Cobordism of algebraic knots defined by Brieskorn polynomials

Osamu Saeki (Kyushu University)

Joint work with **Vincent Blanlœil** (Université de Strasbourg)

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§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

§1. Introduction

§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

Algebraic Knot

Let $f \in \mathbf{C}[z_1, z_2, \ldots, z_{n+1}]$ be a polynomial with $f(\mathbf{0}) = 0$.
We suppose f has an **isolated critical point** at $\mathbf{0}$.

§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

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For $0 < \varepsilon << 1$, $K_f = f^{-1}(0) \cap S_\varepsilon^{2n+1}$ is the ***algebraic knot*** associated with f.

§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

Algebraic Knot

Let $f \in \mathbf{C}[z_1, z_2, \ldots, z_{n+1}]$ be a polynomial with $f(\mathbf{0}) = 0$. We suppose f has an **isolated critical point** at $\mathbf{0}$. For $0 < \varepsilon << 1$, $K_f = f^{-1}(0) \cap S_\varepsilon^{2n+1}$ is the ***algebraic knot*** associated with f.

K_f is a $(2n-1)$-dim. closed manifold embedded in S_ε^{2n+1}.

§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

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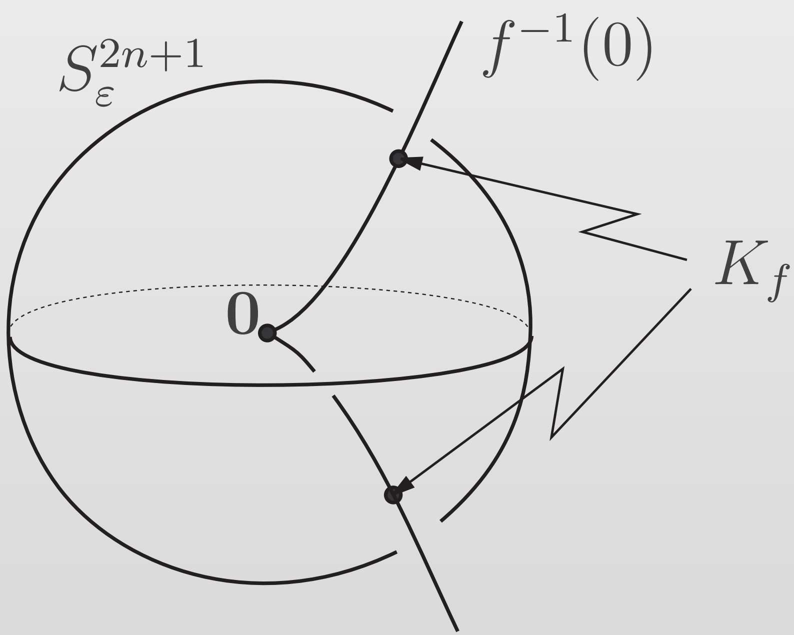

Topological Type

§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

According to Milnor, the homeomorphism class of $(D_\varepsilon^{2n+2}, f^{-1}(0) \cap D_\varepsilon^{2n+2})$ does not depend on $0 < \varepsilon << 1$.

Topological Type

§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

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§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

Topological Type

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Remark 1.1 *For two polynomials f and g, the following three conditions are equivalent.*

- *f and g have the same topological type.*

Topological Type

§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

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Remark 1.1 *For two polynomials f and g, the following three conditions are equivalent.*

- *f and g have the same topological type.*
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§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

Topological Type

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- *f and g have the same topological type.*
- *The algebraic knots K_f and K_g are isotopic.*
- *There exist homeomorphism germs Ψ and ψ which make the following diagram commutative.*

$$\begin{tikzcd} (\mathbf{C}^{n+1}, \mathbf{0}) \arrow[r, "f"] \arrow[d, "\Psi"'] & (\mathbf{C}, 0) \arrow[d, "\psi"] \\ (\mathbf{C}^{n+1}, \mathbf{0}) \arrow[r, "g"] & (\mathbf{C}, 0). \end{tikzcd}$$

Cobordism of Knots

§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

Definition 1.2 An ***m-dimensional knot*** (m-*knot*, for short) is a closed oriented m-dim. submanifold of the oriented S^{m+2}.

Cobordism of Knots

§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

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Cobordism of Knots

§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

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§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

