Topology of quasi-homogeneous isolated hypersurface singularities

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§1. Topology of Complex Hypersurface Singularities

Complex hypersurface

$f = f(z_1, z_2, \ldots, z_{n+1})$ complex polynomial with $f(\mathbf{0}) = 0$
In this talk, we always assume that $\mathbf{0}$ is an **isolated critical point**.

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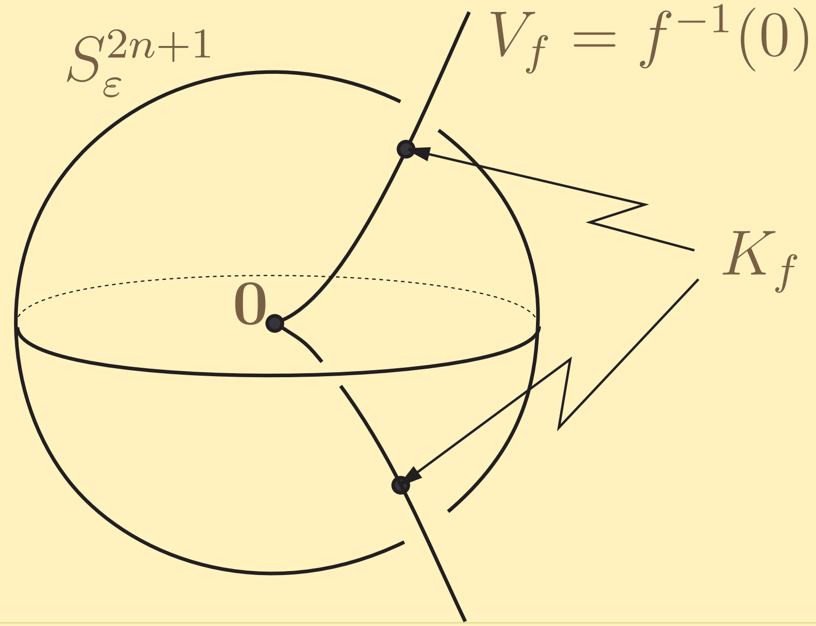

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(1) $\varphi_f = f/|f| : S_\varepsilon^{2n+1} \setminus K_f \to S^1$ *is a locally trivial fibration.*

(2) K_f *is* $(n-2)$*-connected, i.e.,* $\pi_i(K_f) = 0\ \forall i \leq n-2$.

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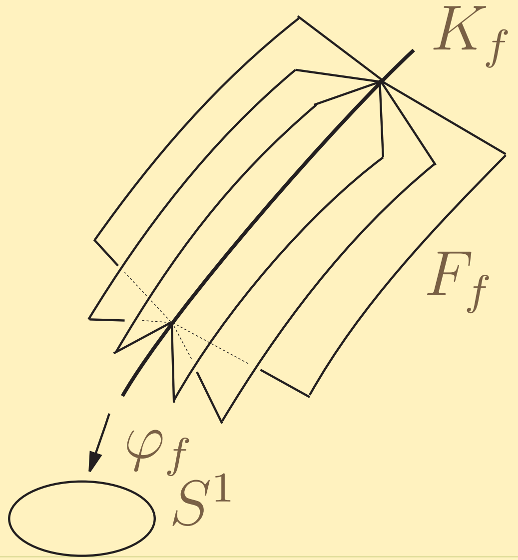

Theorem 1.3 (S., 1989) *For complex polynomials f, g, the following are equivalent.*

1. *The knots $(S^{2n+1}_{\varepsilon}, K_f)$ and $(S^{2n+1}_{\varepsilon'}, K_g)$ are diffeomorphic.*
2. *$\Phi(f^{-1}(0)) = g^{-1}(0)$ for some homeomorphism germ $\Phi : (\mathbf{C}^{n+1}, \mathbf{0}) \to (\mathbf{C}^{n+1}, \mathbf{0})$.*
3. *For some homeomorphism germs Φ and φ, the following diagram commutes.*

$$\begin{array}{ccc} (\mathbf{C}^{n+1}, \mathbf{0}) & \xrightarrow{\quad f \quad} & (\mathbf{C}, 0) \\ \Phi \downarrow & & \downarrow \varphi \\ (\mathbf{C}^{n+1}, \mathbf{0}) & \xrightarrow{\quad g \quad} & (\mathbf{C}, 0) \end{array}$$

