Chapter 2. Presentation of Links

December 13, 2010

Sections

Knots, links and their equivalence

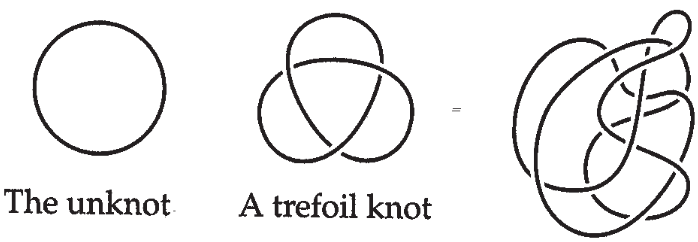
Diagrams of links

Complexity of link diagrams

Braid presentation of links

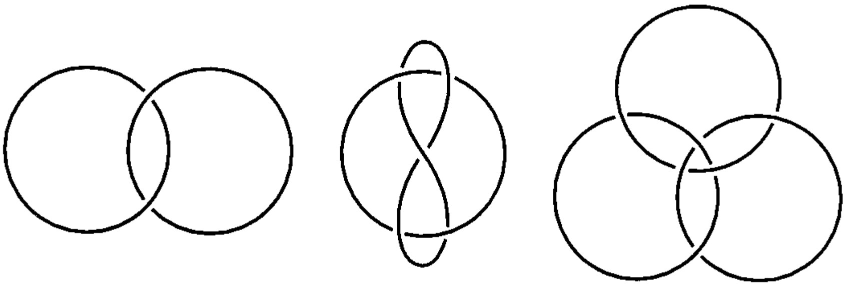
Section

Forming a knot



A closed curve in 3-space $\mathbb{R}^3$ without self intersection is called a knot.

Forming a link



A union of finitely many disjoint knots is called a link.

The individual knots that make up a link are called its components
(So a knot is a link with just one component).

Definition of a knot and link

Definition (2.1)

(1) A link L of μ-components is a smooth imbedding

$$f : S^1_1 \cup S^1_2 \cup \cdots \cup S^1_\mu \longrightarrow S^3, S^1_i \cong S^1$$

of μ mutually disjoint unit circles S^1 into $S^3 := \mathbb{R}^3 \cup \{\infty\}$.
We write

$$L = f(S^1_1) \cup f(S^1_2) \cup \cdots \cup f(S^1_\mu) = K_1 \cup K_2 \cup \cdots \cup K_\mu .$$

Each $K_i (= f(S^1_i))$ is called the i-th component of a link L.
A link L with the only one component is called a knot.

(2) If each K_i of a link L is oriented by an assignment of a direction, then L is called an oriented link of μ components.

(3) The unit circle in $\mathbb{R}^2 \subset \mathbb{R}^3$ is called the trivial knot or the unknot.

Definition (2.2)

(1) Two links L and L' in S^3 are said to be of the same link type, or equivalent, if there exists an orientation preserving homeomorphism $h_1 : S^3 \to S^3$ such that $h_1(L) = L'$, or, equivalently, L and L' are ambient isotopic, i.e., there is a level-preserving isotopy

$$H : S^3 \times [0,1] \to S^3 \times [0,1] \text{ via } H(x,t) = (h_t(x), t)$$

such that

(i) for each $t \in [0,1]$, $h_t : S^3 \to S^3$ is a diffeomorphism,
(ii) h_0 is the identity, and $h_1(L) = L'$.

H is called an ambient isotopy. The equivalence classes of knots (resp. links) are called the knot (resp. link) types.

(2) If two links L and L' are oriented and h preserves the orientation of each component, then we say that L and L' are of the same oriented link type.

Note (2.3)

(1) If two links L and L' in $\mathbb{R}^3$ are equivalent, then the restriction $h = h_1|_{\mathbb{R}^3 - L} : \mathbb{R}^3 - L \to \mathbb{R}^3 - L'$ is a homeomorphism and so it induces an isomorphism $h_* : \pi_1(\mathbb{R}^3 - L, x_0) \cong \pi_1(\mathbb{R}^3 - L', h(x_0))$. Thus, if $\pi_1(\mathbb{R}^3 - L)$ and $\pi_1(\mathbb{R}^3 - L')$ are not isomorphic, then L and L' can not be the same link type.

(2) Let L be any link in $\mathbb{R}^3$. Then the complement $\mathbb{R}^3 - L$ is path connected.

(3) Let L be any link in $\mathbb{R}^3$. Then $\pi_1(\mathbb{R}^3 - L) \cong \pi_1(S^3 - L)$.

(4) Let K be the unknot in $\mathbb{R}^3$. Then $\pi_1(\mathbb{R}^3 - K) \cong \mathbb{Z}$.

Theorem (2.4. C. McA. Gordon and J. Luecke, 1989)

Let K and K' be two prime knots in S^3. Then K and K' are equivalent or K' is the mirror image of K if and only if

$$\pi_1(S^3 - K) \cong \pi_1(S^3 - K').$$

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Regular projection

Let L be a link in $\mathbb{R}^3$ and let $p : \mathbb{R}^3 \to \mathbb{R}^2$ be an orthogonal projection. We call a point $c \in p(L)$ a multiple point if $p^{-1}(c) \cap L$ contains more than one point.

Definition (2.5)

A link L in $\mathbb{R}^3$ is in general position with respect to $p : \mathbb{R}^3 \to \mathbb{R}^2$ or p is a regular projection for L if it satisfies the followings:

(i) The restriction map $p|_L : L \to \mathbb{R}^2$ is an immersion.

(ii) Each multiple point of $p(L)$ is a transverse double point.

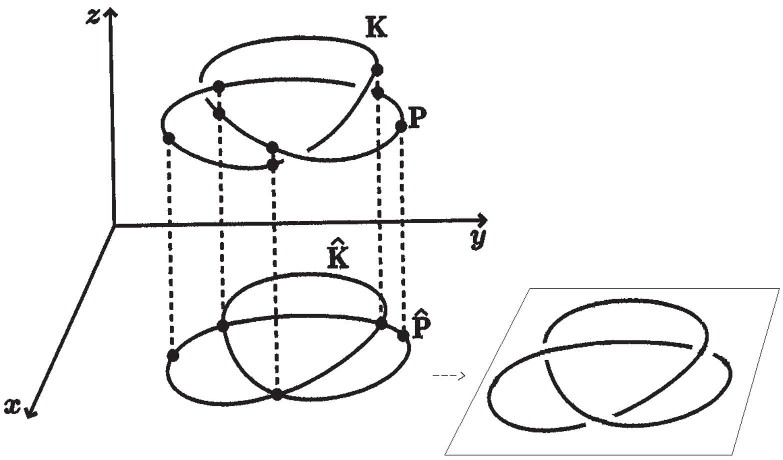
Proposition (2.6)

For any link $L \subset \mathbb{R}^3$, there exists a regular projection for L.

Proof.

See A. Kawauchi, A survey of Knot Theory, Birkhäuser 1996. □

For our convenience, we assume that the projection $p : \mathbb{R}^3 \to \mathbb{R}^2$ given by $p(x, y, z) = (x, y)$ is a regular projection for a link L in $\mathbb{R}^3$.



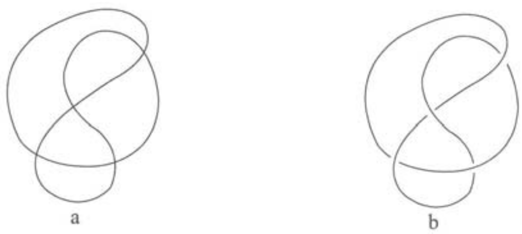
Definition (2.7)

(1) Each double point of $p(L)$ is called a crossing of L.

(2) A diagram D of a link L is a regular projection $p(L) = \widehat{L}$ in $\mathbb{R}^2$ with the information of over/under crossings at each double point of $\widehat{L}$.

Example (2.8)

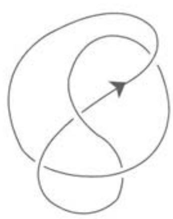
A regular projection of the figure eight knot and its diagram.



Note that if L is an oriented link in $\mathbb{R}^3$, then a diagram of L naturally inherit the orientation of L.

Example (2.9)

Oriented figure eight knot diagram:

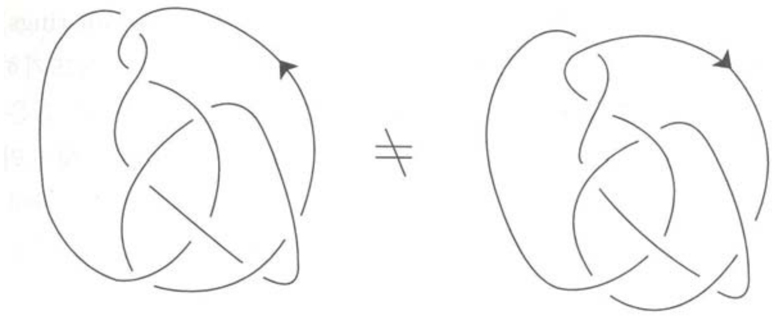


Invertibility of a link

A mirror image of a link L in $\mathbb{R}^3$ is a link which is the image of L by an orientation reversing homeomorphism of $\mathbb{R}^3$, and denoted by L^*. If L is an oriented link, a mirror image of L is also an oriented link.

Definition (2.10)

(1) A link L is said to be acheiral or amphicheiral if L is equivalent to L^*.
(2) A link L is said to be invertible if L is equivalent to $-L$, where $-L$ is the reverse of L, i.e., the link L with the opposite orientation.



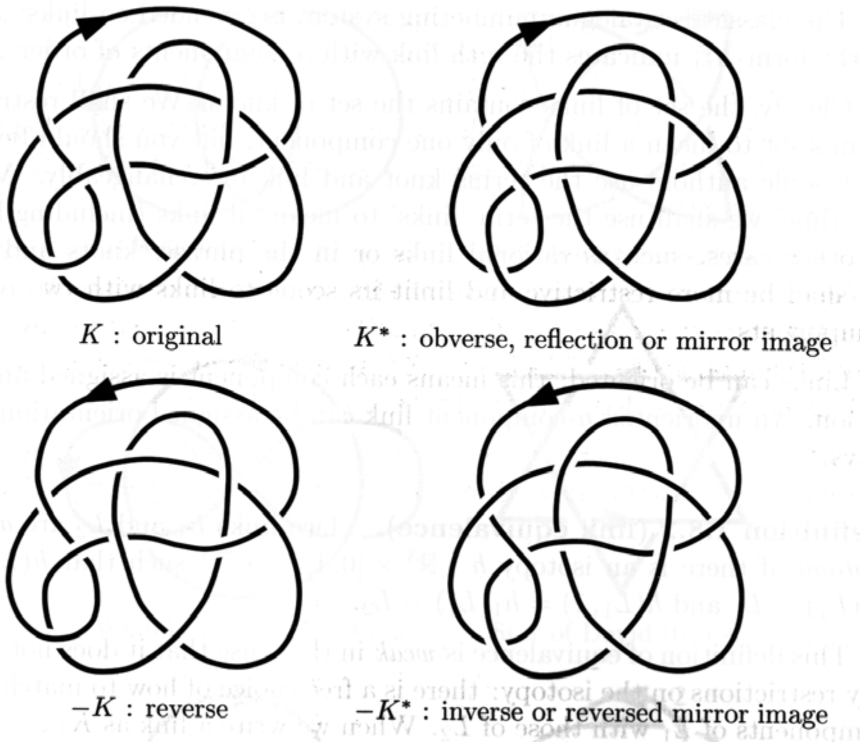


Figure: Non-invertible and cheiral knot 9_{33}

Reidemeister moves

Definition (2.11)

Two link diagrams D_1 and D_2 are said to be equivalent if D_1 can be transformed into D_2 by a finite sequence of Reidemeister moves of the type I, II, III, or plane isotopy.

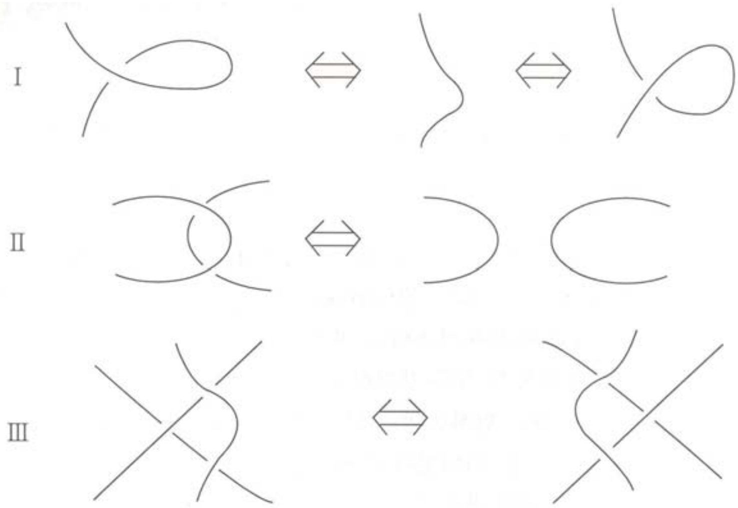


Figure: Reidemeister moves

Fundamental Theorem

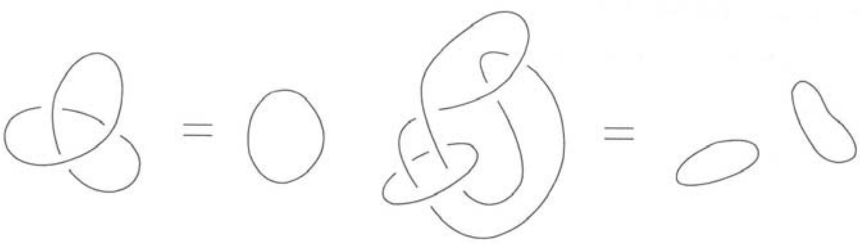
Theorem (2.12)

Two diagrams D_1 and D_2 represent the same link type if and only if D_1 and D_2 are equivalent.

Proof.

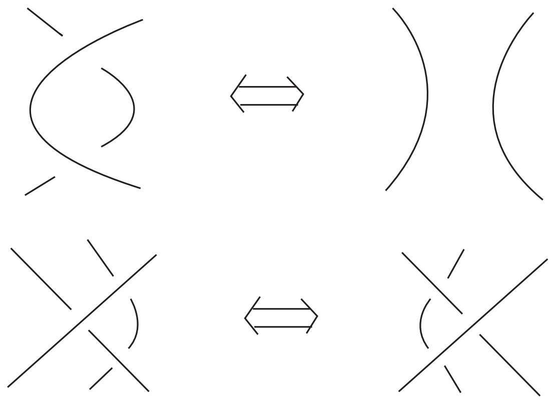
See Appendix A of the book: A. Kawauchi, A survey of Knot Theory, Birkhäuser 1996, or see G. Burde and H. Zieschang, Knots, de Gruter, 1985. □

Example (2.13)

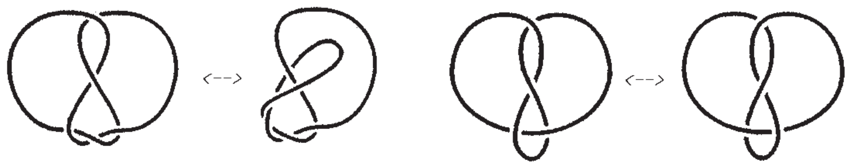


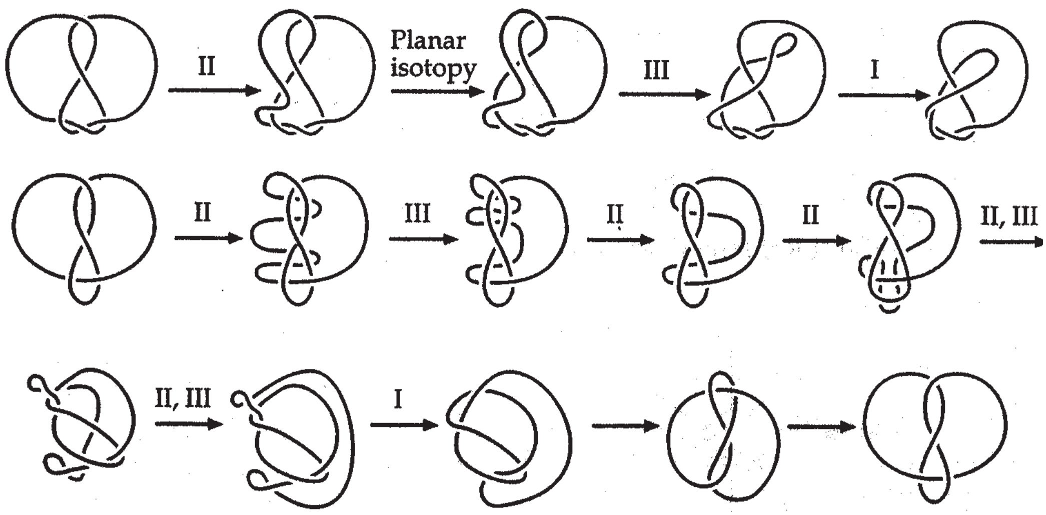
Note (2.14)

The Reidemeister moves I, II, III above generate the following type moves:

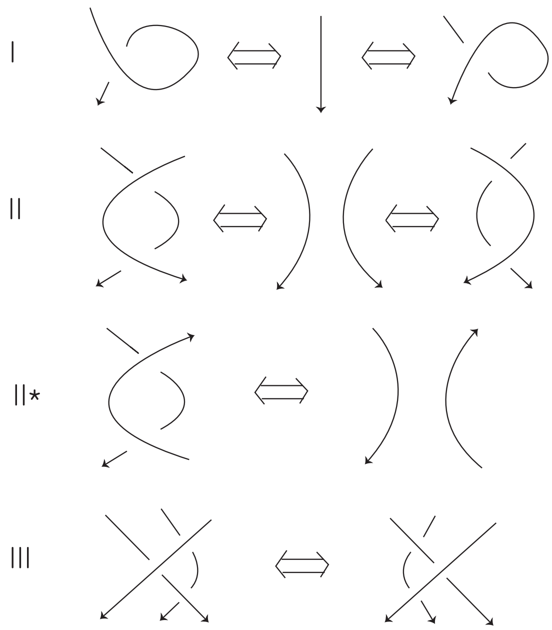


Example (2.15)





The oriented Reidemeister moves



Note (2.16)

The oriented Reidemeister moves $I, II, II*, III$ generate all other possible oriented Reidemeister moves of the type I, II, and III.