Cobordism of Knots

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Cobordism of Knots

§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

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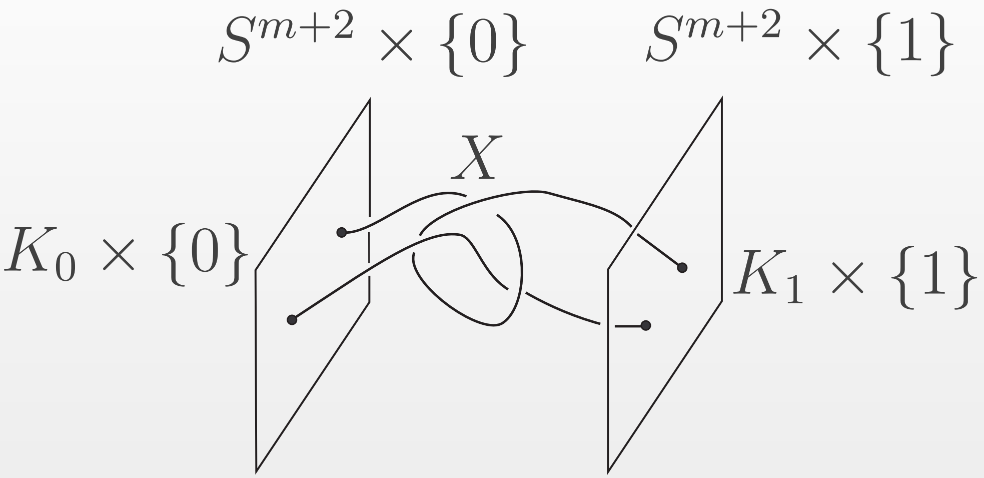
X is called a ***cobordism*** between K_0 and K_1.

§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

Cobordism vs Isotopy

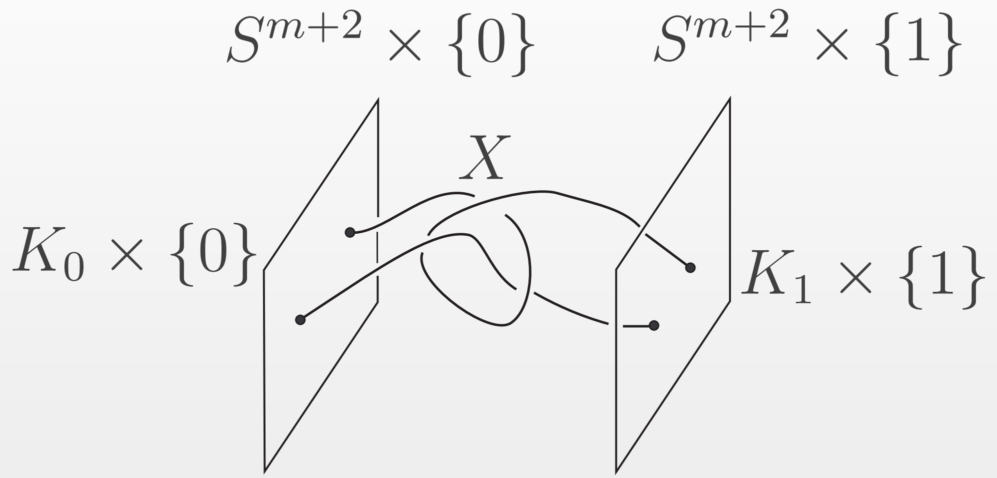


§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

Cobordism vs Isotopy

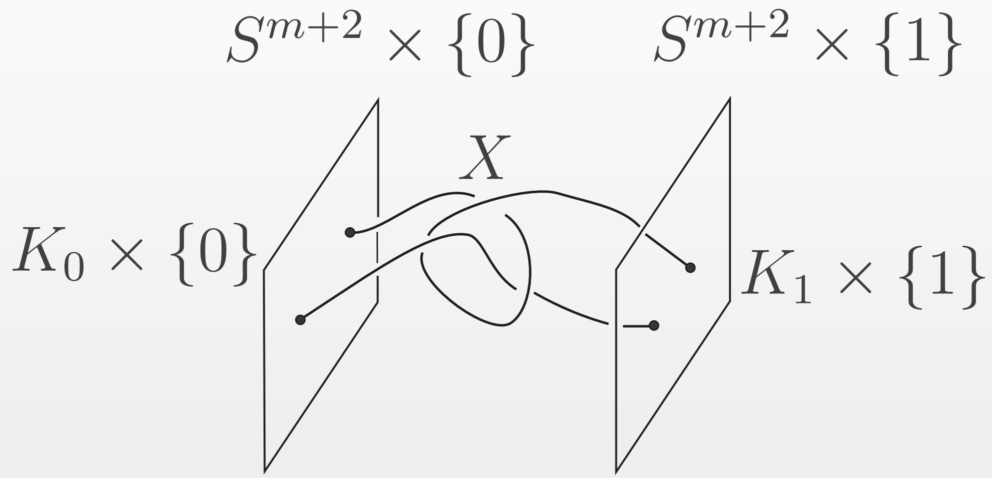


Isotopic

$\Downarrow \quad \not\Uparrow$

Cobordant

Cobordism vs Isotopy



Isotopic

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Cobordant

If two algebraic knots K_f and K_g are **cobordant**, then the topological types of f and g are mildly related.

Problem

§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

Problem 1.3 *Given f and g,*

Problem

§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

Problem 1.3 *Given f and g,*

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Problem

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Problem

§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

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The answers have been given in terms of **Seifert forms**, which are in general **very difficult to compute**.

§1. Introduction
● Algebraic Knot
● Topological Type
● Cobordism of Knots
● Cobordism vs Isotopy
● Problem
§2. Results
§3. Proofs

Problem

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Even if we know the Seifert forms, it is still difficult to check if the corresponding knots are isotopic or cobordant.

