$ denotes the set of vertices of G. Then, we orient each edge of G so that f_1 is increasing with respect to the orientation.

In the following, we denote by $\deg(v)$ the degree of a vertex v of G. For each vertex $v \in V(G)$, its neighborhood satisfies one of the following.

(a) There are at least one edge incident to v which flows out of v, and also at least one edge incident to v which flows into v, and $\deg(v) \geq 3$.

(b) There are exactly one edge incident to v which flows out of v, and also exactly one edge incident to v which flows into v; in particular, $\deg(v) = 2$.

(c) All the edges incident to v flow out of v, or all of them flow into v, and $\deg(v) \geq 2$.
(d) The degree satisfies $\deg(v) = 1$.

In each of the above four cases, we will construct a smooth function $g_v : N_v \to \mathbf{R}$ on a compact surface with boundary as follows.

Case (a). Let ℓ (or m) denote the number of edges incident to v flowing out of v (resp. flowing into v). By assumption, we have $\ell \geq 1$, $m \geq 1$ and $\ell + m \geq 3$.

Let N_v be the 2-sphere with $\ell+m$ open disks removed. Note that N_v is a compact surface which is obtained by attaching $\ell+m-1$ bands to a 2-disk. On this surface, we construct a smooth function $g_v : N_v \to \mathbf{R}$ that satisfies the following properties.

(a1) The function g_v has a unique critical point p in $\operatorname{Int} N_v$ and $g_v(p) = f_1(v)$.
(a2) The function g_v attains its maximum on ℓ circle components of ∂N_v, and attains its minimum on the other m circle components of ∂N_v.
(a3) The inverse image $g_v^{-1}(a)$ is homeomorphic to the bouquet of $\ell + m - 1$ copies of S^1, where $a = f_1(v)$.
(a4) There exist some local coordinates (x, y) around p such that the function g_v can be written as

$$g_v(x,y) = \operatorname{Re}\left((x + \sqrt{-1}y)^{\ell+m-1}\right) + a. \tag{2.1}$$

(a5) The Reeb graph $\mathcal{R}(g_v)$ of g_v is homeomorphic to a small closed neighborhood of v in G.

Such a smooth function $g_v : N_v \to \mathbf{R}$ can be constructed as follows. First, we define g_v by (2.1) on a small disk D centered at the origin. We choose $\varepsilon > 0$ sufficiently small so that the maximum of g_v on ∂D is strictly greater than $a + \varepsilon$, and the minimum of g_v on ∂D is strictly smaller than $a - \varepsilon$. Then, $g_v^{-1}(a)$ consists of $2(\ell + m - 1)$ radial segments of the disk D. Furthermore, the region

$$\Delta = \{(x, y) \in D \,:\, |g_v(x, y) - a| \leq \varepsilon\}$$

intersects with ∂D in $2(\ell + m - 1)$ arcs.

Then, we attach $\ell + m - 1$ copies of a band to Δ along these arcs as in Fig. 1. Let N_v be the resulting compact surface. On the center arc of each band, the value of g_v is constantly equal to a, and along each arc transverse to the center arc, g_v is monotone, its derivative never vanishes, its minimum is equal to $a - \varepsilon$, and its maximum is equal to $a + \varepsilon$. In Fig. 1, $g_v^{-1}(a + \varepsilon)$ is depicted by thick curves, while $g_v^{-1}(a - \varepsilon)$ is depicted by thin curves. Note that $g_v^{-1}(a + \varepsilon)$ consists of ℓ circles, while $g_v^{-1}(a - \varepsilon)$ consists of m circles.

In the following, we take $\varepsilon > 0$ even smaller so that

$$2\varepsilon < \min\{|f_1(v) - f_1(v')| \,:\, v' \in V(G), v' \neq v\}.$$

Lemma 2.2. *The Reeb graph $\mathcal{R}(g_v)$ of g_v is homeomorphic to a small closed neighborhood of v in G.*

Proof. Let G_v be the connected component of $f_1^{-1}([a - \varepsilon, a + \varepsilon])$ containing v. Let us define a continuous map $\rho_v : \mathcal{R}(g_v) \to G_v$ as follows.