Fundamental problems of knot theory

Problem (Recognition Problem)

For given two links L and L′, determine whether or not L and L′ are equivalent.

Problem (Classification Problem)

Create a complete table of knots (or links).

A complete table is one in which no two knots (or links) are equivalent, and a given arbitrary knot (or link) is equivalent to some knot (or link) in this table.

Definition (2.17. Link invariant)

A link invariant is a function from the set of links to some other set whose value depends only on the equivalence class of the link.

Remark (2.18)

(1) Any representative of the equivalence class of a link can be chosen to calculate the invariant.

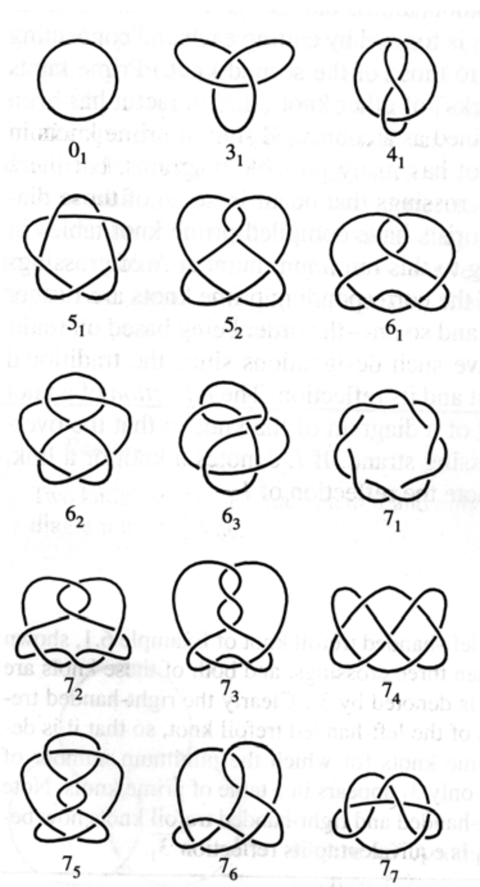
(2) There is no restriction on the kind of objects in the target space. For example, they could be integers, polynomials, matrices, or groups.

Example (2.19)

Link invariants:

- The number $\mu(L)$ of components of a link L.
- The fundamental group $\pi_1(\mathbb{R}^3 - L)$ of the link complement $\mathbb{R}^3 - L$.
- $\cdots\cdots$

Table of all prime knots with 7 or fewer crossings



Splittable link

Definition (2.20)

(1) A link L in $\mathbb{R}^3$ is said to be splittable if there is a 2-sphere S embbedded in the link complement $\mathbb{R}^3 - L$ such that there are some components of L on each side of S. A link that is not splittable is called a non-splittable link.

(2) For a splittable link L, let U_1 and U_2 be two components of $\mathbb{R}^3 - S$ and let $L_i = U_i \cap L$. Then we write $L = L_1 + L_2$, which is called the split union of L.

Fact. Any link can be expressed as a split union of a finite number of non-splittable links.

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Braid presentation of links

The crossing number $c(L)$

Let D be a diagram of a link L. The crossing number of D, denoted by $c(D)$, is defined to be the number of the crossings of D.

Definition (2.21)

The crossing number of a link L, denoted by $c(L)$, is defined to be the minimum

$$c(L) = \min\{c(D) \mid D \text{ is a diagram of } L\}.$$

Note (2.22)

(1) The assignment $L \mapsto c(L)$, the crossing number of L, is a link invariant.

(2) $c(\text{the trefoil knot}) = 3$; $c(\text{the figure-eight knot}) = 4$.

(3) There are no non-trivial knots with the crossing number ≤ 2.

The warp degree $d(D)$

Let $D = D_1 \cup D_2 \cup \cdots \cup D_r$ be a diagram of a link L with r components.

A point of D that is not a crossing is called a single point of D.

A sequence $\mathbf{a} = (a_1, a_2, \ldots, a_r)$ is called a sequence of base points of D if each a_i is a single point of D with $a_i \in D_i$.

Definition (2.23)

Let $\mathbf{a} = (a_1, a_2, \ldots, a_r)$ be a sequence of base points of an oriented diagram D. Then D is said to be monotone with respect to $\mathbf{a}$ if $(D, \mathbf{a})$ satisfies the followings conditions:

(1) For each $i = 1, 2, \ldots, r$, when one travels along D_i staring point a_i following the orientation of D_i, each crossing is first encountered on the over-crossing strand.

(2) If $i < j$, then at each crossing between D_i and D_j, D_i is the over-crossing strand.

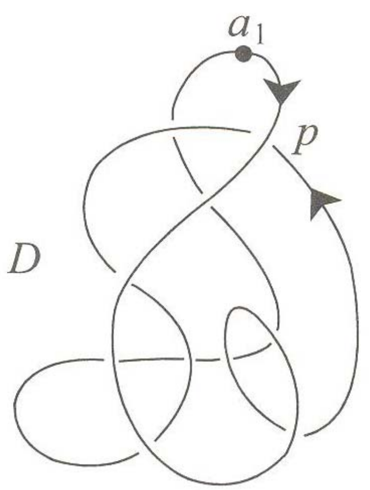


Figure: D is monotone w.r.t. $\mathbf{a} = (a_1)$

Lemma (2.24)

If a diagram D of r components is monotone with respect to a sequence of base points $\mathbf{a} = (a_1, a_2, \ldots, a_r)$ of D, then D can be transformed into a digram D^* by a finite sequence of the Reidemeister moves of type I, II, or III such that $c(D^*) = 0$.

Proof. Suppose that $D = D_1 \cup \cdots \cup D_r$ is a monotone diagram with respect to $\mathbf{a} = (a_1, a_2, \ldots, a_r)$. Then, by Reidemeister moves of type II or III, D is transformed into a split union $D = D_1 + \cdots + D_r$ of the components D_i of D. Hence it suffices to prove that the component D_1 satisfies the assertion. If $c(D_1) = 0$, then we have nothing to prove.

Now suppose that $c(D_1) \geq 1$. The proof will be done by induction on $c(D)$. If $c(D) = 1$, then by Reidemeister moves of type I, D is transformed to D^* with $c(D^*) = 0$.

Assume that $c(D) \geq 2$ and for each monotone diagram D' with $c(D') < c(D)$, the assertion follows.

Let p be the first crossing of D encountered when one travels along D_1 staring point a_1 following the orientation of D_1. Let D' and D'' be as in the figure below. Then D' is a monotone diagram with respect to p and $c(D') < c(D)$. □

By induction hypothesis, D' can be transformed into a digram D'^{*} such that $c(D'^{*}) = 0$ by a finite sequence of the Reidemeister moves of type I, II, or III. Hence D can be transformed into a digram D''. But, D'' is a monotone diagram with respect to a_1 and $c(D'') < c(D)$. By induction hypothesis, D'' can be transformed into a digram D^{*} such that $c(D^{*}) = 0$ by a finite sequence of the Reidemeister moves of type I, II, or III. This completes the proof.

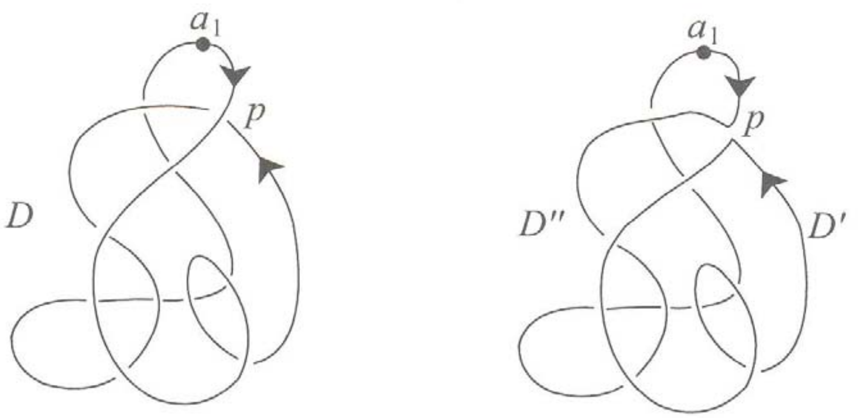


Figure: Diagrams D' and D''

Definition (2.25)

Let $\mathbf{a} = (a_1, a_2, \ldots, a_r)$ be a sequence of base points of an oriented diagram $D = D_1 \cup \cdots \cup D_r$.

(1) The warping degree $d_{\mathbf{a}}(D)$ of D with respect to $\mathbf{a}$ is defined to be the number of crossing changes, called warping crossing point, in D needed to produce the monotone diagram with respect to $\mathbf{a} = (a_1, a_2, \ldots, a_r)$.

(2) The warp degree $d(D)$ of D is the minimum

$$d(D) = \min\{d_{\mathbf{a}}(D) |\ \mathbf{a} \text{ is a sequence of base points of } D\}.$$

(3) The complexity $cd(D)$ of a diagram D is the ordered pair

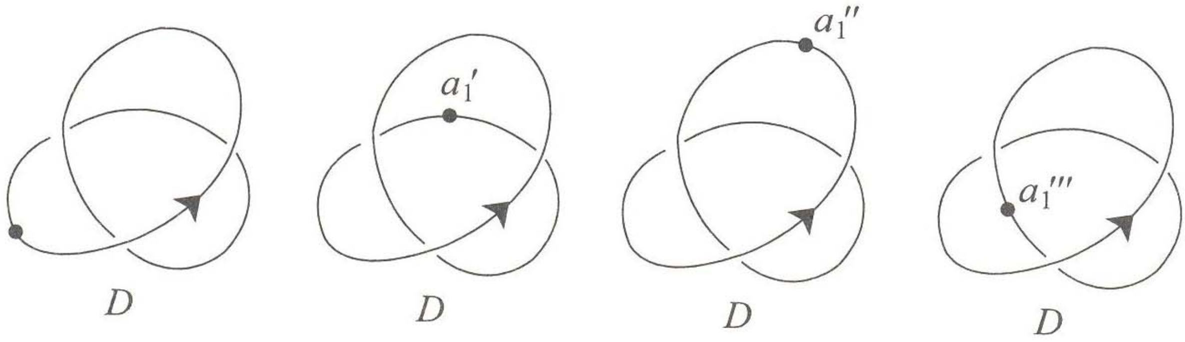
$$cd(D) = (c(D), d(D))$$

considered as the lexicographic order.

Example (2.26)

$D_{\mathbf{a_1}}(D) = 0, D_{\mathbf{a'_1}}(D) = 1, D_{\mathbf{a''_1}}(D) = 2, D_{\mathbf{a'''_1}}(D) = 3$. Hence

$$d(D) = 0 \text{ and } cd(D) = (3,0).$$



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Braid presentation of links

An n-string braid and Braid equivalence

Let $I^3 = \{(x,y,z) \mid 0 \leq x,y,z \leq 1\}$, the cube, and let n be an integer ≥ 1. Take the $2n$ points

$$p_i = (\frac{i}{n+1}, \frac{1}{2}, 1), \quad q_i = (\frac{i}{n+1}, \frac{1}{2}, 0), i = 1, 2, \ldots, n,$$

on the top and bottom of I^3.

Definition (2.27)

An *n-string braid* or simple *n-braid*, is a collection

$$b = s_1 \cup s_2 \cup \cdots \cup s_n$$

of an n mutually disjoint polygonal arcs s_i satisfying the following properties:

(1) $\partial b = \{p_1, \ldots, p_n, q_1, \ldots, q_n,\}$.

(2) Each arc s_i is monoton with respect to the z-coordinate.

We call the arc s_i the *i-th string* of a braid b.

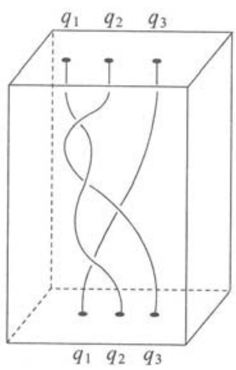


Figure: A 3-string braid

Definition (2.28)

(1) Two braids b_0 and b_1 are said to be *equivalent*, denoted by $b_0 \approx b_1$, if there is an ambient isotopy $f_t : I^3 \to I^3, t \in [0,1]$, such that $f_t|_{\partial I^3} = \mathrm{id}$, $f_0 = \mathrm{id}$, and $f_1(b_0) = b_1$.

(2) Two braids b_0 and b_1 are said to be *strongly equivalent* if there is an ambient isotopy $f_t : I^3 \to I^3, t \in [0,1]$, satisfying the properties of (1) above and the extra condition: for each $t \in [0,1]$, $f_t(b_0)$ is a braid.

The n-braid group B_n

By a regular projection $(x,y,z) \mapsto (x,z)$, we can obtain a (braid) diagram in $I^2 = \{(x,z) \mid 0 \le x, z \le 1\}$ of a braid $b \subset I^3$.

Fact. (1) Two braids b_0 and b_1 are equivalent if and only if they are strongly equivalent. (Artin, 1947)

(2) Two braids b_0 and b_1 are equivalent if and only if their diagrams can be transformed into each other by a finite sequence of Reidemeister moves in the interior of I^2.

For each $n \ge 1$, let

$$B_n = \{[b] \mid b \text{ is an } n\text{-string braid}\}.$$

Let $b_1 \subset I_1^3$ and $b_2 \subset I_2^3$ be two braids. We construct a new braid $b_1 b_2 \subset I_1^3 \cup I_2^3$ by attaching the bottom face of I_1^3 to the top face of I_2^3 naturally and then contracting the height of $I_1^3 \cup I_2^3$ to $\frac{1}{2}$. The resulting braid $b_1 b_2$ is called the product of b_1 and b_2.

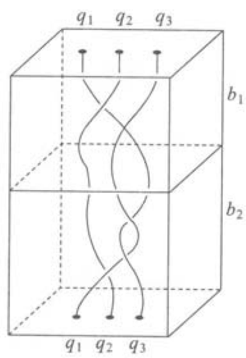


Figure: A 3-string braid

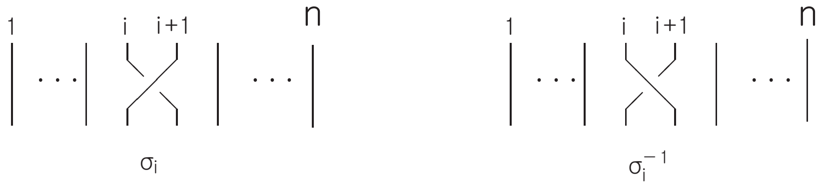
For $[b_1], [b_2] \in B_n$, we define $[b_1][b_2] = [b_1 b_2]$. Then

$$([b_1][b_2])[b_3] = [b_1]([b_2][b_3]),$$
$$[b][1] = [1][b], \text{ and } [b][b^{-1}] = [b^{-1}][b] = [1],$$

where 1 denotes the trivial braid, which is an n-braid in which each p_i is connected to q_i by a straight line seqment, and b^{-1} denotes the mirror image of b with respect to the plane $z = \frac{1}{2}$, called the inverse braid of b.

Under this product operation, B_n form a group, which is called an *n*-string braid group or *n*-braid group.