Problem

§1. Introduction
- Algebraic Knot
- Topological Type
- Cobordism of Knots
- Cobordism vs Isotopy
- Problem

§2. Results

§3. Proofs

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Today's Topic: Problem 1.3 (2) for weighted homogeneous polynomials (in particular, Brieskorn polynomials).

§1. Introduction

§2. Results
- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

§2. Results

Milnor Fibration

§1. Introduction

§2. Results

- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

Let us consider the ***Milnor fibration*** associated with f

$$\varphi_f : S_\varepsilon^{2n+1} \setminus K_f \to S^1$$

defined by $\varphi_f(z) = f(z)/|f(z)|$.

Milnor Fibration

§1. Introduction

§2. Results

- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

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Milnor Fibration

§1. Introduction

§2. Results

- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

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is called the ***Milnor fiber***, which is a compact $2n$-dimensional submanifold of S_ε^{2n+1} with $\partial F_f = K_f$.

It is known

$$F_f \simeq \vee^\mu S^n. \quad \text{(homotopy equivalent)}$$

The number μ of n-spheres is called the ***Milnor number*** of f.

Seifert Form

§1. Introduction

§2. Results

- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

The ***Seifert form*** associated with f is the bilinear form

$$L_f : H_n(F_f; \mathbf{Z}) \times H_n(F_f; \mathbf{Z}) \to \mathbf{Z} \quad \text{define by}$$

$$L_f(\alpha, \beta) = \mathrm{lk}(a_+, b), \quad \text{where}$$

Seifert Form

§1. Introduction

§2. Results

- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

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- a and b are n-cycles representing $\alpha, \beta \in H_n(F_f; \mathbf{Z})$,
- a_+ is obtained by pushing a into the positive normal direction of $F_f \subset S_\varepsilon^{2n+1}$,
- lk is the linking number in S_ε^{2n+1}.

§1. Introduction

§2. Results
- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

Seifert Form

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The isomorphism class of the Seifert form is a **topological invariant** of f.

Seifert Form

§1. Introduction

§2. Results
- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

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Theorem 2.1 (Durfee, Kato, 1974) *For $n \geq 3$, two algebraic knots K_f and K_g are isotopic $\Longleftrightarrow$ the Seifert forms L_f and L_g are isomorphic.*

§1. Introduction

§2. Results
- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

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Let $L_i : G_i \times G_i \to \mathbf{Z}$, $i = 0, 1$, be two bilinear forms defined on free $\mathbf{Z}$-modules of finite ranks.

§1. Introduction

§2. Results
- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

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Algebraic Cobordism

§1. Introduction

§2. Results
- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

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Definition 2.2 Suppose $m = \operatorname{rank} G$ is even.
A direct summand $M \subset G$ is called a ***metabolizer***
if $\operatorname{rank} M = m/2$ and L vanishes on M.

Algebraic Cobordism

§1. Introduction

§2. Results
- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

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Algebraic Cobordism

§1. Introduction

§2. Results

- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

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Theorem 2.3 (Blanlœil–Michel, 1997) *For $n \geq 3$, two algebraic knots K_f and K_g are cobordant $\Longleftrightarrow$ the Seifert forms L_f and L_g are algebraically cobordant.*

Witt Equivalence

§1. Introduction

§2. Results

- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

Remark 2.4 At present, there is no efficient criterion for algebraic cobordism.

It is usually very difficult to determine whether given two forms are algebraically cobordant or not.

§1. Introduction

§2. Results
- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

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Two forms L_0 and L_1 are ***Witt equivalent over*** $\mathbf{R}$ if there exists a metabolizer over $\mathbf{R}$ for $L_0 \otimes \mathbf{R}$ and $L_1 \otimes \mathbf{R}$.

Witt Equivalence

§1. Introduction

§2. Results
- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

Remark 2.4 At present, there is no efficient criterion for algebraic cobordism.
It is usually very difficult to determine whether given two forms are algebraically cobordant or not.

Two forms L_0 and L_1 are ***Witt equivalent over*** $\mathbf{R}$ if there exists a metabolizer over $\mathbf{R}$ for $L_0 \otimes \mathbf{R}$ and $L_1 \otimes \mathbf{R}$.

Lemma 2.5 *If two algebraic knots K_f and K_g are cobordant, then their Seifert forms L_f and L_g are Witt equivalent over $\mathbf{R}$.*

Weighted Homogeneous Polynomials

§1. Introduction

§2. Results

- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

Let f be a ***weighted homogeneous polynomial*** in $\mathbf{C}^{n+1}$,

Weighted Homogeneous Polynomials

§1. Introduction

§2. Results

- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

Let f be a ***weighted homogeneous polynomial*** in $\mathbf{C}^{n+1}$, i.e. $\exists(w_1, w_2, \ldots, w_{n+1}) \in \mathbf{Q}^{n+1}_{>0}$, called ***weights***, such that for each monomial $cz_1^{k_1} z_2^{k_2} \cdots z_{n+1}^{k_{n+1}}$, $c \neq 0$, of f, we have

$$\sum_{j=1}^{n+1} \frac{k_j}{w_j} = 1.$$

Weighted Homogeneous Polynomials

§1. Introduction

§2. Results
- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

Let f be a ***weighted homogeneous polynomial*** in $\mathbf{C}^{n+1}$, i.e. $\exists (w_1, w_2, \ldots, w_{n+1}) \in \mathbf{Q}^{n+1}_{>0}$, called ***weights***, such that for each monomial $cz_1^{k_1} z_2^{k_2} \cdots z_{n+1}^{k_{n+1}}$, $c \neq 0$, of f, we have

$$\sum_{j=1}^{n+1} \frac{k_j}{w_j} = 1.$$

f is ***non-degenerate*** if it has an isolated critical point at $\mathbf{0}$.