There are ℓ segments incident to v in G_v whose values of f_1 lie in $[a, a + \varepsilon]$. We choose any bijective correspondence between such segments and the ℓ circle components of ∂N_v on which g_v takes its maximum. For each $t \in (a, a + \varepsilon]$, $g_v^{-1}(t)$ consists of ℓ circle components. Let C be one of such components, which is an equivalence class and is identified with an element of $\mathcal{R}(g_v)$. Then, we define $\rho_v(C)$

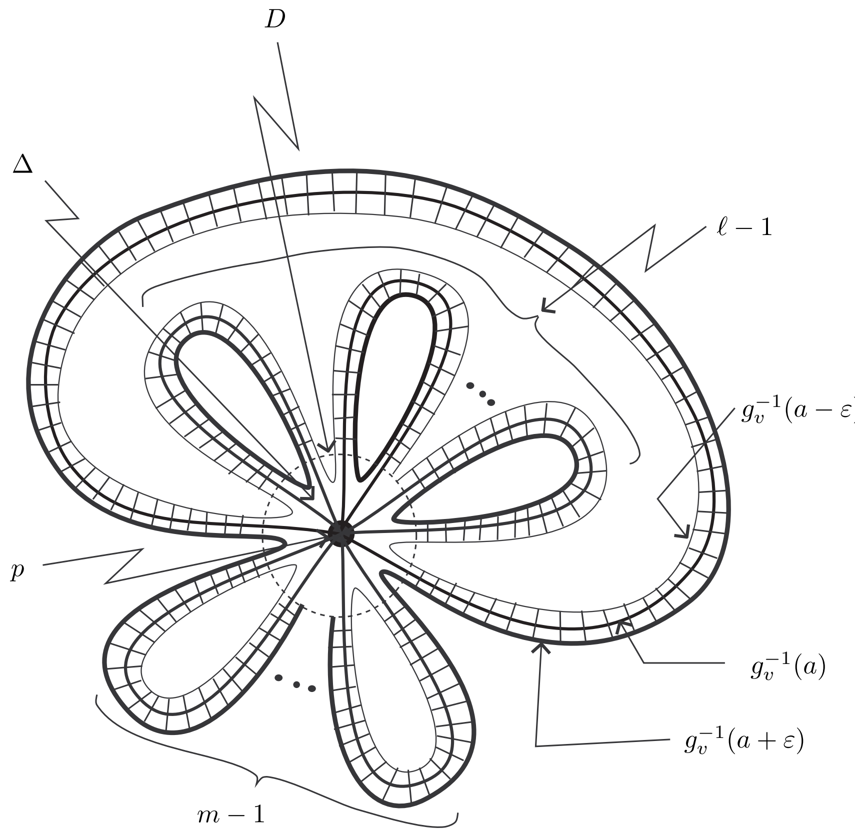

FIGURE 1. Constructing the function $g_v : N_v \to [a-\varepsilon, a+\varepsilon]$ for Case (a)

to be the point $y \in G_v$ such that $f_1(y) = t$ and the segment to which y belongs corresponds to the boundary component of ∂N_v that is in the same component of $g_v^{-1}((a, a+\varepsilon])$ as C. Then, we see easily that ρ_v is continuous and is bijective.

Since $\mathcal{R}(g_v)$ is compact and G_v is Hausdorff, we see that ρ_v is a homeomorphism. This completes the proof of Lemma 2.2. □

Therefore, we have constructed a desired smooth function $g_v : N_v \to \mathbf{R}$ satisfying (a1)–(a5) above.

Case (b). Let N_v be the annulus. We construct a smooth function $g_v : N_v \to \mathbf{R}$ which satisfies conditions similar to (a1) and (a2) above, together with the following conditions.

(b3) The inverse image $g_v^{-1}(a)$ is a circle with a cusp ($a = f_1(v)$).

(b4) There exist some coordinates (x, y) around p such that the function g_v can be written as

$$g_v(x, y) = -x^2 + y^3 + a. \tag{2.2}$$

(b5) The Reeb graph $\mathcal{R}(g_v)$ of g_v is homeomorphic to a small closed neighborhood of v in G.

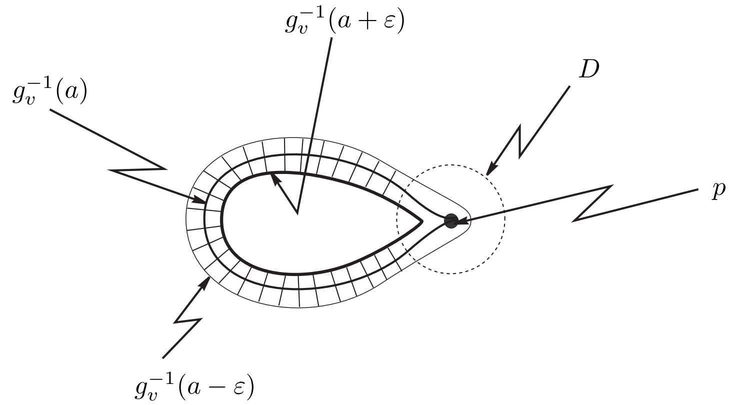

FIGURE 2. Constructing the function $g_v : N_v \to [a-\varepsilon, a+\varepsilon]$ for Case (b)

Such a function can be constructed by the same way as in Case (a). See Fig. 2 for details.