For each $i = 1, 2, \ldots, n-1$, let σ_i be the n-strand braid in B_n represented by the standard digram:



Theorem (2.29)

The n-braid group B_n has the following group presentation:

$$< \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_j = \sigma_j\sigma_i \ (|i-j| \geq 2),$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \ (1 \leq i \leq n-2) > .$$

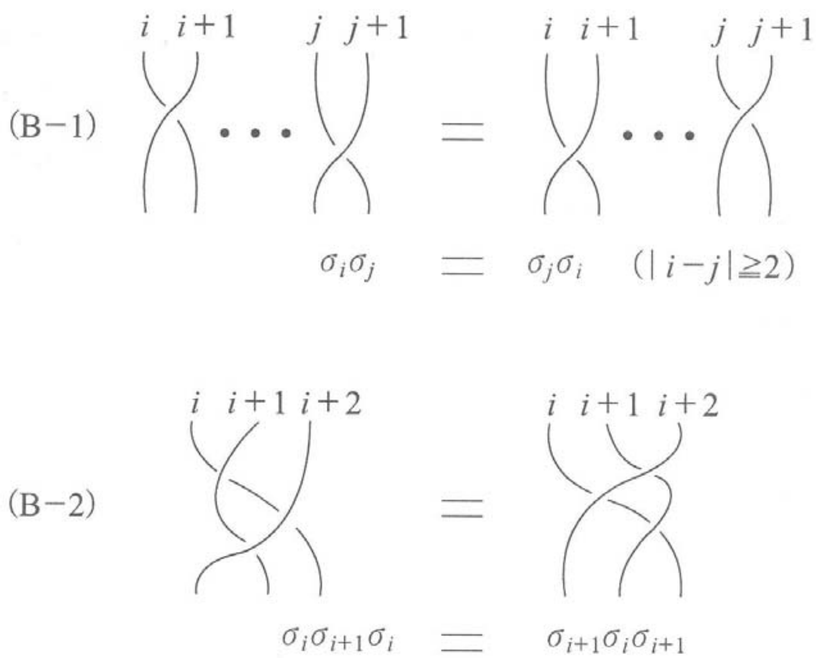


Figure: Relations of B_n

Example (2.30)

Braid words:

$$\sigma_2\sigma_3^{-1}\sigma_2\sigma_3^{-1}\sigma_1^{-2}\sigma_2\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_1^{-1},$$
$$(\sigma_3\sigma_2\sigma_1)^2\sigma_3\sigma_2\sigma_3^2\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3.$$

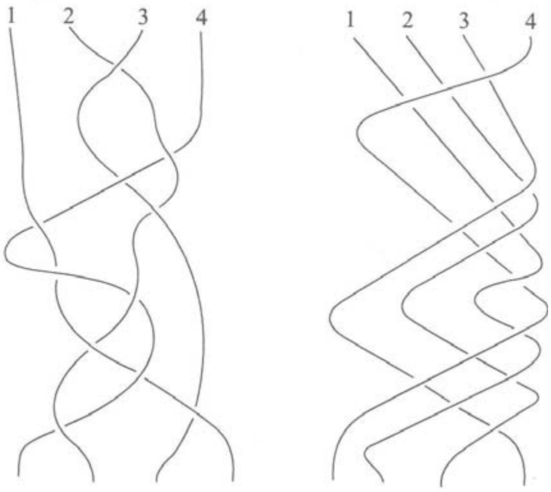


Figure: 4-string braids

A relationship between braids and links

Let $b \subset I^3 \subset \mathbb{R}^3$ be an n-braid in the cube located in $\mathbb{R}^3$. For each $i = 1, \ldots, n$, we connect the end points p_i and q_i by a simple polygonal arc α_i contained in the intersection of the exterior of I^3 and the half plane $y = \frac{1}{2}$ with $x \geq 0$ so that $\alpha_i \cap \alpha_j = \emptyset$ whenver $i \neq j$. Then we obtain a link in $\mathbb{R}^3$, which is called the closure of b or a closed braid, and denoted by $\mathrm{cl}(b)$. We choose an orientation for $\mathrm{cl}(b)$ as shown:

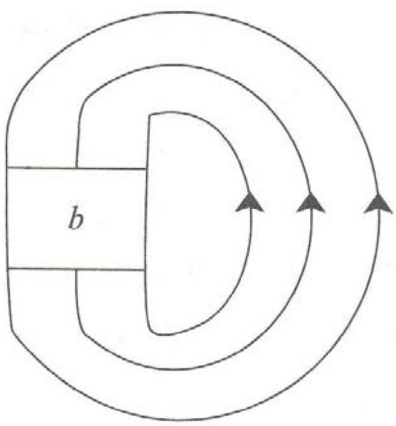


Figure: A diagram of the closure of a braid b

For a given closed n-braid (diagram) β, we can always obtain an n-braid (diagram) b whose closure $\mathrm{cl}(b)$ is equivalent to β as illustrated:

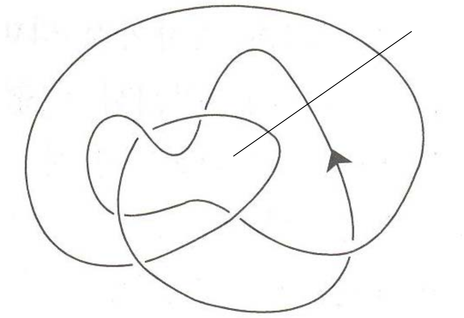


Figure: A closed 3-braid β

Cutting the closed braid β along the straight line as shown in the figure, we obtain a braid

$$b = \sigma_2^3 \sigma_1 \sigma_2^{-1} \sigma_1$$

such that $\mathrm{cl}(b)$ is equivalent to β.

Theorem (2.31. Alexander)

Any link in $\mathbb{R}^3$ and hence in S^3 is equivalent to the closure of an n-braid for some $n \geq 1$.

Proof.

- **Yamada's proof: S. Yamada, The minimal number of Seifert circles equals the braid index of a link, Invent. Math. 89 (1987), 347-356.**
- Birman's proof: J. S. Birman, Braids, links, and mapping class groups, Ann. Math. Studies, 82 (1974).
- Morton's proof: H. R. Morton, Threading knot diagrams, Math. Camb. Phil. Soc., 99 (1986), 247-260.

□

Definition (2.32)

The braid index of a link L is the minimum number of braid strings among all braid presentations for L, i.e., all braids whose closures are equivalent L.

Yamada's proof gives us the following

Corollary (2.33)

The minimum number of Seifert circles of all diagrams of a given link is equal to the braid index of the link.

Markov moves on braids

For any two integers m, n with $m < n$, we consider that $B_m \subset B_n$ by identifying each generator $\sigma_i \in B_m$ with the generator $\sigma_i \in B_n$ $(i = 1, 1, \ldots, m-1)$. For our convenience, we denote an n-string braid b by an ordered pair (b, n). Let

$$\mathbb{B} = \{(b, n) \mid b \in B_n, n = 1, 2, 3, \ldots\}.$$

Definition (2.34)

(1) The Markov moves of type I, or a conjugacy move, is a transformation of braids in $\mathbb{B}$ defined by

$$\text{I. } (b_1 b_2, n) \longleftrightarrow (b_2 b_1, n).$$

(2) The Markov moves of type II, or a stabilization, is a transformation of braids in $\mathbb{B}$ defined by

$$\text{II. } (b, n) \longleftrightarrow (b\sigma_n^{\pm 1}, n+1).$$

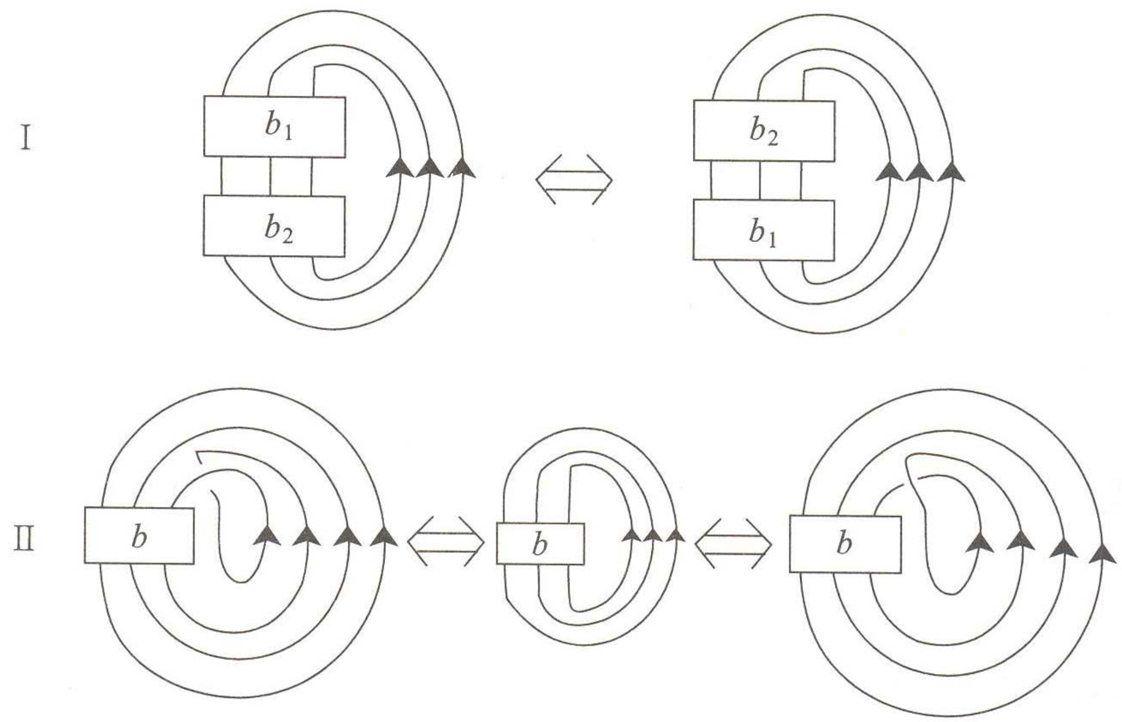


Figure: Markov moves

Definition (2.35)

Two element (b, n) and (b', n') in $\mathbb{B}$ are said to be Markov equivalent if they can be transformed into each other by a finite sequence of Markov moves of type I or II.

Theorem (2.36. Markov)

For two braids (b, n) and (b', n') in $\mathbb{B}$, the closed braids $\mathrm{cl}(b)$ and $\mathrm{cl}(b')$ are the same link type if and only if (b, n) and (b', n') are Markov equivalent.

Proof.

See J. S. Birman, Braids, links, and mapping class groups, Ann. Math. Studies, 82 (1974). □

According to Alexander theorem and Markov theorem, the study of knots and links in S^3 is equivalent to the study of the Markov equivalence classes of the braid groups.

Chapter 3. Elementary Topology of Links

December 14, 2010

Sections

Seifert surfaces

The linking number

Seifert surface and the intersection number

Section

Seifert surfaces

The linking number

Seifert surface and the intersection number

Seifer's Algorithm

Definition (3.1)

(1) A (Seifert) surface of a link L is a connected orientable surface F whose boundary ∂F is ambient isotopic to L.

(2) The genus of an oriented link L, denoted by $g(L)$, is the minimum genus of any Seifert surface of L. The genus of an unoriented link L is the minumum taken over all possible choices of orientation for L.

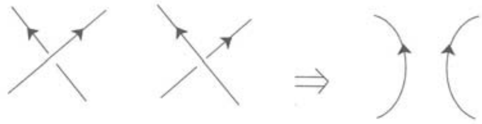
Clearly, the assignment $L \mapsto g(L)$ is a link invariant.

H. Seifert (1934) gave an algorithm which starts with a diagram of a given link L and produces a connected orientable surface F that spans L, i.e., $\partial F = L$, which is now called Seifert Algorithm:

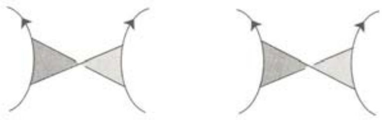
Theorem (3.2. H. Seifert)

Any link L in $\mathbb{R}^3$ has a Seifert surface F in $\mathbb{R}^3$.

Proof. (Sketch) Let D be an oriented diagram of a link L. The Seifert circles of D are simple closed curves obtained from D by smoothing each crossing as



We denote by $s(D)$ the number of the Seifert circles of D. Seifert circles may be nested. In this case, we rearrange the circles so that they have distinct heights in $\mathbb{R}^3$.
Now for each Seifert circle, we take a disc. Finally, attaching a small half-twisted band at the site of each crossing as illustrated in the figure:



we then obtain a surface F that spans L.
It remains to show that F is orientable (Exercise!).

The Canonical Genus

A Seifert surface for a link L constructed via Seifert's algorithm for a diagram D is called the canonical Seifert surface associated with D.

Definition (3.3)

The minimum genus over all canonical Seifert surfaces for L is called the canonical genus for L, denoted by $g_c(L)$.
Clearly, the assignment $L \mapsto g_c(L)$ is a link invariant.

The following theorem (3.4) and its corollary (3.5) are sometimes useful to calculate the canonical genus $g_c(L)$ of a link L.

Theorem (3.4)

Let D be a link diagram. The Euler characteristic $\chi(F)$ of the canonical Seifert surface F associated with D is given by

$$\chi(F) = s(D) - c(D). \tag{1}$$

Proof.

Let F be the canonical Seifert surface associated with D. We may consider F as a disc-band surface. We choose a triangulation of F as follows. A disc at which n bands attached is divided into $2n$ triangles with a vertex in its interior, and each band is divided into two triangles.

Let J denote the total number of joins where a band is attached to a disc. Each rectangle is attached at two ends and so $J = 2c(D)$. There are $2J$ triangles in the discs and two in each band so that the number of faces in the triangulation is $2J + 2c(D)$. It is easily seen that there are $2J + s(D)$ vertices and $4J + 3c(D)$ edges. Hence

$$\chi(F) = (2J + s(D)) - (4J + 3c(D)) + (2J + 2c(D)) = s(D) - c(D).$$

□

Corollary (3.5)

Let D be a link diagram. The genus $g(F)$ of the canonical Seifert surface F associated with D is given by

$$g(F) = \frac{1 - s(D) + c(D)}{2} + \frac{1 - \mu(D)}{2}, \tag{2}$$

where $\mu(D)$ denotes the number of components of D. In particular, if D is a knot diagram, then

$$g(F) = \frac{1 - s(D) + c(D)}{2}. \tag{3}$$

Proof.

Let $\overline{F}$ be the closed orientable surface obtained from F by capping off the boundary circles of F. Then $2g(F) = 2g(\overline{F}) = 2 - \chi(\overline{F}) = 2 - (\chi(F) + \mu(D)) = (1 - \chi(F) + (1 - \mu(D))$. By the previous theorem (3.4), we obtain the equality (2). □

Note (3.6)

(1) In general, the following inequality holds:

$$g(L) \leq g_c(L). \tag{4}$$

(2) A knot K is the trivial knot if and only if $g(K) = 0$. However, this is not true for a link. For example, consider the Hopf link H, which is not a trivial link. But, $g(H) = g_c(H) = 0$.

(3) $g(3_1) = g_c(3_1) = 1;\ g(4_1) = g_c(4_1) = 1.$

Definition (3.7)

An alternating knot is a knot with a diagram that has crossings that alternate between over and under as one travels around the knot in a fixed direction.

It is known (cf. K. Murasugi and D. Gabai) that K is an alternating knot, then the equality in (4) holds, i.e., $g(K) = g_c(K)$.

Connected sum of links

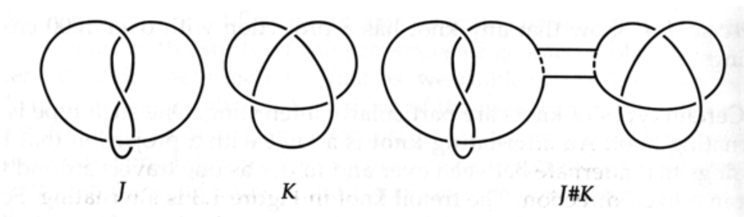


Figure: The connected sum $J \sharp K$ of two knots J and K

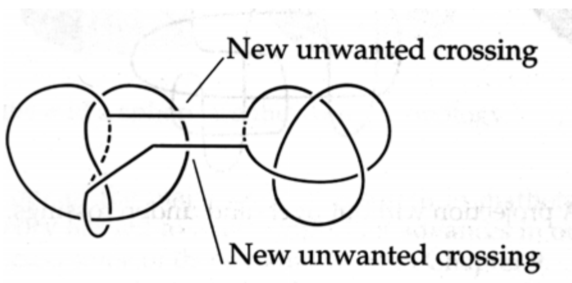


Figure: Not the connected sum of J and K

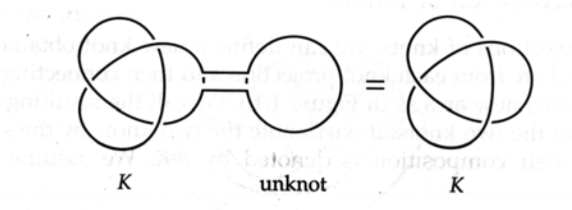


Figure: $K\sharp$ unknot is just K itself

Theorem (3.8. H. Schubert)

For any links L_1 and L_2, $g(L_1 \sharp L_2) = g(L_1) + g(L_2)$.

Proof.

See the text book! □

Example (3.9)

$$g(\underbrace{3_1 \sharp \cdots \sharp 3_1}_{n}) = \underbrace{g(3_1) + \cdots + g(3_1)}_{n} = n.$$

Definition (3.10)

If a link is not the connected sum of any two nontrivial links, we call it a prime link. A non-splittable link that is not prime is called a composite link. The links that make up the composite link are called factor links.

Theorem (3.11. H. Schubert, 1949)

A non-splittable composite link can be uniquely decomposed into a finite number of prime links, excluding the order.

Proof.