Weighted Homogeneous Polynomials

§1. Introduction

§2. Results
- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

Let f be a ***weighted homogeneous polynomial*** in $\mathbf{C}^{n+1}$, i.e. $\exists(w_1, w_2, \ldots, w_{n+1}) \in \mathbf{Q}_{>0}^{n+1}$, called ***weights***, such that for each monomial $cz_1^{k_1} z_2^{k_2} \cdots z_{n+1}^{k_{n+1}}$, $c \neq 0$, of f, we have

$$\sum_{j=1}^{n+1} \frac{k_j}{w_j} = 1.$$

f is ***non-degenerate*** if it has an isolated critical point at $\mathbf{0}$.

According to Saito, if f is non-degenerate, then by an analytic change of coordinates, f can be transformed to a weighted homogeneous polynomial with all weights ≥ 2.

§1. Introduction

§2. Results
- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

Weighted Homogeneous Polynomials

Let f be a ***weighted homogeneous polynomial*** in $\mathbf{C}^{n+1}$, i.e. $\exists(w_1, w_2, \ldots, w_{n+1}) \in \mathbf{Q}^{n+1}_{>0}$, called ***weights***, such that for each monomial $cz_1^{k_1} z_2^{k_2} \cdots z_{n+1}^{k_{n+1}}$, $c \neq 0$, of f, we have

$$\sum_{j=1}^{n+1} \frac{k_j}{w_j} = 1.$$

f is ***non-degenerate*** if it has an isolated critical point at $\mathbf{0}$.

According to Saito, if f is non-degenerate, then by an analytic change of coordinates, f can be transformed to a weighted homogeneous polynomial with all weights ≥ 2.
Furthermore, then the weights ≥ 2 are **analytic invariants** of the polynomial.

Weighted Homogeneous Polynomials

§1. Introduction

§2. Results
- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

Let f be a ***weighted homogeneous polynomial*** in $\mathbf{C}^{n+1}$, i.e. $\exists(w_1, w_2, \ldots, w_{n+1}) \in \mathbf{Q}_{>0}^{n+1}$, called ***weights***, such that for each monomial $cz_1^{k_1} z_2^{k_2} \cdots z_{n+1}^{k_{n+1}}$, $c \neq 0$, of f, we have

$$\sum_{j=1}^{n+1} \frac{k_j}{w_j} = 1.$$

f is ***non-degenerate*** if it has an isolated critical point at $\mathbf{0}$.

According to Saito, if f is non-degenerate, then by an analytic change of coordinates, f can be transformed to a weighted homogeneous polynomial with all weights ≥ 2.
Furthermore, then the weights ≥ 2 are **analytic invariants** of the polynomial.
In the following, we will always assume $\forall$weights ≥ 2.

Criterion for Witt Equivalence over $\mathbf{R}$

§1. Introduction

§2. Results

- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

Set

$$P_f(t) = \prod_{j=1}^{n+1} \frac{t - t^{1/w_j}}{t^{1/w_j} - 1}.$$

$P_f(t)$ is a polynomial in $t^{1/m}$ over $\mathbf{Z}$ for some integer $m > 0$.

Criterion for Witt Equivalence over $\mathbf{R}$

§1. Introduction

§2. Results

- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

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$P_f(t)$ is a polynomial in $t^{1/m}$ over $\mathbf{Z}$ for some integer $m > 0$.

Two non-degenerate weighted homogeneous polynomials f and g have the **same weights** if and only if $P_f(t) = P_g(t)$.

Criterion for Witt Equivalence over $\mathbf{R}$

§1. Introduction
§2. Results
● Milnor Fibration
● Seifert Form
● Algebraic Cobordism
● Witt Equivalence
● Weighted Homogeneous Polynomials
● Criterion for Witt Equivalence over $\mathbf{R}$
● Criterion for Isomorphism over $\mathbf{R}$
● Brieskorn Polynomials
● Cobordism Invariance of Exponents
● Cobordism Invariance of Multiplicities
● Case of 2 or 3 Variables
§3. Proofs

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Two non-degenerate weighted homogeneous polynomials f and g have the **same weights** if and only if $P_f(t) = P_g(t)$.

Theorem 2.6 *Let f and g be non-degenerate weighted homogeneous polynomials in $\mathbf{C}^{n+1}$. Then, their Seifert forms L_f and L_g are* ***Witt equivalent over $\mathbf{R}$*** *iff*

$$P_f(t) \equiv P_g(t) \mod t+1.$$

§1. Introduction

§2. Results
- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

The above theorem should be compared with the following.