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If $1, 2$ or 3 holds, then f and g have the same **topological type**.
If Φ, φ can be chosen to be orientation preserving ($\Leftrightarrow K_f$ and K_g are oriented isotopic), then they have the same **oriented topological type**.

Some invariants

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We denote by $\Delta_f(t)$ the **characteristic polynomial** of

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Milnor number and characteristic polynomial are **topological invariants**; i.e. if f and g have the same topological type, then $\mu_f = \mu_g$ and $\Delta_f(t) = \Delta_g(t)$.

The **Seifert form** associated with f is the bilinear form
$L_f : H_n(F_f;\mathbf{Z}) \times H_n(F_f;\mathbf{Z}) \to \mathbf{Z}$ define by
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- a and b are n-cycles representing $\alpha, \beta \in H_n(F_f; \mathbf{Z})$,
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In general, Seifert forms are very difficult to compute.

Problem 1.5 *For a given f, compute the Seifert form L_f.*

§2. Quasi-homogeneous Polynomials

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In the following, we always assume $\forall$weights ≥ 2.

Milnor–Orlik formulas

For $\Delta(t) = (t - \alpha_1)(t - \alpha_2)\cdots(t - \alpha_k) \in \mathbf{C}[t]$, $\alpha_\ell \in \mathbf{C}^*$, set

$$\operatorname{div}\Delta = \langle\alpha_1\rangle + \langle\alpha_2\rangle + \cdots + \langle\alpha_k\rangle \in \mathbf{QC}^*.$$

Set $\Lambda_m = \operatorname{div}(t^m - 1)$.

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Theorem 2.1 (Milnor–Orlik, 1970) *f: quasi-homogeneous of weights $(w_1, \ldots, w_{n+1}) = (u_1/v_1, \ldots, u_{n+1}/v_{n+1})$, where $v_j > 0$ and $\gcd(u_j, v_j) = 1$. Then, we have the following.*

(1) $\mu_f = (w_1 - 1)(w_2 - 1)\cdots(w_{n+1} - 1)$.

(2) $$\operatorname{div}\Delta_f = \left(\frac{1}{v_1}\Lambda_{u_1} - 1\right)\left(\frac{1}{v_2}\Lambda_{u_2} - 1\right)\cdots\left(\frac{1}{v_{n+1}}\Lambda_{u_{n+1}} - 1\right).$$

μ-constant deformation

Definition 2.2 *Let f_s, $s \in [0,1]$, be an (analytic) family of polynomials.*
(1) *It is a* **μ-constant deformation** *if the Milnor number μ_{f_s}, $s \in [0,1]$, is constant.*
(2) *It is a* **topologially constant deformation** *if f_s have the same topological types for all $s \in [0,1]$.*

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Theorem 2.3 (Steenbrink, Varchenko, etc.)
For quasi-homogeneous polynomials f and g, the following are equivalent.

(1) *They are connected by a* **μ-constant** *deformation.*
(2) *They are connected by a* **topologically constant** *deformation.*
(3) *They have the* **same weights***.*
(4) *They have the* **same "spectrum"***.*

Brieskorn–Pham polynomial

For $a_1, a_2, \ldots, a_{n+1} \geq 2$, set

$$f(z_1, z_2, \ldots, z_{n+1}) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}},$$

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Brieskorn–Pham polynomial $z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}$
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Topology of B-P polynomials

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In fact, we have the following.

Theorem 2.4 (Yoshinaga–Suzuki, 1978)
For two Brieskorn–Pham polynomials f and g, the following three are equivanent.