Omit! □

Section

Seifert surfaces

The linking number

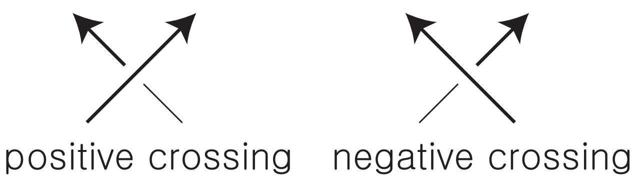
Seifert surface and the intersection number

Crossing sign and the writhe

Let p be a crossing in an oriented diagram D. Define the crossing sign $\varepsilon(p)$ of p by

$$\varepsilon(p) = \begin{cases} +1, & \text{if } p \text{ is a positive crossing;} \\ -1, & \text{if } p \text{ is a negative crossing,} \end{cases}$$

where



If we write "$p \in D$" to mean "p is a crossing in D", then the writhe (or algebraic crossing number) $w(D)$ of D is defined as

$$w(D) = \sum_{p \in D} \varepsilon(p).$$

The linking number

Let $D = D_1 \cup D_2$ be a diagram of the union $L_1 \cup L_2$ of two oriented links L_1 and L_2. Then the crossings of D are of three types: D_1 with itself, D_2 with itself, and D_1 with D_2. We shall concentrate on the last group and denote by $D_1 \cap D_2$.

Definition (3.12)

(1) The linking number $Link(D_1, D_2)$ of D_1 and D_2 is defined to be

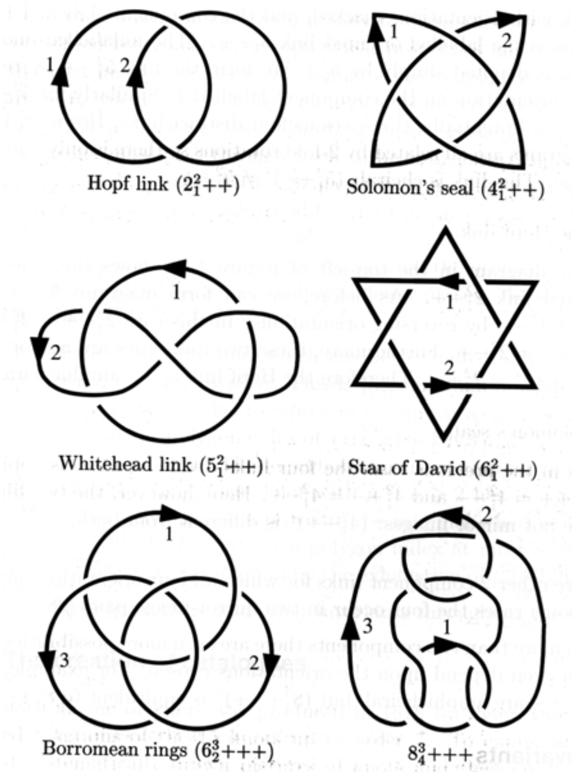
$$Link(D_1, D_2) = \frac{1}{2}(w(D) - w(D_1) - w(D_2)) = \frac{1}{2} \sum_{c \in D_1 \cap D_2} \varepsilon(c).$$

(2) Let $L = D_1 \cup \cdots \cup D_n$ be the union of n oriented links. We define the total linking number $Link(L)$ to be the sum of the linking numbers of all pairs of links, that is,

$$Link(L) = \begin{cases} \sum_{1 \le i < j \le n} Link(D_i, D_j), & \text{if } n \ge 2; \\ 0, & \text{if } n = 1 \text{ and } D_1 \text{ is a knot.} \end{cases}$$

Example (3.13)

Compute the total linking number of the following links:



Theorem (3.14)

Let $D = D_1 \cup D_2$ be a diagram of the union $L_1 \cup L_2$ of two oriented links L_1 and L_2. Then $Link(D_1, D_2)$ is an integer, which is invariant under the oriented Reidemeister moves I, II, III and so the asignment $L_1 \cup L_2 \mapsto Link(D_1, D_2)$ is a link invariant. Hence, we may write $Link(L_1, L_2) = Link(D_1, D_2)$.

The following corollary is immediate from the definition:

Corollary (3.15)

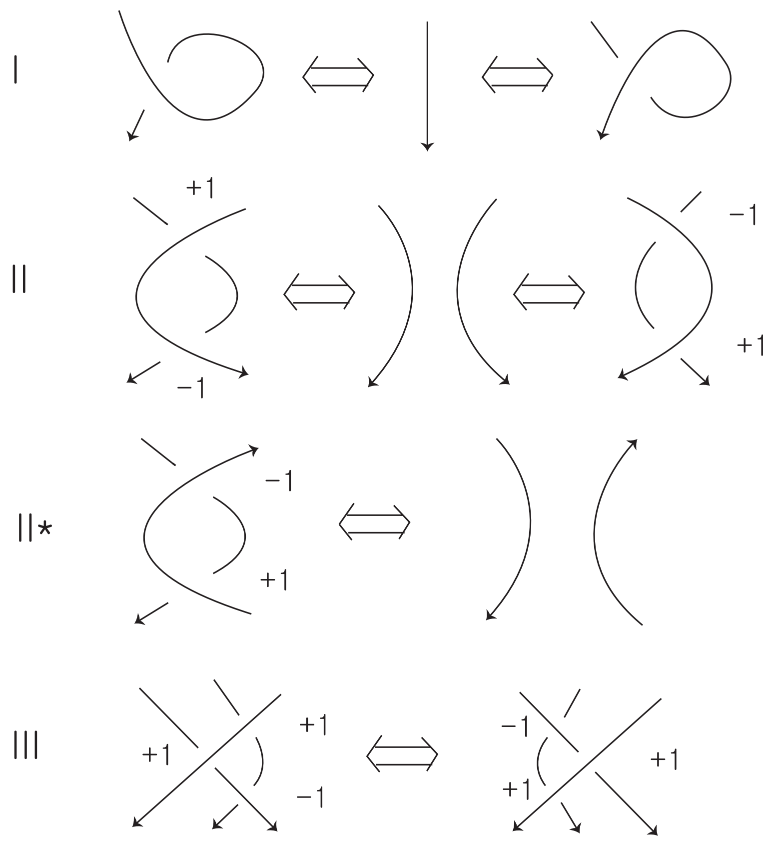
(1) $Link(L_1, L_2) = Link(L_2, L_1)$.

(2) Let $L_i = L_i' \cup L_i''(i = 1, 2)$. Then

$$\begin{aligned} Link(L_1, L_2) &= Link(L_1', L_2) + Link(L_1'', L_2) \\ &= Link(L_1, L_2') + Link(L_1, L_2''). \end{aligned}$$

(3) $Link(-L_1, L_2) = Link(L_1, -L_2) = -Link(L_1, L_2)$, *where* $-L_i$ *denote the link* L_i *with the opposite orientation.*

Proof of Theorem (3.14) (Sketch)

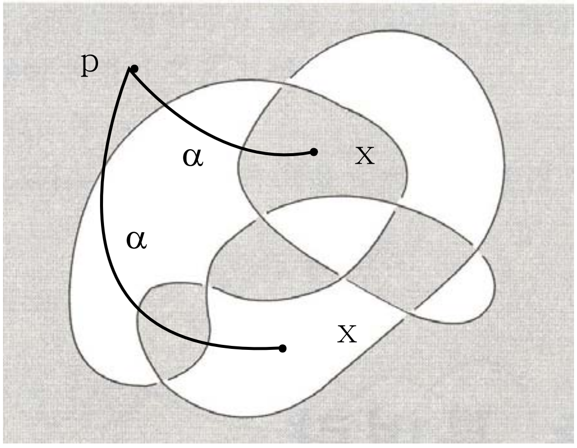


□

Theorem (3.16)

Any link diagram in $\mathbb{R}^2$ is checkerboard colorable.

Proof. First, we color the unbounded region, say X_∞, of $\mathbb{R}^2 - D$ to be a black region and let p be a point of X_∞. Let X be any region of $\mathbb{R}^2 - D$ which not the unbounded region and let x be a point in X. Choose a path α in $\mathbb{R}^2$ from p to x so that α intersetcts with D transversally and each intersection point is not a crossing of D.



If $|D \cap \alpha|$ is odd, we color X to be a white region, and if $|D \cap \alpha|$ is even, we color X to be a black region. Note that this coloring of $\mathbb{R}^2 - D$ does not depends on the choice of α.

Indeed, let α be another such a path from p to x. Then $\alpha \cup \alpha'$ is a simple closed curve intersecting D transversally. If we produce an over or under crossing at each transversal intersection point, we obtain a two component link $L = D \cup (\alpha \cup \alpha')$. We choose an arbitrary orientation for each component of L. Then

$$Link(D, \alpha \cup \alpha') = \frac{1}{2} \sum_{p \in D \cap (\alpha \cup \alpha')} \varepsilon(p)$$

and it is an integer from Theorem (2.14). Hence $|D \cap (\alpha \cup \alpha')| = |D \cap \alpha| + |D \cap \alpha'|$ must be even. This implies that $|D \cap \alpha|$ and $|D \cap \alpha'|$ have the same parity, completing the proof. $\square$

Twisting number

Definition (3.17)

Let $L = K_1 \cup \cdots \cup K_r$ be a r component link and let $L = D_1 \cup \cdots \cup D_r$ be a diagram of L. Then the number

$$t(D) = \sum_{i=1}^{r} w(D_i)$$

is called the twisting number of L.

Section

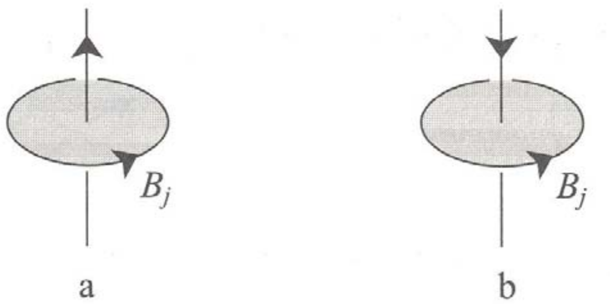
Seifert surfaces

The linking number

Seifert surface and the intersection number

The intersection number

Let L be an oriented link in $\mathbb{R}^3$ and let K be an oriented knot in $\mathbb{R}^3$. Let F be an oriented Seifert surface of L. Then we can perform a small isotopic deformation on F so that F intersects K transversally in a finitely many intersection points, say $x_1, \dots, x_m$. Let $B_1, \dots, B_m$ be sufficiently small mutually disjoint 2-disk neighborhoods of $x_1, \dots, x_m$, respectively. Then we have the following two cases (a) and (b):



The intersection point in (a) is called the positive intersection point ($Link(K, \partial B_j) = +1$). The intersection point in (b) is called the negative intersection point ($Link(K, \partial B_j) = -1$).

Definition (3.18)

Let p and q denote the number of the transversal positive and negative intersection points of K with F, respectively ($p + q = m$). The the number

$$\mathrm{Int}(F, K) = p - q = \sum_{j=1}^{m} Link(K, \partial B_j)$$

is called the intersection number of K with F.

Theorem (3.19)

$\mathrm{Int}(F, K) = Link(L, K)$.

Theorem (3.20)

Let L be an oriented link and let K be an oriented knot. T.A.E.

(1) $Link(L, K) = 0$.

(2) *There exists a Seifert surface F of L such that $F \cap K = \emptyset$.*

(3) *There exists a 2-chain c in $\mathbb{R}^3 - K$ such that $\partial c = L$.*

Chapter 4. Standard Examples of Links

December 15, 2010

Sections

Section

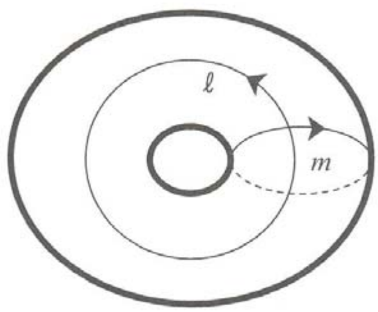
Torus Knots

Two-bridge links

Pretzel links

Meridean and Longitude

Let $T = S^1 \times S^1$ be the standard torus in $\mathbb{C}^2$ and let $m = S^1 \times \{1\}$ and $\ell = \{1\} \times S^1$ be the simple closed curves on T, which are called a meridian and a longitude of T, respectively. Choose the orientations on m and ℓ as shown:



Then

$$H_1(T) = \mathbb{Z} \oplus \mathbb{Z} = <[m]> \oplus <[\ell]>,$$

where $[m]$ and $[\ell]$ denote the the first homology class of m and ℓ.

The torus links

Definition (4.1)

(1) A torus knot of type (a,d), denoted by $T(a,d)$, is a knot K embedded in the the standard torus $T = S^1 \times S^1 \subset S^3$ such that $[k]$ is homologous to $a[m] + d[\ell]$ for some coprime intgers a and d.

(2) A torus link of type (na,nd) $(n \geq 2)$, denoted by $T(na,nd)$, is an n-component paralle link L embedded in the the standard torus $T = S^1 \times S^1 \subset S^3$ such that $[L]$ is homologous to $na[m] + nd[\ell]$ for some coprime intgers a and d.

Remark (4.2)

(1) If $|a| \leq 1$ or $|d| \leq 1$, then $T(a,d)$ is the trivial knot, and so we usually assume that $|a| \geq 2$ or $|d| \geq 2$.

(2) By the oriented torus link of type (na,nd) we mean the torus link of type (na,nd) in which all parallel components are oriented in the same direction.

Classification of the torus links

Theorem (4.3)

Let $T(a,d)$ be the torus link of type (a,d). Then

(1) *$T(a',d') = T(a,d)$ if and only if (a',d') is equal to one of $(a,d),(d,a),(-a,-d)$, and $(-d,-a)$.*

(2) *The mirror image of $T(a,d)$ is $T(a,-d)$.*

(3) *If $d \geq 2$, then torus link $T(a,d)$ is equivalent to the closure of the d-braid*

$$(\sigma_1\sigma_2\cdots\sigma_{d-1})^a.$$

Corollary (4.4)

(1) *The torus links are invertible.*

(2) *The torus links are not amphicheiral.*

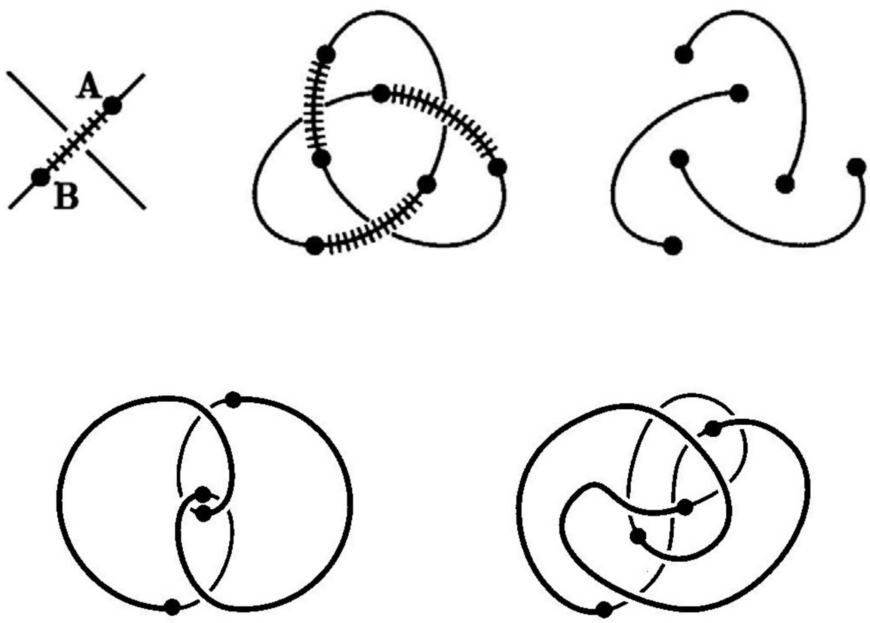
Section

Torus Knots

Two-bridge links

Pretzel links

The *n*-bridge links



Definition (4.5)

Let D be a diagram of a knot or link L. If we can divide up D into $2n$ curves $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$, i.e.,

$$D = \alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_n \cup \beta_1 \cup \beta_2 \cup \ldots \cup \beta_n,$$

that satisfy the conditions:

(1) $\alpha_1, \alpha_2, \ldots, \alpha_n$ are mutually disjoint, simple curves.
(2) $\beta_1, \beta_2, \ldots, \beta_n$ are mutually disjoint, simple curves.
(3) At the crossing points of D, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are arcs that pass over the crossing points, called the bridges of D. While at the crossing points of D, $\beta_1, \beta_2, \ldots, \beta_n$ are arcs that pass under the crossing points.

Then the bridge number of D, $\mathrm{br}(D)$, is said to be at most n. If $\mathrm{br}(D) \leq n$ but $\mathrm{br}(D) \nleq n-1$, then we define $\mathrm{br}(D) = n$.

Definition (4.6)

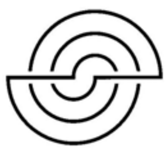
For a knot (or link) L, the least bridge number $br(L)$ defined by

$$\mathrm{br}(L) = \min_{D \text{ is a diagram of } L} \mathrm{br}(D)$$

is called the bridge number (or the bridge index) of L. If L is a knot (or link) in S^3 with $\mathrm{br}(L) = n$, then it is called an n-bridge knot (or link).