Remark 2.7 *The Seifert forms L_f and L_g associated with non-degenerate weighted homogeneous polynomials f and g are* ***isomorphic over $\mathbf{R}$*** *iff*

$$P_f(t) \equiv P_g(t) \mod t^2 - 1.$$

Brieskorn Polynomials

§1. Introduction

§2. Results
- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

Proposition 2.8 *Let*

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$

be Brieskorn polynomials.

Brieskorn Polynomials

§1. Introduction

§2. Results
- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

Proposition 2.8 *Let*

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$

be Brieskorn polynomials.
Then, their Seifert forms are ***Witt equivalent over*** $\mathbf{R}$ *iff*

$$\prod_{j=1}^{n+1} \cot \frac{\pi\ell}{2a_j} = \prod_{j=1}^{n+1} \cot \frac{\pi\ell}{2b_j}$$

holds for all odd integers ℓ.

Cobordism Invariance of Exponents

§1. Introduction

§2. Results
- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

Theorem 2.9 *Suppose that for each of the Brieskorn polynomials*

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j},$$

no exponent is a multiple of another one.

§1. Introduction
§2. Results
● Milnor Fibration
● Seifert Form
● Algebraic Cobordism
● Witt Equivalence
● Weighted Homogeneous Polynomials
● Criterion for Witt Equivalence over $\mathbf{R}$
● Criterion for Isomorphism over $\mathbf{R}$
● Brieskorn Polynomials
● Cobordism Invariance of Exponents
● Cobordism Invariance of Multiplicities
● Case of 2 or 3 Variables
§3. Proofs

Cobordism Invariance of Exponents

Theorem 2.9 *Suppose that for each of the Brieskorn polynomials*

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j},$$

no exponent is a multiple of another one.
Then, the knots K_f and K_g are ***cobordant*** *iff*

$$a_j = b_j, \quad j = 1, 2, \ldots, n+1,$$

up to order.

Cobordism Invariance of Multiplicities

§1. Introduction

§2. Results
- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

Proposition 2.10 *Suppose that for each of the Brieskorn polynomials*

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$

the exponents are pairwise distinct.

Cobordism Invariance of Multiplicities

§1. Introduction

§2. Results

- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

Proposition 2.10 *Suppose that for each of the Brieskorn polynomials*

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$

the exponents are pairwise distinct.
*If K_f and K_g are **cobordant**, then the **multiplicities** of f and g coincide.*

§1. Introduction

§2. Results
- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

Case of 2 or 3 Variables

Proposition 2.11 *Let f and g be weighted homogeneous polynomials of two variables with weights (w_1, w_2) and (w'_1, w'_2), respectively, with $w_j, w'_j \geq 2$.*
If their Seifert forms are Witt equivalent over $\mathbf{R}$, then $w_j = w'_j$, $j = 1, 2$, up to order.

Case of 2 or 3 Variables

§1. Introduction

§2. Results

- Milnor Fibration
- Seifert Form
- Algebraic Cobordism
- Witt Equivalence
- Weighted Homogeneous Polynomials
- Criterion for Witt Equivalence over $\mathbf{R}$
- Criterion for Isomorphism over $\mathbf{R}$
- Brieskorn Polynomials
- Cobordism Invariance of Exponents
- Cobordism Invariance of Multiplicities
- Case of 2 or 3 Variables

§3. Proofs

Proposition 2.11 *Let f and g be weighted homogeneous polynomials of two variables with weights (w_1, w_2) and (w_1', w_2'), respectively, with $w_j, w_j' \geq 2$.*
If their Seifert forms are Witt equivalent over $\mathbf{R}$, then $w_j = w_j'$, $j = 1, 2$, up to order.

Proposition 2.12 *Let $f(z) = z_1^{a_1} + z_2^{a_2} + z_3^{a_3}$ and $g(z) = z_1^{b_1} + z_2^{b_2} + z_3^{b_3}$ be Brieskorn polynomials of three variables.*
If the Seifert forms L_f and L_g are Witt equivalent over $\mathbf{R}$, then $a_j = b_j$, $j = 1, 2, 3$, up to order.

§1. Introduction

§2. Results

§3. Proofs

- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

§3. Proofs

Proof of Theorem 2.6

§1. Introduction
§2. Results
§3. Proofs
- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

Theorem 2.6 Let f and g be non-degenerate weighted homogeneous polynomials in $\mathbf{C}^{n+1}$. Then, their Seifert forms L_f and L_g are **Witt equivalent over $\mathbf{R}$** iff

$$P_f(t) \equiv P_g(t) \mod t+1.$$

Proof of Theorem 2.6

§1. Introduction

§2. Results

§3. Proofs

- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

Theorem 2.6 Let f and g be non-degenerate weighted homogeneous polynomials in $\mathbf{C}^{n+1}$. Then, their Seifert forms L_f and L_g are **Witt equivalent over $\mathbf{R}$** iff

$$P_f(t) \equiv P_g(t) \mod t+1.$$

Proof. For simplicity, we consider the case of n even.