(1) f *and* g *have the same topological type.*
(2) f *and* g *have the same set of exponents.*
(3) $\Delta_f(t) = \Delta_g(t)$.

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Conjecture 2.5 (Folklore)

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1. $n = 1$: **Yoshinaga–Suzuki** 1979, Nishimura 1986
2. $n = 2$: **S.** 1988, **Xu-Yau** 1989, S. 2000
3. When f has weights of the form $(u_1/v_1, \ldots, u_{n+1}/v_{n+1})$ with $u_1 = \cdots = u_{n+1}$ even: **S.** 1998.

Example

Set $f = z_1^2 z_2 + z_1 z_2^6$, $g = z_1^3 z_2 + z_1 z_2^4$, which are quasi-homogeneous of weights $(11/5, 11)$ and $(11/3, 11/2)$, respectively.
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S. (1987): For $n = 2$, every quasi-homogeneous polynomial $h(z_1, z_2, z_3)$ with $\Delta_h(1) = 1$ has the topological type of a Brieskorn-Pham polynomial.

§3. Cobordism

Definition 3.1 Oriented $(2n-1)$-knots K_0 and K_1 in S^{2n+1} are **cobordant** if $\exists X (\cong K_0 \times [0,1]) \subset S^{2n+1} \times [0,1]$, a properly embedded oriented $2n$-dim. submanifold, such that

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Cobordism of knots

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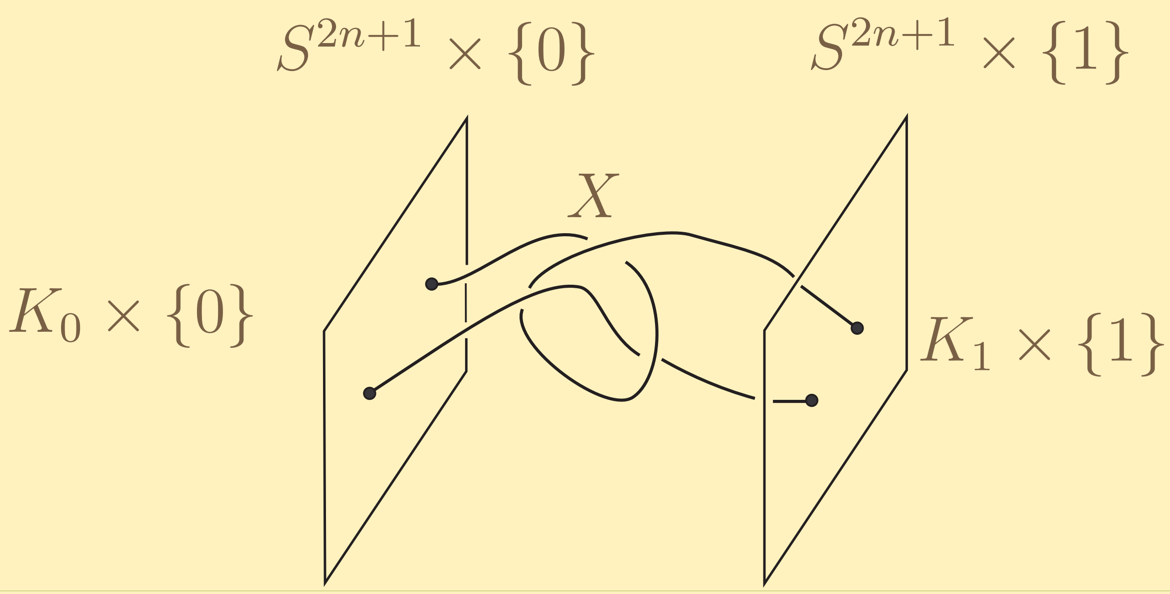

Problem

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Problem 3.2 *Given f and g, determine whether K_f and K_g are cobordant.*

Known results

Theorem 3.3 (Lê, 1972)

f, g: 2-variable polynomials, irreducible at $\mathbf{0}$.
The following are equivalent.
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It has long been conjectured that cobordant algebraic knots would be isotopic for all n.