Note (4.7)

(1) The assignment $L \mapsto \mathrm{br}(L)$ is clearly a link invariant.
(2) $\mathrm{br}(K) = 1 \iff K$: the unknot. If K: the unknot, then $\mathrm{br}(K) \geq 2$.
(3) $\mathrm{br}(3_1) = 2$ and $\mathrm{br}(4_1) = 2$.

Conway normal form of 2-bridge links

For an integer $n \geq 1$, let $a_1, a_2, \dots, a_n$ be a finite sequence of nonzero integers and let b_n be a 3-braid given by

$$b_n = \begin{cases} \sigma_1^{a_1}\sigma_2^{-a_2}\sigma_1^{a_3}\cdots\sigma_2^{-a_n}, & n \text{ is even;} \\ \sigma_1^{a_1}\sigma_2^{-a_2}\sigma_1^{a_3}\cdots\sigma_1^{a_n}, & n \text{ is odd.} \end{cases}.$$

Let $C(a_1, a_2, \dots, a_n)$ be the link obtained from b_n by closing the braid string as shown:

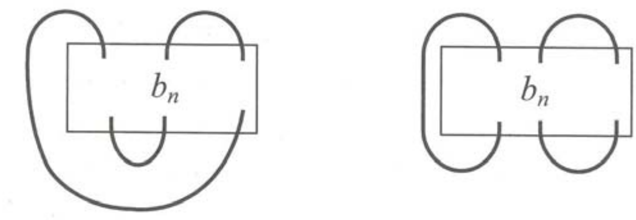


Figure: n:even n:odd

Then $C(a_1, a_2, \dots, a_n)$ becomes a 2-bridge link, which is called the Conway normal form.

Lemma (4.8)

(1) $C(a_1, a_2, \dots, a_n) = C(a_1, a_2, \dots, a_{n-1}, a_n + 1, -1)$.

(2) $C(a_1, a_2, \dots, a_n) = C(a_1, a_2, \dots, a_{n-1}, a_n - 1, +1)$.

(2) $C(a_n, \dots, a_2, a_1) = C((-1)^{n-1}a_1, (-1)^{n-1}a_2, \dots, (-1)^{n-1}a_n)$.

Definition (4.9)

A continued fraction is a finite formal expression of the form:

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}$$

where a_i are integers and $a_n \neq 0$. We denote it by $[a_1, a_2, \dots, a_n]$, i.e., $[a_1, a_2, \dots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \frac{1}{a_n}}}}$

Each continued fraction can be evaluated to give a rational number and, conversely, every rational number can be expanded as a continued fraction by using Euclid's division algorithm:

Input: two integers p and q with $\gcd(p, q) = 1$.

$r_{-1} := p$
$r_0 := q$
$i := -1$
repeat
  $i := i + 1$
  find a_{i+1} and r_{i+1} such that $r_{i-1} = a_{i+1} r_i + r_{i+1}$ and $0 \leq r_{i+1} < r_i$
  print a_{i+1}
until $(r_{i+1} = 0)$

Output: a sequence of integers $a_1, a_2, \ldots, a_n$ with

$$\frac{p}{q} = [a_1, a_2, \ldots, a_n].$$

$$\begin{aligned} p &= a_1 q + r_1 \\ q &= a_2 r_1 + r_2 \\ r_1 &= a_3 r_2 + r_3 \\ &\vdots \\ r_{n-3} &= a_{n-1} r_{n-2} + r_{n-1} \\ r_{n-2} &= a_n r_{n-1} + 0. \end{aligned}$$

$$\begin{aligned} \frac{p}{q} &= a_1 + \frac{r_1}{q} = a_1 + \frac{1}{q/r_1} \\ \frac{q}{r_1} &= a_2 + \frac{r_2}{r_1} = a_2 + \frac{1}{r_1/r_2} \\ \frac{r_1}{r_2} &= a_3 + \frac{r_3}{r_2} = a_3 + \frac{1}{r_2/r_3} \\ &\vdots \\ \frac{r_{n-3}}{r_{n-2}} &= a_{n-1} + \frac{r_{n-1}}{r_{n-2}} = a_{n-1} + \frac{1}{r_{n-2}/r_{n-1}} \\ \frac{r_{n-2}}{r_{n-1}} &= a_n. \end{aligned}$$

This gives a continued fraction expansion of $\frac{p}{q}$:

$$\frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}} = [a_1, a_2, \ldots, a_n].$$

Example (4.10)

Input $p = 159,\ q = 46.$

$$\begin{aligned} 159 &= 3 \cdot 46 + 21 \\ 46 &= 2 \cdot 21 + 4 \\ 21 &= 5 \cdot 4 + 1 \\ 4 &= 4 \cdot 1 + 0 \end{aligned}$$

$$\begin{aligned} \frac{159}{46} &= 3 + \frac{21}{46} = 3 + \frac{1}{46/21} \\ \frac{46}{21} &= 2 + \frac{4}{21} = 2 + \frac{1}{21/4} \\ \frac{21}{4} &= 5 + \frac{1}{4} = 5 + \frac{1}{4/1} \\ \frac{4}{1} &= 4 \end{aligned}$$

This gives

$$\frac{159}{46} = 3 + \frac{1}{2 + \frac{1}{5 + \frac{1}{4}}} = [3, 2, 5, 4].$$

Remark (4.11)

A continued fraction expansion is not unique. For an example,

$$\begin{aligned} \frac{159}{46} &= 3 + \frac{21}{46} = 4 - \frac{25}{46} = 4 - \frac{1}{46/25} \\ \frac{46}{25} &= 2 - \frac{4}{25} = 2 - \frac{1}{25/4} \\ \frac{25}{4} &= 6 + \frac{1}{4} = 6 + \frac{1}{4/1} \\ \frac{4}{1} &= 4 \end{aligned}$$

This gives

$$[3, 2, 5, 4] = \frac{159}{46} = 3 + \frac{1}{2 + \frac{1}{5 + \frac{1}{4}}} = 4 + \frac{1}{-2 + \frac{1}{6 + \frac{1}{4}}} = [4, -2, 6, 4].$$

Note (4.12)

If

$$\frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}} = [a_1, a_2, \ldots, a_n],$$

then

$$-\frac{p}{q} = -a_1 + \cfrac{1}{-a_2 + \cfrac{1}{-a_3 + \cfrac{1}{\ddots + \cfrac{1}{-a_n}}}} = [-a_1, -a_2, \ldots, -a_n].$$

For an example,

$$-\frac{159}{46} = -3 + \cfrac{1}{-2 + \cfrac{1}{-5 + \cfrac{1}{-4}}} = [-3, -2, -5, -4].$$

Convention. $\frac{1}{0} = \infty, \frac{1}{\infty} = 0, \; a + \infty = \infty \; (a \in \mathbb{Z})$.

Definition (4.13)

A continued fraction expansion is called regular if all the coefficients are positive, with the possible exception of a_1.

The following Lagrange's formula:

$$x + \frac{1}{-y} = (x-1) + \cfrac{1}{1 + \cfrac{1}{(y-1)}}$$

can be used to reduce the number of negative coefficient in a continued fraction expansion. Hence, by repeated use of Lagrange's formula, any continued fraction expansion can be converted into a regular expansion.

Theorem (4.14)

Let $C(a_1, a_2, \ldots, a_n)$ and $C(b_1, b_2, \ldots, b_m)$ be regular continued fractions that evaluate to the same rational number and $n \leq m$. Then

(i) *$m = n$ and $b_i = a_i$ for all i, or*

(ii) *$m = n + 1$ and $b_i = a_i$ for all $i < n$, and $b_n = a_n - 1$ and $b_m = 1$.*

Classification of 2-bridge links

Definition (4.15)

Let $L = C(a_1, a_2, \ldots, a_m)$ be a Conway normal form of a 2-bridge link L. Let $\frac{a}{p} = [a_1, a_2, \ldots, a_m]$, called the slop of L. If $p \geq 0$ and $(p, a) = 1$, then the ordered pair (p, a) is called the type of L.

Theorem (4.16)

(1) *A 2-bridge link $L = C(a_1, a_2, \ldots, a_n)$ is the unknot if and only if the type of L is $(1, m)$ for some $m \in \mathbb{Z}$.*

(2) *A 2-bridge link $L = C(a_1, a_2, \ldots, a_n)$ is the link unlink if and only if the type of L is $(0, 1)$.*

(3) *Two non trivial 2-bridge links $L = C(a_1, a_2, \ldots, a_n)$ and $L' = C(a'_1, a'_2, \ldots, a'_m)$ without orientation are equivalent if and only if the types (p, a) and (p', a') of L and L' satisfy that either $p = p'$ and $a \equiv a' \pmod{p}$ or $p = p'$ and $aa' \equiv 1 \pmod{p}$. Moreover, if p is odd, then L is a knot. If p is even, then L is a link with 2-components.*

Note (4.17)

(1) The 2-bridge link of type $(p, -a)$ is the mirror image of 2-bridge link of type (p, a).

(2) The 2-bridge link L of type $(p, p-a)$ is the mirror image of 2-bridge link L' of type (p, a). For, $p - a \equiv -a \pmod{p}$ and so it follows from Theorem (5.15) that L is equivalent to the 2-bridge link of type $(p, -a)$, which is the mirror image of L' by (1).

(3) Let $K_n = C(2, n)$. Then K_n is 2-bridge knot of type $(2n + 1|, |n|)$, which is called the twist knot.

Let K_0 denote the unknot. Then $K_0 = K_{-1}$ and $K_2 = K_2^* = K_{-3}$, the figure eight knot. In general, $K_n = K_{-n-1}$ and $K_n \neq K_m$ if $n \neq m$, excluding the cases above.

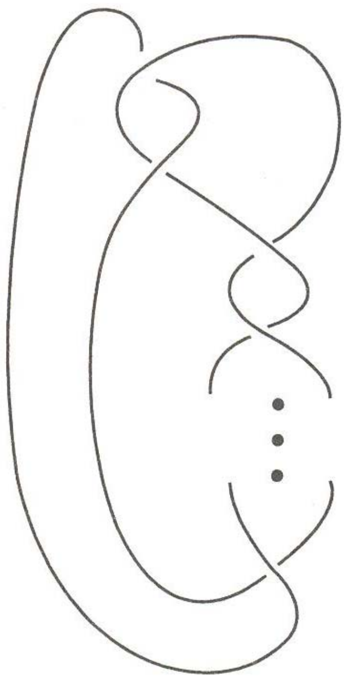


Figure: The twist knot

Section

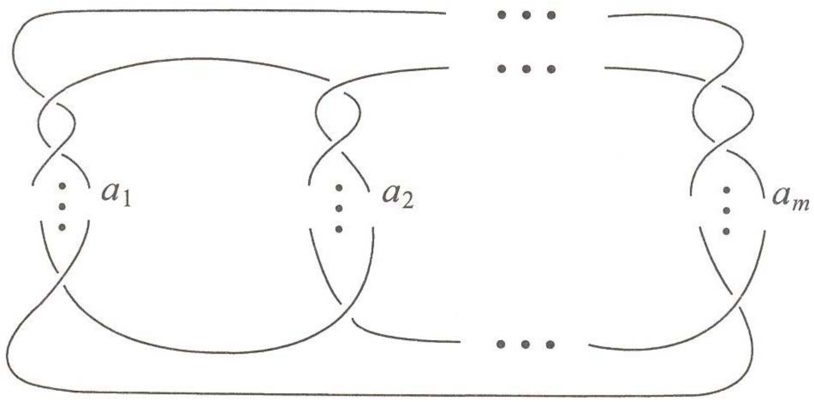
Torus Knots

Two-bridge links

Pretzel links

Pretzel knots and links

Let $a_1, a_2, \ldots, a_m$ be nonzero integers. The link with a diagram



is called the pretzel link, and denoted by $P(a_1, a_2, \ldots, a_m)$.

If $(a'_1, a'_2, \ldots, a'_m)$ is a cyclic permutation of $(a_1, a_2, \ldots, a_m)$, then it is clear that $P(a'_1, a'_2, \ldots, a'_m) = P(a_1, a_2, \ldots, a_m)$.

If $a_i = \pm 1$ for some $1 \le i \le m$, then

$$P(a_1, a_2, \ldots, a_m) = P(a_i, a_1, a_2, \ldots, a_{i-1}, a_{i+1}, a_m).$$

This implies that every pretzel link L is of the form:

$$P(\varepsilon_1, \ldots, \varepsilon_m, d_1, d_2, \ldots, d_n) \ (\varepsilon_i = \pm 1, |d_j| > 1).$$

Let $c = -(\varepsilon_1 + \cdots + \varepsilon_m)$, Then L has a diagram which is denoted by $P(c; d_1, d_2, \ldots, d_n)$:

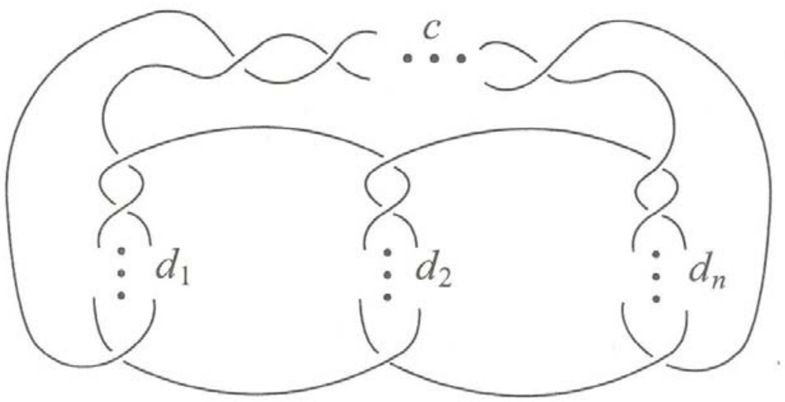


Figure: $P(c; d_1, d_2, \ldots, d_n)$

Lemma (4.18)

The prezel link $P(c; d_1, d_2, \ldots, d_n)$ is a knot if and only if it satisfies one of the following two conditions.

(1) *$n \geq 0$ and $d_1, d_2, \ldots, d_n, n+c$ are all odd, which is called a pretzel knot of odd type.*

(2) *$n \geq 1$ and exactly one of $d_1, d_2, \ldots, d_n$ is even, which is called a pretzel knot of even type.*

Let $P(c; d_1, d_2, \ldots, d_n)$ be a pretzel link. Suppose that $d_i = 2\varepsilon, \varepsilon = \pm 1$. Then $P(c; d_1, d_2, \ldots, d_n)$ can be transformed into $P(c+\varepsilon; d_1, d_2, \ldots, -d_i, \ldots, d_n)$ by applying the following flyping operations:

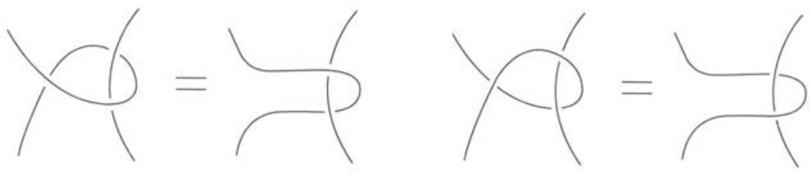


Figure: $P(c; d_1, d_2, \ldots, d_n)$

Classification of pretzel links

Thus, any given pretzel link diagram $P(c; d_1, d_2, \ldots, d_n)$ can always be deformed without changing the link type so that $|c|$ is minimal. With this type presentation, we have the following classification theorem for pretzel links:

Theorem (4.19)

(1) *$P(c; d_1, d_2, \ldots, d_n)$ is a 2-bridge link if and only if $n \leq 2$. In this case, if $c = 0$, then $P(c; d_1, d_2) = C(d_1 + d_2)$, and if $c \neq 0$, then $P(c; d_1, d_2) = C(d_1, -c, d_2)$.*

(2) *For $n \geq 3$, $P(c; d_1, d_2, \ldots, d_n)$ and $P(c'; d'_1, d'_2, \ldots, d'_{n'})$ are equivalent without orientation if and only if $n = n', c = c'$ and $(d'_1, d'_2, \ldots, d'_{n'})$ is a cyclic permutation of $(d_1, d_2, \ldots, d_n)$ or $(d_n, \ldots, d_2, d_1)$.*

(3) *For $n \geq 3$, $P(c; d_1, d_2, \ldots, d_n)$ is non-invertible if and only if it is odd type and $(d_n, \ldots, d_2, d_1)$ is not appear in any cyclic permutation of $(d_1, d_2, \ldots, d_n)$.*

Note (4.20)

(1) The mirror image of $P(c; d_1, d_2, \ldots, d_n)$ is $P(-c; -d_1, -d_2, \ldots, -d_n)$.

(2) Non-invertible pretzel knots with the least crossing number are the eight pretzel knots $P(0; \pm 3, \pm 5, \pm 7)$ of the crossing number 15.