Case (c). Let N_v be the 2-sphere with n open disks removed, where $n = \deg(v)$. For simplicity, we assume that all the edges incident to v flow out of v. (The other case can be treated similarly.) We construct a smooth function $g_v : N_v \to [a, a+\varepsilon]$ which satisfies the following properties.

(c1) The function g_v has a unique critical value a, which is the minimum.
(c2) The function attains its maximum on ∂N_v.
(c3) The inverse image $g_v^{-1}(a)$ is homeomorphic to the bouquet of $n-1$ copies of S^1.
(c4) There exist some coordinates (x, y) around p such that the function g_v can be written as

$$g_v(x, y) = \left(\mathrm{Re}\left((x + \sqrt{-1}y)^{n-1}\right)\right)^2 + a. \tag{2.3}$$

(c5) The Reeb graph $\mathcal{R}(g_v)$ of g_v is homeomorphic to a small closed neighborhood of v in G.

The construction of such a function is similar to Case (a). The essential difference is that we define g_v so that along each arc transverse to the center arc of a band, g_v behaves like the function $t \mapsto t^2 + a$, $t \in [-\varepsilon^{1/2}, \varepsilon^{1/2}]$. Thus, the resulting function has positive dimensional critical point set corresponding to the minimum a, and it forms a bouquet of $n-1$ circles. For details, see Fig. 3.

Case (d). Let N_v be a small 2-disk centered at the origin. Then, we define $g_v : N_v \to \mathbf{R}$ by $g_v(x, y) = \pm(x^2 + y^2) + f_1(v)$, where we choose the sign "+"if the edge incident to v flows out of v, and the sign "$-$" otherwise. Note that the Reeb graph $\mathcal{R}(g_v)$ is homeomorphic to a small closed neighborhood of v in G.

Let us now construct a desired smooth function $f : M \to \mathbf{R}$ on a closed surface M by gluing the above-constructed smooth functions $g_v : N_v \to \mathbf{R}$. Let us take an edge e of G, and we denote by v and v' its end points. We may assume $f_1(v) < f_1(v')$. Set $a = f_1(v)$, $a_\pm = f_1(v) \pm \varepsilon$, $a' = f_1(v')$, and $a'_\pm = f_1(v') \pm \varepsilon$. Note that $a_- < a < a_+ < a'_- < a' < a'_+$. Let $g_v : N_v \to [a_-, a_+]$ and $g_{v'} : N_{v'} \to [a'_-, a'_+]$ be the smooth functions constructed above corresponding to the vertices v and v',

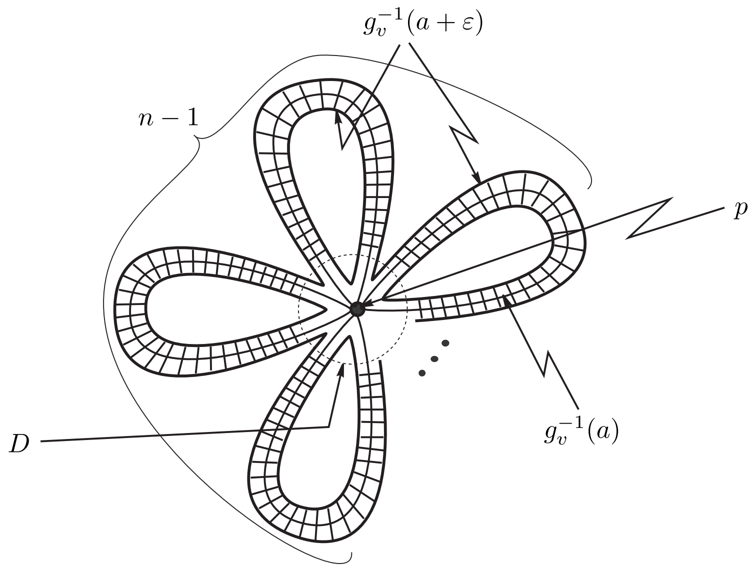

FIGURE 3. Constructing the function $g_v : N_v \to [a, a+\varepsilon]$ for Case (c)

respectively. There is a unique boundary circle C (or C') of N_v (resp. $N_{v'}$) with $g_v(C) = a_+$ (resp. $g_{v'}(C') = a'_-$) which corresponds to the edge e.