§1. Introduction
§2. Results
§3. Proofs
● Proof of Theorem 2.6
● Proof of Theorem 2.6 (Continued)
● Proof of Theorem 2.6 (Continued)
● Proof of Theorem 2.6 (Continued)
● Proof of Theorem 2.6 (Continued)
● Proof of Proposition 2.8
● Proof of Proposition 2.8 (Continued)
● Proof of Theorem 2.9
● Open problem
● Cobordism and Isotopy for Brieskorn Polynomials

Proof of Theorem 2.6

Theorem 2.6 Let f and g be non-degenerate weighted homogeneous polynomials in $\mathbf{C}^{n+1}$. Then, their Seifert forms L_f and L_g are **Witt equivalent over $\mathbf{R}$** iff

$$P_f(t) \equiv P_g(t) \mod t+1.$$

Proof. For simplicity, we consider the case of n even.

Let $\Delta_f(t)$ be the **characteristic polynomial of the monodromy**

$$h_* : H_n(F_f; \mathbf{C}) \longrightarrow H_f(F_f; \mathbf{C}),$$

where $h : F_f \longrightarrow F_f$ is the characteristic diffemorphism of the Milnor fibration $\varphi_f : S_\varepsilon^{2n+1} \setminus K_f \longrightarrow S^1$.

Proof of Theorem 2.6 (Continued)

§1. Introduction

§2. Results

§3. Proofs

- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

We have

$$H^n(F_f; \mathbf{C}) = \oplus_\lambda H^n(F_f; \mathbf{C})_\lambda,$$

where λ runs over all the roots of $\Delta_f(t)$, and $H^n(F_f; \mathbf{C})_\lambda$ is the eigenspace of h_* corresponding to the eigenvalue λ.

Proof of Theorem 2.6 (Continued)

§1. Introduction

§2. Results

§3. Proofs

- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

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where λ runs over all the roots of $\Delta_f(t)$, and $H^n(F_f; \mathbf{C})_\lambda$ is the eigenspace of h_* corresponding to the eigenvalue λ.

The **intersection form** $S_f = L_f + L_f^T$ of F_f on $H^n(F_f; \mathbf{C})$ decomposes as the orthogonal direct sum of $(S_f)|_{H^n(F_f;\mathbf{C})_\lambda}$.

Proof of Theorem 2.6 (Continued)

§1. Introduction
§2. Results
§3. Proofs
- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

We have

$$H^n(F_f; \mathbf{C}) = \oplus_\lambda H^n(F_f; \mathbf{C})_\lambda,$$

where λ runs over all the roots of $\Delta_f(t)$, and $H^n(F_f; \mathbf{C})_\lambda$ is the eigenspace of h_* corresponding to the eigenvalue λ.

The **intersection form** $S_f = L_f + L_f^T$ of F_f on $H^n(F_f; \mathbf{C})$ decomposes as the orthogonal direct sum of $(S_f)|_{H^n(F_f;\mathbf{C})_\lambda}$.

Let $\mu(f)^+_\lambda$ (resp. $\mu(f)^-_\lambda$) denote the number of positive (resp. negative) eigenvalues of $(S_f)|_{H^n(F;\mathbf{C})_\lambda}$.

Proof of Theorem 2.6 (Continued)

§1. Introduction

§2. Results

§3. Proofs

- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

We have

$$H^n(F_f;\mathbf{C}) = \oplus_\lambda H^n(F_f;\mathbf{C})_\lambda,$$

where λ runs over all the roots of $\Delta_f(t)$, and $H^n(F_f;\mathbf{C})_\lambda$ is the eigenspace of h_* corresponding to the eigenvalue λ.

The **intersection form** $S_f = L_f + L_f^T$ of F_f on $H^n(F_f;\mathbf{C})$ decomposes as the orthogonal direct sum of $(S_f)|_{H^n(F_f;\mathbf{C})_\lambda}$.

Let $\mu(f)_\lambda^+$ (resp. $\mu(f)_\lambda^-$) denote the number of positive (resp. negative) eigenvalues of $(S_f)|_{H^n(F;\mathbf{C})_\lambda}$.

The integer

$$\sigma_\lambda(f) = \mu(f)_\lambda^+ - \mu(f)_\lambda^-$$

is called the ***equivariant signature*** of f with respect to λ.

Proof of Theorem 2.6 (Continued)

§1. Introduction

§2. Results

§3. Proofs

- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

Lemma 3.1 (Steenbrink, 1977)

Set $P_f(t) = \sum c_\alpha t^\alpha$. Then we have

$$\sigma_\lambda(f) = \sum_{\substack{\lambda=\exp(-2\pi i\alpha) \\ \lfloor\alpha\rfloor:\ \text{even}}} c_\alpha - \sum_{\substack{\lambda=\exp(-2\pi i\alpha), \\ \lfloor\alpha\rfloor:\ \text{odd}}} c_\alpha$$

for $\lambda \neq 1$, where $i = \sqrt{-1}$, and $\lfloor\alpha\rfloor$ is the largest integer not exceeding α.