Chapter 5. Goeritz Invariant

December 15, 2010

Sections

Section

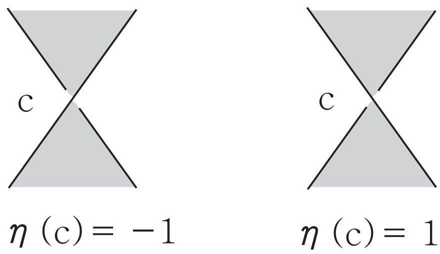
Computation of Goeritz Invariant

Modified Goeritz matrix and its signature

Goeritz Invariant

Let L be an oriented link in S^3 and let D be a connected link diagram in the plane $\mathbb{R}^2$. Let U be the diagram D forgetting the orientation. By Theorem (3.16), we can color the regions of $\mathbb{R}^2 - U$ alternately black and white, which is called a checkerboard coloring of U. Denote the black regions by $X_0, X_1, \cdots, X_m$. (We always take the unbounded region to be black and denote it by X_0.)

Let $C(U)$ denote the set of all crossings of U. Assign an connecting index $\eta(c) = \pm 1$ to each crossing $c \in C(U)$ as shown in the figure (a) below.



Let

$$g_{ij} = \begin{cases} \displaystyle\sum_{c \in C_U(X_i, X_j)} \eta(c), & \text{for } i \neq j; \\ \displaystyle -\sum_{j=0, j\neq i}^{m} g_{ij}, & \text{for } i = j. \end{cases}$$

where

$$C_U(X_i, X_j) = \{c \in C(U) | c \text{ is incident to both } X_i \text{ and } X_j\}.$$

Definition (5.1)

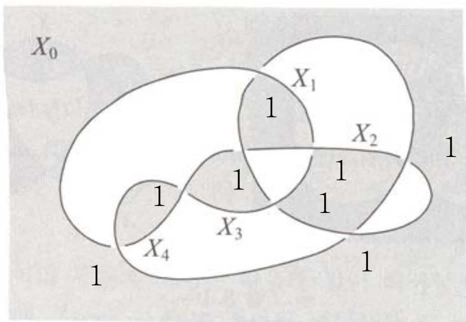
Let L be an oriented link in S^3 and let U be a connected diagram of L in $\mathbb{R}^2$ forgetting the orientation. Then the integral symmetric matrix given by

$$G = G_U(L) = (g_{ij})_{0 \leq i,\ j \leq m}$$

is called the Goeritz matrix of L associated with U.

Example (5.2)

Let L be a link with a connected checkerboard colored diagram U:



Then

$$G_U(L) = \begin{pmatrix} -4 & 1 & 2 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 2 & 1 & -4 & 1 & 0 \\ 0 & 1 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

Elementary transformations for integral matrices

Let G and G' be free abelian groups with bases $B = \{a_1, a_2, \ldots, a_n\}$ and $B' = \{a'_1, a'_2, \ldots, a'_m\}$, respectively. If $h : G \to G'$ is a homomorphism, then

$$\begin{aligned} h(a_1) &= \lambda_{11} a'_1 + \lambda_{21} a'_2 + \cdots + \lambda_{m1} a'_m, \\ h(a_2) &= \lambda_{12} a'_1 + \lambda_{22} a'_2 + \cdots + \lambda_{m2} a'_m, \\ &\vdots \\ h(a_n) &= \lambda_{1n} a'_1 + \lambda_{2n} a'_2 + \cdots + \lambda_{mn} a'_m \end{aligned}$$

for unique integers $\lambda_{ij}, 1 \le i \le n, 1 \le j \le m$.
The $m \times n$ integral matrix

$$A = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{m1} & \lambda_{m2} & \cdots & \lambda_{mn} \end{pmatrix}$$

is called the <u>matrix of h</u> relative to the given bases B and B'.

Let $f_A : G \to G'$ be the homomorphism defined by

$$f_A(x) = A[x]$$

for each $x = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n \in G$, identified with the column vector $[x] = (x_1, x_2, \ldots, x_n)^T$, the transpose of the row vector $(x_1, x_2, \ldots, x_n)$. Then

$$h = f_A.$$

Now consider the following operations on an integral matrix A.

(1) Elementary row operations:
- (ER1) Exchange row i and row k.
- (ER2) Multiply row i by -1.
- (ER3) Replace row i by (row i)+q(row k), $q \in \mathbb{Z}, i \neq k$.

(2) Elementary column operations:
- (EC1) Exchange column i and column k.
- (EC2) Multiply column i by -1.
- (EC3) Replace column i by (column i)+q(column k), $q \in \mathbb{Z}, i \neq k$.

Theorem (5.3)

Let G and G' be free abelian groups of ranks n and m, respectively, and let $h: G \to G'$ be a homomorphism. Then there are bases S and S' for G and G', respectively, such that the matrix of h relative to S and S' has the form

$$A = \left(\begin{array}{c|cccc|c} I_\beta & & O & & & O \\ \hline & k_1 & 0 & \cdots & 0 & \\ & 0 & k_2 & \cdots & 0 & \\ O & \vdots & \vdots & \ddots & \vdots & O \\ & 0 & 0 & \cdots & k_d & \\ \hline O & & O & & & O_{(m-\beta-d)\times(n-\beta-d)} \end{array}\right), \qquad (1)$$

where I_β is the $\beta \times \beta$ identity matrix, $k_i \geq 2, i = 1, 2, \ldots, d$, and $k_1 | k_2 | \cdots | k_d$. This matrix A is uniquely determined by h and called the normal form of h.

Definition (5.4)

In Theorem (5.3), the sequence $k_* = (k_1, k_2, \ldots, k_d)$ is called the torsion invariant of A, d is called the depth of A, and the number $m - \beta - d$ of the zero rows in A is called the nullity of A.

Remark 5.5. In Theorem (5.3), let $S = \{a_1, \ldots, a_n\}$ and $S' = \{e_1, \ldots, e_m\}$. Then

$$h(a_i) = e_i, i = 1, \ldots, \beta,$$
$$h(a_{\beta+i}) = k_i e_{\beta+i}, i = 1, \ldots, d,$$
$$h(a_{\beta+d+i}) = 0, i = 1, \ldots, n - (\beta + d).$$

Then $\{e_1, \ldots, e_\beta, k_1 e_{\beta+1}, \ldots, k_d e_{\beta+d}\}$ is a basis of $h(G)$ and $\{a_{\beta+d+1}, \ldots, a_n\}$ is a basis for $\mathrm{Ker}(h)$. Hence

$$G \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n}, \quad \mathrm{Ker}(h) = \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n-(\beta+d)},$$

$$\therefore\ G/\mathrm{Ker}(h) \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\beta+d}.$$

Furthermore,

$$G' \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{m-d} \oplus \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{d},$$

$$h(G) \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\beta} \oplus k_1\mathbb{Z} \oplus \cdots \oplus k_d\mathbb{Z}.$$

$$\therefore\ G'/h(G) \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{m-(\beta+d)} \oplus \mathbb{Z}/k_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/k_d\mathbb{Z}$$

$$\cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{m-(\beta+d)} \oplus \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_d}.$$

Hence the number $\beta + d$ is the rank of the free abelian group $h(G) \subset G'$. The integers $k_1, \ldots, k_d$ are the torsion coefficients of the quotient group $G'/h(G)$, called the cokernel of h.

Goeritz invariant

Theorem (5.6)

Let U_1 and U_2 be any two checkerboard colored diagrams of an oriented link L. Then the associated Goeritz matrices $G_{U_1}(L)$ and $G_{U_2}(L)$ can be transformed into each other by a finite number of transformations of the following types and their inverses:

(I) $G \to UGU^T$, *where U is a unimodular matrix of integers,*

(II) $G \to \begin{pmatrix} G & 0 \\ 0 & \pm 1 \end{pmatrix}$,

(III) $G \to \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix}$.

Proof.

See the text book!! □

Let L be an oriented link in S^3 and let U be a connected diagram of L in $\mathbb{R}^2$ forgetting the orientation. Let $G_U(L)$ be the Goeritz matrix of L associated with U and let

$$G'_U(L) = (g_{ij})_{1 \le i,j \le m}$$

be the principal minor of Goeritz matrix $G_U(L)$. By Theorem (5.6), we have

Theorem (5.7)

The torsion invariant $k_*(G'_U(L))$ is a topological invariant of the link L.

Definition (5.8)

(1) The torsion invariant $k_*(G'_U(L))$ is called the Goeritz invariant of L and denoted by $k_*(L)$.
(2) $d(G'_U(L))$ is called the depth of L and denoted by $d(L)$.
(3) $n(G'_U(L))$ is called the nullity of L and denoted by $n(L)$.

Corollary (5.9)

Let L be an oriented link in S^3. Then

$$d(L) = d(L') = d(L^*),$$
$$k_*(L) = k_*(L') = k_*(L^*),$$
$$n(L) = n(L') = n(L^*),$$

where L' is the link L in which the orientations of some components are reversed.

Example (5.10)

Let $G'_U(L)$ be the Goeritz matrix in Example (5.2). Then $G'_U(L)$ has the normal form $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 37 \end{pmatrix}$. Hence $k_*(L) = 37, d(L) = 1, n(L) = 0$.

Section

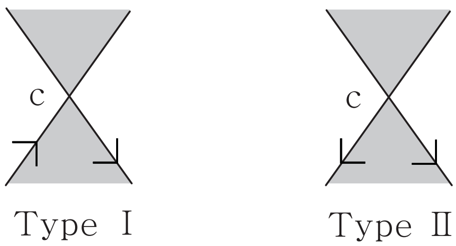
Computation of Goeritz Invariant

Modified Goeritz matrix and its signature

Modified Goeritz matrix

Let L be an oriented link in S^3, let D be any oriented diagram of L, and let U be the diagram D forgetting the orientation. Let $G'_U(L) = (g_{ij})_{1 \leq i,j \leq m}$ be the principal minor of the Goeritz matrix $G_U(L)$ associated with U.

First, we define a crossing $c \in C(D)$ in the oriented checkerboard colored diagram D to be of type I or type II as indicated in (b) of the figure below.



Let $C_{\mathrm{I}}(D) = \{c_1, c_2, \cdots, c_s\}$ denote the set of all crossings of type I in D and let

$$A(D) = \mathrm{diag}(\eta(c_1), \eta(c_2), \cdots, \eta(c_s)),$$

an $s \times s$ diagonal matrix.

Now let $S(D)$ denote the compact surface with boundary L, which is built up out of disks and bands. Each disk lies in $S^2 = \mathbb{R}^2 \cup \{\infty\}$ and is a closed white region in the checkerboard colored diagram U less a small neighborhood of each crossing. Each crossing gives a small half-twisted band. Let $\beta(D)$ denote the number of connected components of the surface $S(D)$.

Definition (5.11)

The modified Goeritz matrix $H_D(L)$ of L associated to D by

$$H_D(L) = \begin{pmatrix} G'_D(L) & & \\ & A_D(L) & \\ & & O_{\beta(D)-1} \end{pmatrix},$$

where $O_{\beta(D)-1}$ denotes the $(\beta(D)-1) \times (\beta(D)-1)$ zero matrix.

Theorem (5.12)

Let D_1 and D_2 be any two checkerboard colored diagrams of an oriented link L. Then the associated modified Goeritz matrices $H_{D_1}(L)$ and $H_{D_2}(L)$ can be transformed into each other by a finite sequence of the following matrix transformations $\Lambda_i (i = 1, 2)$ and their inverses:

$\Lambda_1 :\ H \longmapsto UHU^T,$

$$\Lambda_2 :\ H \longmapsto \begin{pmatrix} H & O & O \\ O & 1 & 0 \\ O & 0 & -1 \end{pmatrix},$$

where U is a unimodular matrix of integers, O a zero matrix, and U^T denotes the transpose of U.

Recall that any real symmetric matrix A is congruent to a diagonal matrix M by an invertible matrix P, that is, $PAP^T = M$. The signature, denoted by $\sigma(A)$, is the difference of the number of positive diagonal entries and the number of negative diagonal entries of M. The nullity, denoted by $\mathcal{N}(A)$, is the number of zero diagonal entries of M.

It is well known by Sylvester's law that even though a symmetric matrix A may be congruent to various diagonal matrices, the signature and the nullity of A do not change no matter what the diagonalizing matrices are.

Definition (5.13)

Let D be an oriented diagram of an oriented link L.

(1) The determinant of D, denoted by $\det(D)$, is defined by

$$\det(D) = |\det(H_D(L))|.$$

(2) The signature of D, denoted by $\sigma(D)$, is defined by

$$\sigma(D) = \sigma(H_D(L)).$$

(3) The nullity of D, denoted by $\mathscr{N}(D)$, is defined by

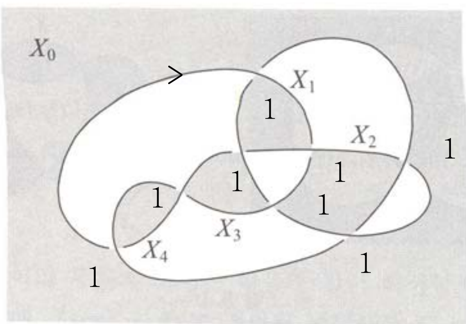
$$\mathscr{N}(D) = \mathscr{N}(H_D(L)) + 1.$$

Theorem (5.14)

Let L be an oriented link in S^3 and let D be any oriented diagram of L. Then the determinant $\det(D)$, the signature $\sigma(D)$ and the nullity $\mathscr{N}(D)$ are all invariants of L, and denoted by $\det(L), \sigma(L)$ and $\mathscr{N}(L)$, respectively.

Example (5.15)

Let L be a link with an oriented checkerboard colored diagram D:



$$A_D(L) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $S(U)$ is a connected surface, $O_{\beta(D)-1}$ is the empty and so it follows from Example (5.2) that

$$
\begin{aligned}
H_D(L) &= G'_U(L) \oplus A_D(L) \\
&= \begin{pmatrix} -3 & 1 & 1 & 0 \\ 1 & -4 & 1 & 0 \\ 1 & 1 & -3 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= U \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 37 \end{pmatrix} U^T \oplus \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Hence

$$\det(L) = 37,\ \sigma(L) = 8,\ \mathscr{N}(L) = 1.$$

Chapter 6. The Jones Polynomial

December 16, 2010

Sections

Section

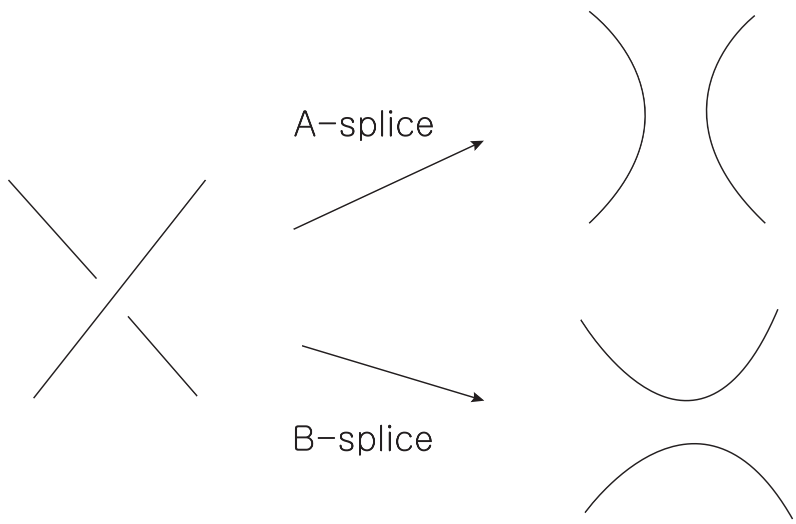
The Kauffmann bracket polynomial

The existence of the Jones polynomial

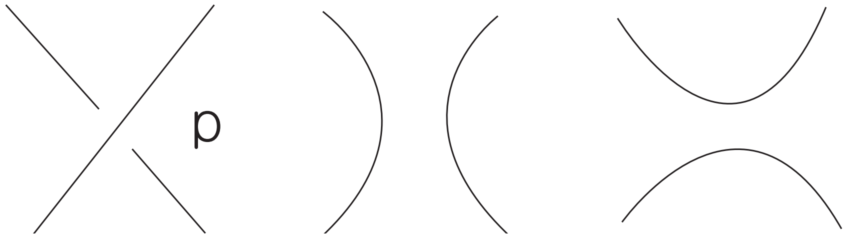
Skein relation and some calculations for the Jones polynomial

Crossing splice

Let D be an orineted link diagram and let U be the diagram D forgetting the orientation. For each croosing p, we consider the folloing two splices, say *A*-splice and *B*-splice:



Denote



U $\qquad$ $U_0(\text{or } U_0^p)$ $\qquad$ $U_\infty(\text{or } U_\infty^p)$

Definition of the bracket polynomial

Definition (6.1)

Let U be an unoriented link diagram with $n \geq 0$ crossings. The (Kauffmann) bracket polynomial of U is an integral polynomial $< U > \in \mathbb{Z}[A, B, \delta]$ with the three variables A, B and δ inductively defined by the three axioms:

(0) If $n = 0$, then $< U > = \delta^{r-1}$, where r is the number of the simple loops in U.

(1) If $n = 1$, then

$$< U > = A < U_0 > + B < U_\infty > .$$

(2) Assume that $n \geq 2$ and for all diagram U' with the crossings $\leq n - 1$, the polynomial $< U' >$ are defined. Then for a diagram U with n crossings and a crossing p of U, we define

$$< U > = A < U_0^p > + B < U_\infty^p > .$$

Lemma (6.2)

In (2) of Definition (6.1),

$$< U >= A < U_0^p > + B < U_\infty^p >$$

does not depend on the choice of a croosing p in U. Hence, $< U >$ is a uniquely defined polynomial for U.