Set $N_e = S^1 \times [a_+, a'_-]$ and let

$$g_e : N_e \to [a_+, a'_-]$$

be the projection to the second factor. Let us glue N_v, N_e and $N_{v'}$ by identifying $S^1 \times \{a_+\}$ with C and $S^1 \times \{a'_-\}$ with C', where any identifications by diffeomorphisms are allowed. Then the functions g_v, $g_{v'}$ and g_e extend over this glued surface.

Let us perform the above mentioned gluing operation for each edge of G. Then we get a smooth closed surface M and a smooth function $f : M \to \mathbf{R}$ such that $f|N_v = g_v$ for each vertex v of G and $f|N_e = g_e$ for each edge e of G.

Finally, by an argument similar to that in the proof of Lemma 2.2, we can show that the Reeb graph $\mathcal{R}(f)$ of f is naturally homeomorphic to G so that the function $f_1 : G \to \mathbf{R}$ can be identified with the function $\bar{f} : \mathcal{R}(f) \to \mathbf{R}$. This completes the proof of Theorem 2.1. □

Remark 2.3. In the above construction, the connected components of the inverse image of critical values that contain critical points are in one-to-one correspondence with the set of vertices of G. In this sense, the above proof implies that any finite graph without loops can be realized as the Reeb graph of a smooth function on a closed surface with finitely many critical values, not only as a topological space, but also as a graph with its combinatorial structure.

Remark 2.4. If G is a finite oriented graph without an oriented cycle, then it is well known that there exists a continuous function $f_1 : G \to \mathbf{R}$ that is increasing on each oriented edge. Then, the above proof shows that we can realize G as the *oriented Reeb graph* of a smooth function with finitely many critical values on a closed surface. Here, each edge of a Reeb graph is oriented so that the naturally induced function is increasing with respect to the orientation.

Remark 2.5. In the above proof, the resulting surface M may not be orientable. The orientability depends on the gluing maps between the circle components corresponding to an edge contained in a cycle of G. However, if G is a tree, then the surface M constructed above is orientable: in fact, it is diffeomorphic to the 2-sphere.

Remark 2.6. In Theorem 2.1, we have constructed a smooth function $f : M \to \mathbf{R}$ on a closed 2-dimensional manifold that realizes the given graph as its Reeb graph. For each integer $n \geq 3$, let us consider the composition

$$\tilde{f} : S^{n-2} \times M \xrightarrow{\quad \pi \quad} M \xrightarrow{\quad f \quad} \mathbf{R},$$

where π is the projection to the second factor. Then, we see that $\tilde{f}$ is a smooth function with finitely many critical values, and that $\mathcal{R}(\tilde{f})$ can naturally be identified with $\mathcal{R}(f)$, since for each $a \in \mathbf{R}$, we have $\tilde{f}^{-1}(a) = S^{n-2} \times f^{-1}(a)$. Therefore, in Theorem 2.1, any finite graph without loops can be realized as the Reeb graph of a smooth function with finitely many critical values on a smooth closed manifold of any given dimension $n \geq 2$.

Remark 2.7. When a finite graph G contains a loop, we can show that it is realized as the Reeb graph of a smooth map $f : M \to S^1$ of a closed surface into the circle with finitely many critical values. The proof is similar to that of Theorem 2.1. When G is a finite oriented graph which may contain oriented cycles, we can realize it as the oriented Reeb graph of a smooth map into the circle as above (see Remark 2.4).

3. Realization by a height function

In this section, we show that any finite graph without loops or isolated vertices can be realized as the Reeb graph of a height function associated with an immersion of a closed surface into $\mathbf{R}^3$.

Let $\iota : M \to \mathbf{R}^3$ be an immersion of a closed surface M into the 3-space. Let $Z : \mathbf{R}^3 \to \mathbf{R}$ be the projection defined by $Z(x, y, z) = z$. Then, the smooth function h on M defined by the composition $h = Z \circ \iota : M \to \mathbf{R}$ is called a *height function* associated with the immersion ι.