§1. Introduction
§2. Results
§3. Proofs
● Proof of Theorem 2.6
● Proof of Theorem 2.6 (Continued)
● Proof of Theorem 2.6 (Continued)
● Proof of Theorem 2.6 (Continued)
● Proof of Theorem 2.6 (Continued)
● Proof of Proposition 2.8
● Proof of Proposition 2.8 (Continued)
● Proof of Theorem 2.9
● Open problem
● Cobordism and Isotopy for Brieskorn Polynomials

Proof of Theorem 2.6 (Continued)

Lemma 3.1 (Steenbrink, 1977)
Set $P_f(t) = \sum c_\alpha t^\alpha$. Then we have

$$\sigma_\lambda(f) = \sum_{\substack{\lambda=\exp(-2\pi i\alpha) \\ \lfloor\alpha\rfloor:\ \text{even}}} c_\alpha - \sum_{\substack{\lambda=\exp(-2\pi i\alpha), \\ \lfloor\alpha\rfloor:\ \text{odd}}} c_\alpha$$

for $\lambda \neq 1$, where $i = \sqrt{-1}$, and $\lfloor\alpha\rfloor$ is the largest integer not exceeding α.

Remark 3.2 The equivariant signature for $\lambda = 1$ is always equal to zero.

Proof of Theorem 2.6 (Continued)

§1. Introduction

§2. Results

§3. Proofs

- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

Seifert forms L_f and L_g are Witt equivalent over $\mathbf{R}$.

$\Longrightarrow \quad \sigma_\lambda(f) = \sigma_\lambda(g)$ for all λ.

Proof of Theorem 2.6 (Continued)

§1. Introduction
§2. Results
§3. Proofs
- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

Seifert forms L_f and L_g are Witt equivalent over $\mathbf{R}$.
$\Longrightarrow \quad \sigma_\lambda(f) = \sigma_\lambda(g)$ for all λ.

Set

$$P_f(t) = P_f^0(t) + P_f^1(t), \quad \text{where}$$

$$P_f^0(t) = \sum_{\lfloor\alpha\rfloor \equiv 0 \pmod 2} c_\alpha t^\alpha,$$

$$P_f^1(t) = \sum_{\lfloor\alpha\rfloor \equiv 1 \pmod 2} c_\alpha t^\alpha.$$

We define $P_g^0(t)$ and $P_g^1(t)$ similarly.

Proof of Theorem 2.6 (Continued)

§1. Introduction
§2. Results
§3. Proofs
- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

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$$\begin{aligned} P_f^0(t) &= \sum_{\lfloor \alpha \rfloor \equiv 0 \pmod 2} c_\alpha t^\alpha, \\ P_f^1(t) &= \sum_{\lfloor \alpha \rfloor \equiv 1 \pmod 2} c_\alpha t^\alpha. \end{aligned}$$

We define $P_g^0(t)$ and $P_g^1(t)$ similarly.
Since the equivariant signatures of f and g coincide, we have

$$\begin{aligned} tP_f^0(t) - P_f^1(t) &\equiv tP_g^0(t) - P_g^1(t) \mod t^2 - 1, \\ tP_f^1(t) - P_f^0(t) &\equiv tP_g^1(t) - P_g^0(t) \mod t^2 - 1. \end{aligned}$$

Proof of Theorem 2.6 (Continued)

§1. Introduction
§2. Results
§3. Proofs
- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

Adding up these two congruences we have

$$(t-1)P_f(t) \equiv (t-1)P_g(t) \mod t^2-1, \tag{1}$$

Proof of Theorem 2.6 (Continued)

§1. Introduction

§2. Results

§3. Proofs

- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

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which implies

$$P_f(t) \equiv P_g(t) \mod t+1. \tag{2}$$

Proof of Theorem 2.6 (Continued)

§1. Introduction
§2. Results
§3. Proofs
- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

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Conversely, suppose that (2) holds.

$\Longrightarrow$ (1) holds.

$\Longrightarrow$ f and g have the same equivariant signatures.

Proof of Theorem 2.6 (Continued)

§1. Introduction

§2. Results

§3. Proofs

- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

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which implies

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Conversely, suppose that (2) holds.

$\Longrightarrow$ (1) holds.

$\Longrightarrow$ f and g have the same equivariant signatures.

Then, we can prove that they are Witt equivalent over $\mathbf{R}$.

This completes the proof.

Proof of Proposition 2.8

§1. Introduction

§2. Results

§3. Proofs

- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

Proposition 2.8 Let

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$

be Brieskorn polynomials. Then, their Seifert forms are Witt equivalent over $\mathbf{R}$ iff

$$\prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2a_j} = \prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2b_j}$$

holds for all odd integers ℓ.

Proof of Proposition 2.8 (Continued)

§1. Introduction

§2. Results

§3. Proofs

- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

Proof.

$P_f(t)$ and $P_g(t)$ are polynomials in $s = t^{1/m}$ for some m.
Put $Q_f(s) = P_f(t)$ and $Q_g(s) = P_g(t)$.

Then, $P_f(t) \equiv P_g(t) \mod t+1$ holds
$\Longleftrightarrow \quad Q_f(\xi) = Q_g(\xi)$ for all ξ with $\xi^m = -1$.