Proof. Let U be an unoriented link diagram with $n \geq 1$. The case of $n = 1$ is obvious. Assume that $n \geq 2$. Let p and q be any two crossings of U.

Claim: $A < U_0^p > + B < U_\infty^p >= A < U_0^q > + B < U_\infty^q >$.

Since U_0^p, U_∞^p, U_0^q, and U_∞^q are diagrams with $n-1$ crossings, By induction hypothesis, we have

$$\begin{aligned}
< U_0^p > &= A < (U_0^p)_0^q > + B < (U_0^p)_\infty^q >, \\
< U_\infty^p > &= A < (U_\infty^p)_0^q > + B < (U_\infty^p)_\infty^q >, \\
< U_0^q > &= A < (U_0^q)_0^p > + B < (U_0^q)_\infty^p >, \\
< U_\infty^q > &= A < (U_\infty^q)_0^p > + B < (U_\infty^q)_\infty^p > .
\end{aligned}$$

Now

$$\begin{aligned}
A < U_0^p > + B < U_\infty^p > &= A\Big(A < (U_0^p)_0^q > + B < (U_0^p)_\infty^q >\Big) \\
&\quad + B\Big(A < (U_\infty^p)_0^q > + B < (U_\infty^p)_\infty^q >\Big) \\
&= A^2 < (U_0^p)_0^q > + AB < (U_0^p)_\infty^q > \\
&\quad + AB < (U_\infty^p)_0^q > + B^2 < (U_\infty^p)_\infty^q > \\
A < U_0^q > + B < U_\infty^q > &= A\Big(A < (U_0^q)_0^p > + B < (U_0^q)_\infty^p >\Big) \\
&\quad + B\Big(A < (U_\infty^q)_0^p > + B < (U_\infty^q)_\infty^p >\Big) \\
&= A^2 < (U_0^q)_0^p > + AB < (U_0^q)_\infty^p > \\
&\quad + AB < (U_\infty^q)_0^p > + B^2 < (U_0^q)_\infty^p > .
\end{aligned}$$

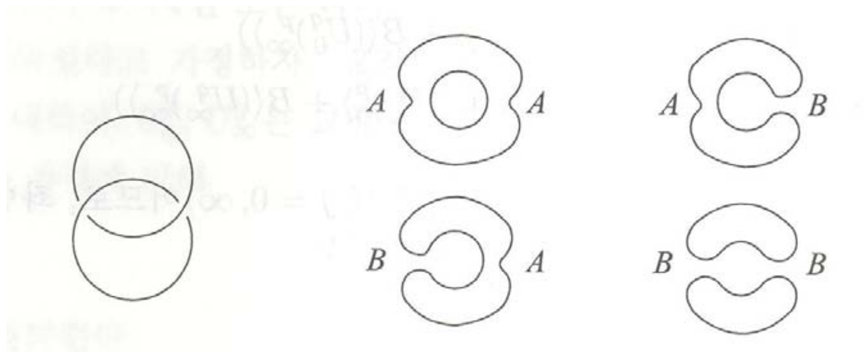
Since $(U_i^p)_j^q = (U_j^q)_i^p$ for each $i, j = 0.\infty$, we obtain the claim, completing the proof. □

State model for the bracket polynomial

A state for U is a diagram obtained from U by applying A-splice or B-splice for each crossing. Let S denote the set of all states for U. Note that a state $s \in S$ is a disjoint union of simple loops. The number of these simple loops in a state s is denoted by $|s|$.

Example (6.3)

Hopf link diagram H and its all states:



For a given state $s \in S$, we define the weight of U with respect to s, denoted by $< U/s >$, by

$$< U/s > = A^p B^q,$$

where $p = p(s)$ and $q = q(s)$ denote the number of A-splice and B-splice applied to U to obtain the state s (cf. $p + q = n$).

Lemma (6.4. State Model)

$$< U > = \sum_{s \in S} < U/s > \delta^{|s|-1} = \sum_{s \in S} A^{p(s)} B^{q(s)} \delta^{|s|-1}. \qquad (1)$$

Example (6.3, continued)

$$\begin{aligned} < H > &= \sum_{s \in S} < U/s > \delta^{|s|-1} \\ &= A^2\delta + AB + AB + B^2\delta \\ &= A^2\delta + 2AB + B^2\delta. \end{aligned}$$

Proof of Lemma (6.4). The case of $n=0$ is clear. Assume that $n \geq 1$ and for any diagram U with the crossings $\leq n-1$, the identity (1) holds.

Let U be an unoriented link diagram with n crossings ($n \geq 1$) and let p be a crossing of U. Then $<U> = A <U_0^p> + B <U_\infty^p>$. Now, let S, S_0, and S_∞ denote the set of all states of U, U_0^p and U_∞^p, respectively. Then $S = S_0 \cup S_\infty$. By the hypothesis of the induction,

$$<U_0^p> = \sum_{s \in S_0} <U_0^p/s> \delta^{|s|-1}, \quad <U_\infty^p> = \sum_{s \in S_\infty} <U_\infty^p/s> \delta^{|s|-1}.$$

On the other hand, if $s \in S_0$, then $<U/s> = A <U_0^p/s>$, and if $s \in S_\infty$, then $<U/s> = B <U_\infty^p/s>$. Thus

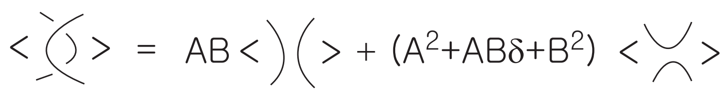
$$\begin{aligned}\sum_{s \in S} <U/s> \delta^{|s|-1} &= \sum_{s \in S_0} <U/s> \delta^{|s|-1} + \sum_{s \in S_\infty} <U/s> \delta^{|s|-1} \\ &= \sum_{s \in S_0} A <U_0^p/s> \delta^{|s|-1} + \sum_{s \in S_\infty} B <U_\infty^p/s> \delta^{|s|-1} \\ &= A <U_0^p> + B <U_\infty^p> = <U>.\end{aligned}$$

□

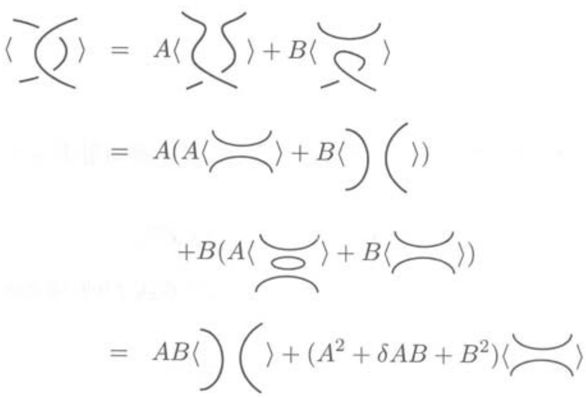
Theorem (6.5)

(1) $<U+U'> = \delta <U><U'>, \quad <O+U> = \delta <U>$.

(2)



Proof. (1) Exercise! (2)



□

Note (6.6)

It follws from Theorem (6.5) that $< U >$ is invariant under the Reidemeister move of type II if and only if

$$AB = 1,\ A^2 + AB\delta + B^2 = 0.$$

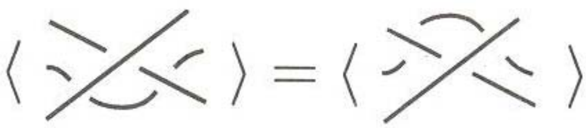
Taking $B = A^{-1}$ and $\delta = -A^2 - A^{-2}$, we have

$$AB = AA^{-1} = 1,\ A^2 + AB\delta + B^2 = A^2 + (-A^2 - A^{-2}) + A^{-2} = 0.$$

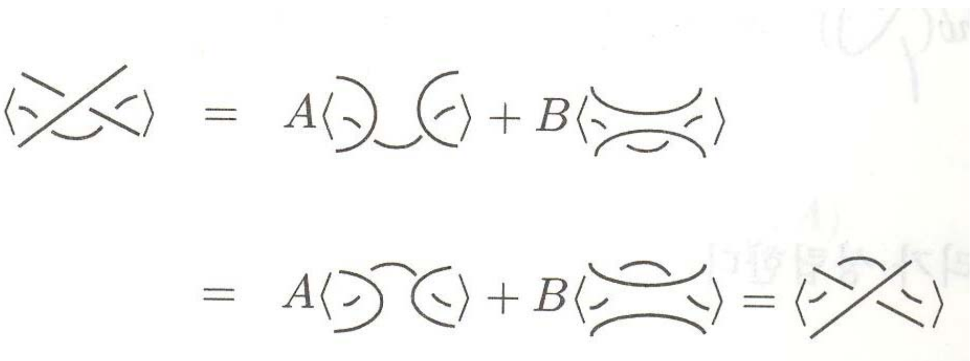
Let $< U > (A) :=< U > |_{B=A^{-1};\delta=-A^2-A^{-2}} \in \mathbb{Z}[A, A^{-1}]$.

Theorem (6.7)

$< U > (A)$ is invariant under the Reidemeister move of type III.



Proof. Let $B = A^{-1}$ and $\delta = -A^2 - A^{-2}$. By Note (6.6), $< \ \ >$ is invariant under Reidemeister move of type II. Hence



$\square$

Section

The Kauffmann bracket polynomial

The existence of the Jones polynomial

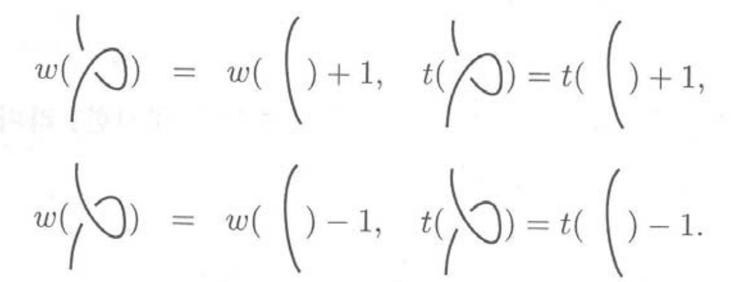
Skein relation and some calculations for the Jones polynomial

The normalized bracket polynomial

Let $D = D_1 \cup \cdots D_r$ be an oriented link diagram of r components. Then

$$\begin{aligned} w(D) &= \sum_{p \in D} \varepsilon(p) \\ &= \sum_{i=1}^{r} \left(\sum_{p \in D_i} \varepsilon(p) \right) + \sum_{1 \le i < j \le r} \left(\sum_{p \in D_i \cap D_j} \varepsilon(p) \right) \\ &= t(D) + 2Link(D). \end{aligned}$$

Note (6.8)



Theorem (6.9)

Let $D = D_1 \cup \cdots D_r$ be an oriented diagram of an oriented link L with r components and let U be the same diagram D without the orientation.

(1) *The Laurent polynomial*

$$J(U;A) = (-A)^{-3t(U)} < U > (A)$$

is an invariant of the unoriented link L. We denote $J(U;A)$ by $J(L;A)$.

(2) *The Laurent polynomial*

$$V(D;A) = (-A)^{-6Link(D)} J(U;A) = (-A)^{-3w(D)} < U > (A)$$

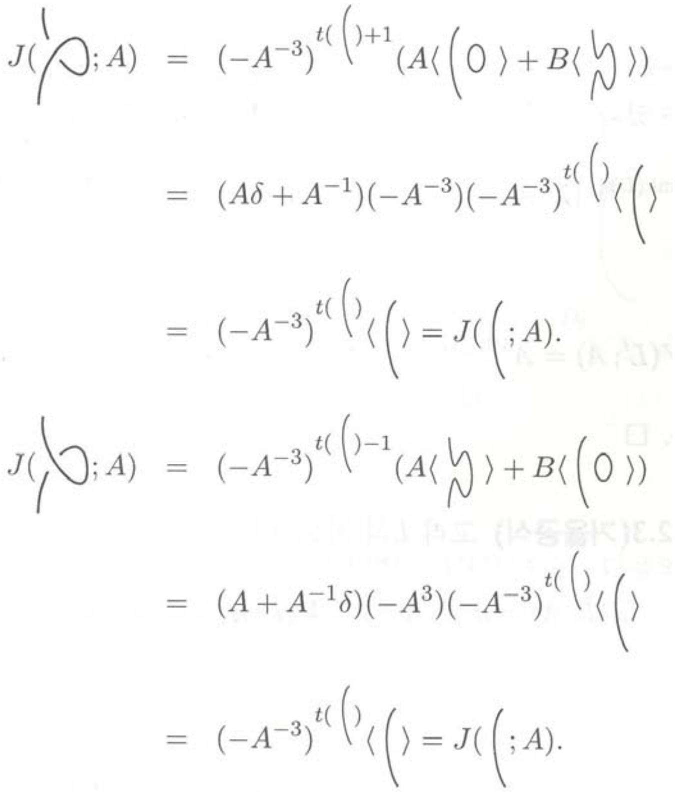
is an invariant of the oriented link L. We denote $V(D;A)$ by $V(L;A)$, which is called the normalized (Kauffmann) bracket polynomial of L.

Proof. Let $D = D_1 \cup \cdots D_r$ be an oriented diagram of an oriented link L with r components and let U be the same diagram D without the orientation. By Note (6.6) and Theorem (6.7), $< U > (A)$ is invariant under the Reidemeister moves of type II and III.

On the other hand, since each D_i is a knot diagram, the writhe $w(D_i)$ is independent on the orientation of D_i. This show that the twist number $t(D)$ is also invariant under the Reidemeister moves of type II and III and so is $J(U;A)$.

Now, since $Link(D)$ is an invariant for oriented links, $(-A)^{-6Link(D)}$ is an invariant of the oriented link L. Thus if we have proved the invariance of $J(U;A)$ under the Reidemeister move of type I, then $V(D;A) = (-A)^{-6Link(D)} J(U;A)$ becomes an invariant for oriented links. This gives the assertion (2).

Finally, for the Reidemeister move of type I, we have



□

Corollary (6.10)

(1) $V(-L;A) = V(L;A)$.

(2) $V(L';A) = A^{12\lambda}V(L;A)$, *where L' is the link L in which the orientation of one component K is reversed and $\lambda = Link(K, L-K)$.*

(3) $J(L^*;A) = J(L, A^{-1})$.

(4) $V(L^*;A) = V(L, A^{-1})$.

Definition (6.11)

Let L be an oriented link. The Laurent polynomial

$$V_L(t) = V(L;A)\big|_{A=t^{-\frac{1}{4}}}$$

is called the Jones polynomial of L.

In what follows, we also call $V(D;A)$, or equivalently, $V(L;A)$, the Jones polynomial of L.

Section

The Kauffmann bracket polynomial

The existence of the Jones polynomial

Skein relation and some calculations for the Jones polynomial

Skein relation

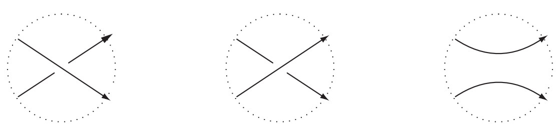
Theorem (6.12)

Let D be an oriented link diagram. The Jones polynomial $V(D; A)$ satisfies the following three properties (1), (2) and (3) that can be used to calculate the polynomial $V(D; A)$.

(1) *If D and D' can be transformed into each other by a finite sequence of the Reidemeister moves of type I, II, III, then $V(D; A) = V(D'; A)$.*

(2) *If D is a trivial knot diagram, then $V(D; A) = 1$.*

(3) $A^4 V(D_+; A) - A^{-4} V(D_-; A) = (A^{-2} - A^2) V(D_0; A)$.



D_+ D_- D_0

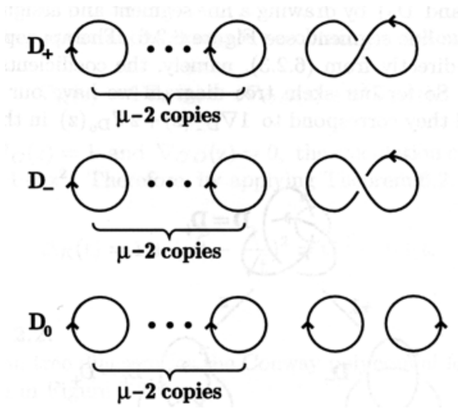
Figure: Skein triple

Lemma (6.13)

Let O^μ denote the unlink of μ components. Then

$$V(O^\mu; A) = (-A^2 - A^{-2})^{\mu-1}.$$

Proof. Consider



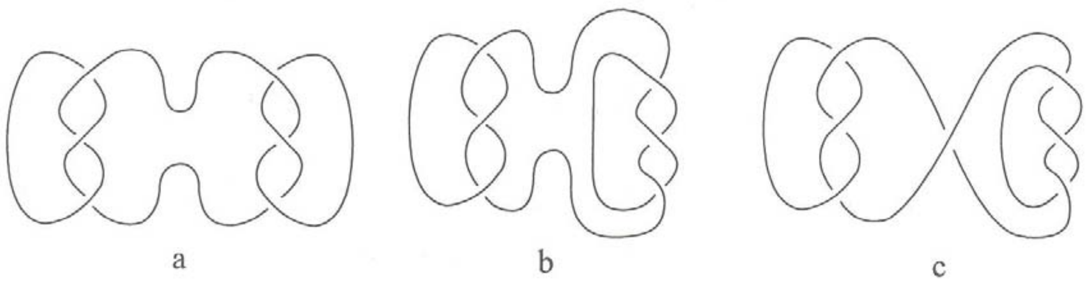
By (1) and (3),
$A^4 V(O^{\mu-1}; A) - A^{-4} V(O^{\mu-1}; A) = (A^{-2} - A^2) V(O^\mu; A)$. So
$V(O^\mu; A) = (-A^{-2} - A^2) V(O^{\mu-1}; A)$. By induction on μ and (2),
$V(O^\mu; A) = (-A^2 - A^{-2})^{\mu-1}$.