Recall that there exist smooth functions on surfaces that can never be realized as a height function (for example, see [1, 9]).

Our second result of this paper is the following refinement of Theorem 2.1.

Theorem 3.1. *Let G be a finite graph without loops or isolated vertices. Then, there exists an immersion $\iota : M \to \mathbf{R}^3$ of a closed surface M into $\mathbf{R}^3$ such that the Reeb graph $\mathcal{R}(h)$ of the height function $h = Z \circ \iota$ associated with ι is homeomorphic to G.*

Proof. We proceed as in the proof of Theorem 2.1. We use the same notations.

Let us first consider Case (a) and construct an embedding $\iota_v : N_v \to \mathbf{R}^2 \times [a - \varepsilon, a+\varepsilon]$ as follows. First, we regard N_v to be embedded in the plane $\mathbf{R}^2$ as in Fig. 1. Then, let ι_v be the graph of g_v: i.e., $\iota_v(q) = (q, g_v(q))$ for $q \in N_v \subset \mathbf{R}^2$. Then, its associated height function $h_v = Z \circ \iota_v$ clearly coincides with g_v.

For the other cases, we construct the embedding $\iota_v : N_v \to \mathbf{R}^3$ by exactly the same way.

Let us now glue the embeddings. For each edge e of G, we consider the cylinder $N_e = S^1 \times [a_+, a'_-]$. Since $\iota_v|C : C \to \mathbf{R}^2 \times \{a_+\}$ and $\iota_{v'}|C' : C' \to \mathbf{R}^2 \times \{a'_-\}$ are embeddings, they are isotopic with respect to an appropriate identification between C and C'. Using these identifications together with the isotopy, we can construct an embedding of $N_v \cup N_e \cup N_{v'}$ into $\mathbf{R}^2 \times [a_-, a'_+]$.

We can perform this operation for each edge of G consistently so as to obtain an immersion $\iota : M \to \mathbf{R}^3$ of a closed surface M. Note that the resulting map ι may not be an embedding any more, since the images of $\iota|N_e$ for various edges e may intersect with each other.

Finally, we see that the Reeb graph $\mathcal{R}(Z \circ \iota)$ is homeomorphic to G as in the proof of Theorem 2.1. In fact, if we choose the identification diffeomorphisms appropriately in the proof of Theorem 2.1, the height function $Z \circ \iota$ coincides with the smooth function f constructed in the proof of Theorem 2.1.

This completes the proof of Theorem 3.1. □

Note that the above construction is based on the method developed in [1].

Remark 3.2. When a finite graph G has a loop (but does not have any isolated vertices), we can still construct an immersion $\iota : M \to \mathbf{R}^2 \times S^1$ of a closed surface M such that the Reeb graph of $\pi' \circ \iota : M \to S^1$ is homeomorphic to G, where $\pi' : \mathbf{R}^2 \times S^1 \to S^1$ is the projection to the second factor. The proof is similar to that of Theorem 3.1. See also Remark 2.7.

Remark 3.3. For each integer $n \geq 3$, let us consider the immersion

$$\begin{aligned}\tilde{\iota} : S^{n-2} \times M \xrightarrow{\mathrm{id}_{S^{n-2}} \times \iota} S^{n-2} \times \mathbf{R}^3 = (S^{n-2} \times \mathbf{R}) \times \mathbf{R}^2 \\ \xrightarrow{j \times \mathrm{id}_{\mathbf{R}^2}} \mathbf{R}^{n-1} \times \mathbf{R}^2 = \mathbf{R}^{n+1},\end{aligned}$$

where $\iota : M \to \mathbf{R}^3$ is the immersion given in Theorem 3.1, id_* denotes the identity map, and $j : S^{n-2} \times \mathbf{R} \to \mathbf{R}^{n-1}$ is an embedding[1]. Then, we see that for the projection $pr_{n+1} : \mathbf{R}^{n+1} \to \mathbf{R}$ to the last coordinate, the height function $\tilde{h} = pr_{n+1} \circ \tilde{\iota}$ has finitely many critical values, and that $\mathcal{R}(\tilde{h})$ is isomorphic to the given graph G. Therefore, in Theorem 3.1, any finite graph without loops or isolated vertices can be realized as the Reeb graph of a height function with finitely many critical values on a smooth closed manifold of any given dimension $n \geq 2$ immersed in $\mathbf{R}^{n+1}$. Compare this with Remark 2.6.

If G is a tree, then we have the following refinement.