Proof of Proposition 2.8 (Continued)

§1. Introduction
§2. Results
§3. Proofs
- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

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$P_f(t)$ and $P_g(t)$ are polynomials in $s = t^{1/m}$ for some m.
Put $Q_f(s) = P_f(t)$ and $Q_g(s) = P_g(t)$.

Then, $P_f(t) \equiv P_g(t) \mod t+1$ holds
$\iff \quad Q_f(\xi) = Q_g(\xi)$ for all ξ with $\xi^m = -1$.

Note that ξ is of the form

$$\exp(\pi\sqrt{-1}\ell/m)$$

with ℓ odd and that

$$\frac{-1-\exp(\pi\sqrt{-1}\ell/a_j)}{\exp(\pi\sqrt{-1}\ell/a_j)-1} = \sqrt{-1}\cot\frac{\pi\ell}{2a_j}.$$

Then, we immediately get Proposition 2.8.

Proof of Theorem 2.9

§1. Introduction
§2. Results
§3. Proofs
● Proof of Theorem 2.6
● Proof of Theorem 2.6 (Continued)
● Proof of Theorem 2.6 (Continued)
● Proof of Theorem 2.6 (Continued)
● Proof of Theorem 2.6 (Continued)
● Proof of Proposition 2.8
● Proof of Proposition 2.8 (Continued)
● Proof of Theorem 2.9
● Open problem
● Cobordism and Isotopy for Brieskorn Polynomials

Theorem 2.9 Suppose that for each of the Brieskorn polynomials

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j},$$

no exponent is a multiple of another one.
Then, the knots K_f and K_g are **cobordant** iff

$$a_j = b_j, \quad j = 1, 2, \ldots, n+1,$$

up to order.

Proof of Theorem 2.9

§1. Introduction
§2. Results
§3. Proofs
- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

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$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j},$$

no exponent is a multiple of another one.
Then, the knots K_f and K_g are **cobordant** iff

$$a_j = b_j, \quad j = 1, 2, \ldots, n+1,$$

up to order.

This is a consequence of the "**Fox–Milnor type relation**" for the characteristic polynomials of cobordant algebraic knots.

Open problem

§1. Introduction
§2. Results
§3. Proofs
- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

Problem 3.3 *Are the exponents cobordism invariants for Brieskorn polynomials?*

Open problem

§1. Introduction

§2. Results

§3. Proofs

- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

Problem 3.3 *Are the exponents cobordism invariants for Brieskorn polynomials?*

Proposition 2.8 reduces the above problem to a number theoretical problem involving cotangents.

Open problem

§1. Introduction
§2. Results
§3. Proofs
- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

Problem 3.3 *Are the exponents cobordism invariants for Brieskorn polynomials?*

Proposition 2.8 reduces the above problem to a number theoretical problem involving cotangents.

$$\prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2a_j} = \prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2b_j} \quad \forall \text{odd integers } \ell$$

$$\Longrightarrow \quad a_j = b_j \quad \text{up to order ?}$$

§1. Introduction
§2. Results
§3. Proofs
● Proof of Theorem 2.6
● Proof of Theorem 2.6 (Continued)
● Proof of Theorem 2.6 (Continued)
● Proof of Theorem 2.6 (Continued)
● Proof of Theorem 2.6 (Continued)
● Proof of Proposition 2.8
● Proof of Proposition 2.8 (Continued)
● Proof of Theorem 2.9
● Open problem
● Cobordism and Isotopy for Brieskorn Polynomials

Cobordism and Isotopy for Brieskorn Polynomials

Remark 3.4 Theorem 2.9 implies that two algebraic knots K_f and K_g associated with certain Brieskorn polynomials are **isotopic** if and only of they are **cobordant**.

Cobordism and Isotopy for Brieskorn Polynomials

§1. Introduction

§2. Results

§3. Proofs

- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

Remark 3.4 Theorem 2.9 implies that two algebraic knots K_f and K_g associated with certain Brieskorn polynomials are **isotopic** if and only of they are **cobordant**.

According to **Yoshinaga–Suzuki**, two algebraic knots associated with Brieskorn polynomials in general are isotopic if and only if they have the same set of exponents.

Cobordism and Isotopy for Brieskorn Polynomials

§1. Introduction
§2. Results
§3. Proofs
- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

Remark 3.4 Theorem 2.9 implies that two algebraic knots K_f and K_g associated with certain Brieskorn polynomials are **isotopic** if and only of they are **cobordant**.

According to **Yoshinaga–Suzuki**, two algebraic knots associated with Brieskorn polynomials in general are isotopic if and only if they have the same set of exponents.

In fact, they showed that the **characteristic polynomials coincide if and only if the Brieskorn polynomials have the same set of exponents**.

§1. Introduction

§2. Results

§3. Proofs

- Proof of Theorem 2.6
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Theorem 2.6 (Continued)
- Proof of Proposition 2.8
- Proof of Proposition 2.8 (Continued)
- Proof of Theorem 2.9
- Open problem
- Cobordism and Isotopy for Brieskorn Polynomials

Thank you!