Corollary (6.14)

$$V(L \sharp L'; A) = V(L; A) V(L'; A).$$

Definition

A link diagram D is called a reduced diagram if for each crossing of D, the A-spliced diagram and B-spliced diagram are all connnected.
A diagram which is not a reduced diagram is called reducible diagram.



a, b: reduced diagrams; c: a reducible diagram

Theorem (6.15. K. Murasugi, 1987)

Let D be a connected link diagram. Then

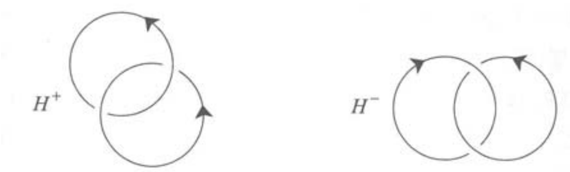
$$\operatorname{maxdeg} V(D;A) - \operatorname{mindeg} V(D;A) \leq 4c(D).$$

In particular, if D is a connected reduced alternating diagram, then

$$\operatorname{maxdeg} V(D;A) - \operatorname{mindeg} V(D;A) = 4c(D).$$

Example (6.16)

$$J(H^+;A) = -(A^4 + A^{-4}),$$
$$V(H^+;A) = -A^{-6}(A^4 + A^{-4}), V(H^-;A) = -A^{6}(A^4 + A^{-4}).$$



Chapter 7. Seifert Matrices and Derived Invariants

December 17, 2010

Sections

Construction of Seifert matrices

Invariants from Seifert matrices

Section

Construction of Seifert matrices

Invariants from Seifert matrices

Construction of a Seifert matrix

Let L be an oriented link in S^3 with r components and let F be a connected Seifert surface for L of genus g. Then F is homeomorphic to a disc-band surface

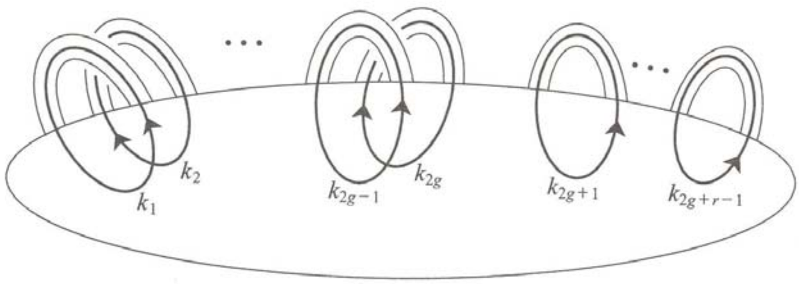


Figure: 7.1

and $H_1(F) \cong \mathbb{Z}^n$, where $n = 2g + r - 1$.

Let $h : F \times [-1, 1] \to S^3$ be an embedding such that $h(F \times \{0\}) = F$ and $f(F \times \{1\})$ is in the positive normal direction of F.

Let $\ell_1, \ell_2, \dots, \ell_n$ be 1-cycles in F such that the set $\{[\ell_1], [\ell_2], \dots, [\ell_n]\}$ of the homology classes forms a basis for $H_1(F)$. Denote

$$\ell_i^+ = h(\ell_i \times \{1\}),\ \ell_i^- = h(\ell_i \times \{-1\}),$$

and, for each pair i, j with $1 \le i, j \le n$,

$$v_{ij} = Link(\ell_i^+, \ell_j) = Link(\ell_i, \ell_j^-) = Link(\ell_i^+, \ell_j^-).$$

Definition (7.1)

(1) The mapping $\phi : H_1(F) \times H_1(F) \to \mathbb{Z}$ defined by

$$\phi([\ell_i], [\ell_j]) = v_{ij} = Link(\ell_i^+, \ell_j)$$

is called a Seifert form or a Seifert pairing of L associated with F.

(2) The matrix $V = (v_{ij})_{1 \le i,j \le n} = (Link(\ell_i^+, \ell_j))_{1 \le i,j \le n}$ is called a Seifert matrix of L associated with F.

Convention:

$$V = \Big(\phi([\ell_i], [\ell_j])\Big)_{1 \le i,j \le n} = \phi\Big(([\ell_1], [\ell_2], \dots, [\ell_n])^T ([\ell_1], [\ell_2], \dots, [\ell_n])\Big).$$

Lemma (7.2)

Let $\{[\ell_1'], [\ell_2'], \dots, [\ell_n']\}$ be an another basis for $H_1(F)$ and let V' be the corresponding Seifert matrix. Then there exists an unomodular integral matrix P such that

$$V' = P^T V P.$$

(We say that V' is obtained from V by a change of basis.)

Proof. Since $\{[\ell_1], [\ell_2], \dots, [\ell_n]\}$ is a basis for $H_1(F)$, we have

$$\begin{aligned}
[\ell_1'] &= p_{11}[\ell_1] + p_{21}[\ell_2] + \cdots + p_{n1}[\ell_n], \\
[\ell_2'] &= p_{12}[\ell_1] + p_{22}[\ell_2] + \cdots + p_{n2}[\ell_n], \\
&\vdots \\
[\ell_n'] &= p_{1n}[\ell_1] + p_{2n}[\ell_2] + \cdots + p_{nn}[\ell_n],
\end{aligned}$$

where $p_{ij} \in \mathbb{Z}$ such that $P = (p_{ij})_{1 \le i,j \le n}$ is an invertible integral matrix and so is unimodular. That is,

$$([\ell'_1], [\ell'_2], \dots, [\ell'_n]) = ([\ell_1], [\ell_2], \dots, [\ell_n])P.$$

Then

$$\begin{aligned} V' &= \Big(\phi([\ell'_i], [\ell'_j]) \Big)_{1 \le i,j \le n} \\ &= \phi\Big(([\ell'_1], [\ell'_2], \dots, [\ell'_n])^T ([\ell'_1], [\ell'_2], \dots, [\ell'_n]) \Big) \\ &= \phi\Big((([\ell_1], [\ell_2], \dots, [\ell_n])P)^T ([\ell_1], [\ell_2], \dots, [\ell_n])P \Big) \\ &= \phi\Big(P^T ([\ell_1], [\ell_2], \dots, [\ell_n])^T ([\ell_1], [\ell_2], \dots, [\ell_n])P \Big) \\ &= P^T \phi\Big(([\ell_1], [\ell_2], \dots, [\ell_n])^T ([\ell_1], [\ell_2], \dots, [\ell_n]) \Big) P \\ &= P^T V P. \end{aligned}$$

Calculation of Seifert matrices

Let $k_1, k_2, \dots, k_n$ be simple closed curves in F as in Figure 7.1 above. Then the homology classes $[k_1], [k_2], \dots, [k_n]$ forms a basis for $H_1(F)$. For each $i = 1, \dots, n$, let k'_i be a parallel translation of k_i in sufficiently small neighborhood in F such that $k'_i \cap k_j = \emptyset$ and $k'_i \cap k'_j = \emptyset$. Then

$$v_{ii} = Link(k_i^+, k_i) = Link(k'_i, k_i),$$

which can be calculated as illustrated in the figure 7.2 below.

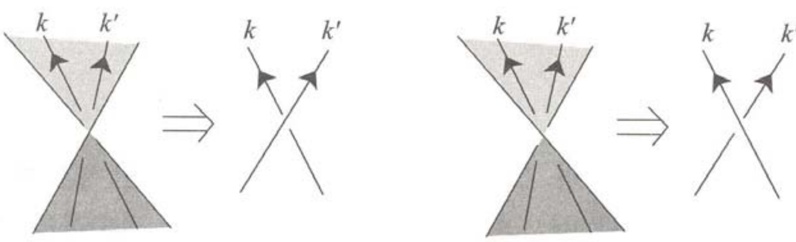


Figure: 7.2

Example (7.3)

Let H^+ be the right-handed Hopf link. Consider the seifert surface below. Then
$V = (v_{11}) = (Link(k^+, k)) = (Link(k', k)) = (-1)$.

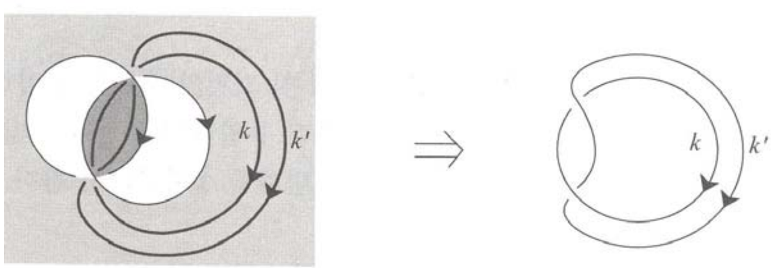


Figure: 7.3

Now, to compute $v_{ij} = Link(k_i^+, k_j)$ for $i \neq j$, we may assume that the intersection of k_i and k_j is transversal. First, we draw a diagram of the union $k_i \cup k_j$ and choose a small disk neighborhood of the transversal intersection point (cf. Figure 7.4).

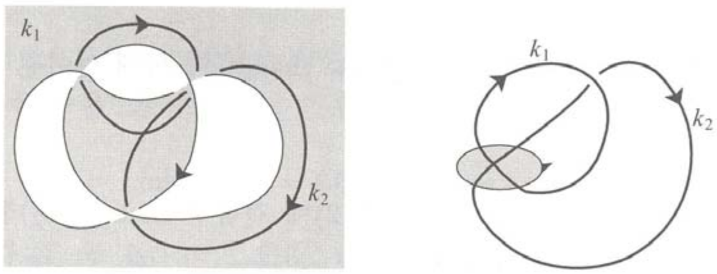


Figure: 7.4

Next, let k_i'' be the simple closed curve obtained by lifting k_i in the chosen small disc in the positive normal direction, leaving the outside of the disc fixed. Then

$$v_{ij} = Link(k_i^+, k_j) = Link(k_i'', k_j).$$

Example (7.4)

Let K^+ be the right-handed trefoil knot. Consider the Seifert surface as shown in Figure 7.4. Then

$$V = \begin{pmatrix} Link(k_1^+, k_1) & Link(k_1^+, k_2) \\ Link(k_2^+, k_1) & Link(k_2^+, k_2) \end{pmatrix} = \begin{pmatrix} Link(k_1', k_1) & Link(k_1'', k_2) \\ Link(k_2'', k_1) & Link(k_2', k_2) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}.$$

Example (7.5)

Let K be the figure eight knot. Consider the Seifert surface as shown in Figure 7.5. Then

$$V = \begin{pmatrix} Link(k_1^+, k_1) & Link(k_1^+, k_2) \\ Link(k_2^+, k_1) & Link(k_2^+, k_2) \end{pmatrix} = \begin{pmatrix} Link(k_1', k_1) & Link(k_1'', k_2) \\ Link(k_2'', k_1) & Link(k_2', k_2) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}.$$

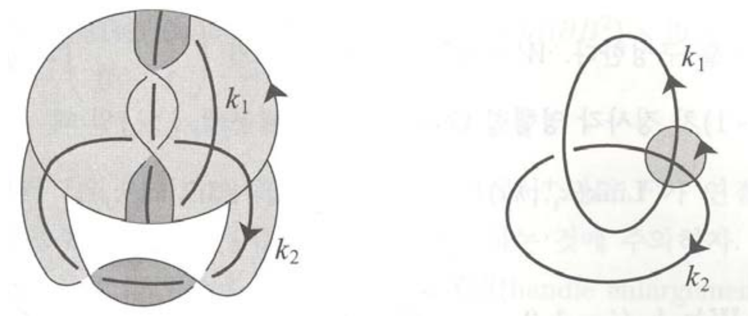


Figure: 7.5

Characterization of Seifert matrices

Theorem (7.6)

An $n \times n$ integral matrix V is a Seifert matrix for an oriented link L with r components if and only if

(1) $g = \frac{n-r+1}{2}$ *is a nonnegative integer,*

(2) *there exists an unimodular matrix P such that*

$$P^T(V - V^T)P = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{g} \oplus O_{r-1},$$

where O_{r-1} is the $(r-1) \times (r-1)$ zero matrix.

S-equivalence of Seifert matrices

Definition (7.7)

Two integral square matrices are *S*-equivalent if they are related by a finite sequence of the following transformations or their inverses:

(1) $M \longrightarrow P^T VP$, (P : an unimodular matrix).

(2) $M \longrightarrow \begin{pmatrix} & & & * & 0 \\ & M & & \vdots & \vdots \\ & & & * & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ (row enlargement).

(3) $M \longrightarrow \begin{pmatrix} & & & 0 & 0 \\ & M & & \vdots & \vdots \\ & & & 0 & 0 \\ * & \cdots & * & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ (column enlargement).

Theorem (7.8)

All Seifert matrices for an oriented link L in S^3 are S-equivalent.

Section

The Alexander polynomial

Let V be an $n \times n$ Seifert matrix for an oriented link L. Define

$$A(L;t) = \det(tV^T - V) \in \mathbb{Z}[t, t^{-1}],$$

a Laurent polynomial in variable t. By Theorem (7.8), $A(L;t)$ is an invariant of L up to multiplication by powers of $\pm t$, which is called the Alexander polynomial of L.

Theorem (7.9)

(1) *If L is an oriented link with r components, then there exists $m \in \mathbb{Z}$ such that*

$$A(L;t^{-1}) = (-1)^{r-1} t^m A(L;t).$$

(2) *If L is a knot, then $A(L;1) = 1$ and $\deg A(L;t) = \max\deg A(L;t) - \min\deg A(L;t)$ is even.*

(3) *If L is an oriented link with r components and L has a connected Seifert surface of genus g, then $\deg A(L;t) = \max\deg A(L;t) - \min\deg A(L;t) \leq 2g + r - 1$.*

Let V be an $n \times n$ Seifert matrix for an oriented link L. Define

$$C(L;x) = \det(xV^T - x^{-1}V) \in \mathbb{Z}[x],$$

an integral polynomial in variable x. Then

$$C(L;x) = \det(x^{-1}(x^2V^T - V)) = x^{-n}A(L;x^2).$$

Since $A(L;t)$ is an invariant of L, $C(L;x)$ is also an invariant of L, up to multiplication by powers of $\pm t$.

Lemma (7.10)

Let L be an oriented link and let D be its diagram. Then $C(D;x) := C(L;x)$ satisfies the following three properties (1), (2) and (3) that can be used to calculate the polynomial $C(D;x)$.

(1) *$C(D;x)$ is invariant under the Reidemeister moves I, II and III.*

(2) *If L is the unknot, then $C(D;x) = 1$.*

(3) *For the skein triple (D_+, D_-, D_0),*

$$C(D_+;x) - C(D_-;x) = (x^{-1} - x)C(D_0;x).$$

The Conway polynomial

In Lemma (7.10), let $z = x^{-1} - x$. Then $\nabla(L; z) := C(L; z) = C(D; x)$ is an integral polynomial in variable z that has no terms of negative exponent, which is called the Conway polynomial of L.

Theorem (7.11)

Let L be an oriented link and let D be its diagram. Then the Conway polynomial $\nabla(D; z) := \nabla(L; z)$ satisfies the following three properties (1), (2) and (3) that can be used to calculate the polynomial $\nabla(D; z)$.

(1) *$\nabla(D; z)$ is invariant under the Reidemeister moves I, II and III.*

(2) *If L is the unknot, then $\nabla(D; z) = 1$.*

(3) *For the skein triple (D_+, D_-, D_0),*

$$\nabla(D_+; z) - \nabla(D_-; z) = z\nabla(D_0; z).$$

Corollary (7.12)

Let L be an oriented link of r components.

(1) *The Conway polynomial of L is of the form:*

$$\nabla(L; z) = z^{r-1}(a_0 + a_2 z^2 + \cdots + a_{2m} z^{2m}).$$

Moreover,

$$m \leq g(L).$$

(2) *If $r = 1$, then $a_0 = 1$. If $r = 2$, then $a_0 = Link(L)$.*

Determinant and signature

Let L be an oriented link and let V be any Seifert matrix for L. The determinant $\det(L)$ of L is defined by

$$\det(L) = |\det(V^T + V)|.$$

The signature $\sigma(L)$ of L is defined by

$$\sigma(L) = \sigma(V^T + V)|.$$

Theorem (7.13)

Let L be an oriented link. Then $\det(L)$ and $\sigma(L)$ are topological invariants of L.