Proposition 3.4. *Let G be a finite tree which is not just a point. Then, there exists an embedding $\eta : S^2 \to \mathbf{R}^3$ such that the Reeb graph $\mathcal{R}(Z \circ \eta)$ of the associated height function is homeomorphic to G.*

Proof. We proceed by induction on the number of vertices of G. If G has exactly two vertices, then G is an interval, and the standard embedding $S^2 \hookrightarrow \mathbf{R}^3$ gives the desired height function. Note that this embedding can be regarded as being constructed by the procedure described in the proof of Theorem 3.1. Note also that this height function has only local minima and maxima as its critical points.

[1]The authors are indebted to the referee for this construction.

Suppose that G contains three or more vertices. There exists a vertex v of G of degree one, and let G' denote the tree obtained by taking off v and the edge incident to it from G. By our induction hypothesis, there exists an embedding $\eta' : S^2 \to \mathbf{R}^3$ such that the Reeb graph $\mathcal{R}(Z \circ \eta')$ is homeomorphic to G' and that the height function $h' = Z \circ \eta'$ has only local minima and maxima as its critical points. Let $\bar{h}' : \mathcal{R}(h') \to \mathbf{R}$ be the continuous function naturally induced from h'. In the following, we will identify $\mathcal{R}(h')$ with G'. We assume that the embedding η' is constructed by the procedure described in the proof of Theorem 3.1, and will use the same notation as in that proof, with the exception that we use η instead of ι.

Let v' be the unique vertex adjacent to v in G. If v' is a local minimum, then we modify the embedding $\iota_{v'} : N_{v'} \to \mathbf{R}^3$ as follows. If the degree of v' in G' is equal to one, then h' has a unique critical point in $N_{v'}$, and we replace the embedding $\iota_{v'}$ by an embedding obtained from a small center circle in the construction for Case (c) in the proof of Theorem 3.1 (with $n = 2$), together with a 2-disk with a local maximum attached to the inside circle parallel to the center circle. If the degree of v' in G' is greater than or equal to two, then we increase the number of circles in the bouquet of circles contained in $(h')^{-1}(\bar{h}'(v'))$ by one. We may assume that the new born circle is very small, and we attach a 2-disk along this new born circle so that it has a local maximum at its center. If we choose the value of the local maximum very close to $\bar{h}'(v')$, then the resulting immersion $\eta : S^2 \to \mathbf{R}^3$ remains to be an embedding.

Note that the resulting embedding η can be regarded as being constructed by the procedure described in the proof of Theorem 3.1.

If v' is a local maximum, then we construct the embedding η similarly.

Then, it is easy to see that the Reeb graph of the height function associated with the new embedding η is homeomorphic to G.

This completes the proof. □

Compare the above proposition with Remark 2.5.

We end this paper by posing some related problems. Let $\iota : M \to \mathbf{R}^3 \setminus \{0\}$ be an immersion and let $D : \mathbf{R}^3 \setminus \{0\} \to (0, \infty)$ be the function defined by $D(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, which measures the distance from the origin. The composition $d = D \circ \iota : M \to (0, \infty)$ is called the *distance function* associated with the immersion ι.

Problem 3.5. Let G be an arbitrary finite graph without loops or isolated vertices.

(1) Is there an *embedding* $\eta : M \to \mathbf{R}^3$ of a closed orientable surface such that the Reeb graph $\mathcal{R}(Z \circ \eta)$ of the associated height function is homeomorphic to G ?
(2) Is there an *embedding* $\eta : M \to \mathbf{R}^3 \setminus \{0\}$ of a closed orientable surface such that the Reeb graph $\mathcal{R}(D \circ \eta)$ of the associated distance function is homeomorphic to G ?

Note that if the answer to (1) is affirmative, then so is the answer to (2). We expect that the class of graphs for which the answer to (2) is affirmative is somewhat larger than that for (1), since $\mathbf{R}^3 \setminus \{0\}$ is diffeomorphic to $S^2 \times (0, \infty)$ and S^2 gives more freedom than $\mathbf{R}^2$ to avoid self-intersections.

Remark 3.6. Let G be a finite graph. We can embed it in $\mathbf{R}^3$ in such a way that the height function is nonsingular on each edge. Let us consider the closed orientable surface M obtained as the boundary of a small regular neighborhood of the embedded graph. Then, one might think that the height function on M has G as its Reeb graph; however, this is not correct. The Reeb graph does not coincide with the original graph around the vertices. Therefore, the above problems make sense.

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