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(1) f and g have the same topological type.
(2) K_f and K_g are cobordant.
(3) $\Delta_f(t) = \Delta_g(t)$.

It has long been conjectured that cobordant algebraic knots would be isotopic for all n.
This conjecture was negatively answered almost twenty years later.

Known results

Theorem 3.3 (Lê, 1972)
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du Bois–Michel (1993)
Examples of two algebraic knots that are cobordant, but are not isotopic, $n \geq 3$.

Let $L_i : G_i \times G_i \to \mathbf{Z}$, $i = 0, 1$, be two bilinear forms defined on free $\mathbf{Z}$-modules of finite ranks.

Algebraic cobordism

Let $L_i : G_i \times G_i \to \mathbf{Z}$, $i = 0, 1$, be two bilinear forms defined on free $\mathbf{Z}$-modules of finite ranks.
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Theorem 3.5 (Blanlœil–Michel, 1997) *For $n \geq 3$, two algebraic knots K_f and K_g are cobordant $\Longleftrightarrow$ Seifert forms L_f and L_g are algebraically cobordant.*

Remark 3.6

At present, there is no efficient criterion for algebraic cobordism. It is usually very difficult to determine whether given two forms are algebraically cobordant or not.

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Witt equivalence

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Lemma 3.7 *If two algebraic knots K_f and K_g are cobordant, then their Seifert forms L_f and L_g are Witt equivalent over $\mathbf{R}$.*

Criterion for Witt equiv. over $\mathbb{R}$

Theorem 3.8 (Blanlœil–S., 2011) *Let f and g be quasi-homogeneous polynomials. Seifert forms L_f and L_g are* **Witt equivalent over $\mathbb{R}$** *iff $P_f(t) \equiv P_g(t) \mod t+1$.*

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Here, $$P_f(t) = \prod_{j=1}^{n+1} \frac{t - t^{1/w_j}}{t^{1/w_j} - 1}.$$

The above theorem should be compared with the following.

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The above theorem should be compared with the following.

Theorem 3.9 (S., 2000) *For f, g as above, L_f and L_g are* **isomorphic over $\mathbb{R}$** *iff $P_f(t) \equiv P_g(t) \mod t^2 - 1$.*

Proposition 3.10 (Blanlœil–S., 2011) *Let*

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$

be Brieskorn–Pham polynomials.
Then, their Seifert forms are **Witt equivalent over $\mathbf{R}$** *iff*

$$\prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2a_j} = \prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2b_j}$$

holds for all odd integers ℓ.

Cobordism invariance of exponents

Theorem 3.11 (Blanlœil–S., 2011) *Suppose that for each of the Brieskorn–Pham polynomials*

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j},$$

no exponent is a multiple of another one.
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Problem 3.12 *Are the exponents cobordism invariants for Brieskorn–Pham polynomials in general?*

Case of two or three variables

Proposition 3.13 (Blanlœil–S., 2011) *Let f and g be quasi-homogeneous polynomials of two variables with weights (w_1, w_2) and (w'_1, w'_2).*
If their Seifert forms are Witt equivalent over $\mathbf{R}$, then $w_j = w'_j$, $j = 1, 2$, up to order.

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Let $f(z) = z_1^{a_1} + z_2^{a_2} + z_3^{a_3}$ and $g(z) = z_1^{b_1} + z_2^{b_2} + z_3^{b_3}$ be Brieskorn–Pham polynomials of three variables.
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These imply that weights and exponents are cobordism invariants!

Example

Set $f = z_1^5 + z_2^{31} + z_2 z_3^{75}$, $g = z_1^7 + z_2^{11} + z_3^{154}$, which are quasi-homogeneous of weights $(5, 31, 155/2)$ and $(7, 11, 154)$, respectively.

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Problem 3.15 *Are weights cobordism invariants for quasi-homogeneous polynomials of 3 variables?*

鈴木先生，還暦おめでとうございます！