Lindström's theorem, both syntax and semantics free

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Abstract

Lindström's theorem characterises first-order logic in terms of its essential model theoretic properties. One cannot gain expressive power extending first-order logic without losing at least one of compactness or downward Löwenheim-Skolem property. We cast this result in an abstract framework of institution theory, which does not assume any internal structure either for sentences or for models, so it is more general than the notion of abstract logic usually used in proofs of Lindström's theorem, indeed, it can be said that institutional model theory is both *syntax and semantics free*. Our approach takes advantage of the methods of institutional model theory to provide a *structured* proof of Lindström's theorem at a level of abstraction applicable to any logical system that is strong enough to describe its own concept of isomorphism, and its own concept of elementary equivalence. We apply our results to some logical systems formalised as institutions and widely used in computer science practice.

1 Introduction

Traditionally, abstract model theory is understood as abstracting from the internal structure of sentences. As noted by Barwise in [2], one of its tasks is to determine the relationship between gaining expressive power in extending first-order logic and losing some of its important properties. A paramount result in this direction is Lindström's theorem, which characterises first-order logic among its extensions by two major properties: the downward Lowenheim-Skolem Property and compactness. In any proper extension of first-order logic at least one of the two fails.

To make the notion of a proper extension general enough, one is forced to abstract (at least) from what the sentences are. But abstract model theory itself is not a single theory. There are a number of abstract frameworks that can serve the purpose, including the one originally proposed in [2]. Another framework, particularly attractive to the authors, is institution theory, initiated by Goguen and Burstall in [13].

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In institution theory one studies meta-theoretic properties of logical systems without commitment to the internal structure of either sentences or models. Working in this framework can be cumbersome, unintuitive, and heavily laden with non-standard notation, but for this price one gets structured, modular proofs applicable to a wide variety of logical systems without need of adaptation. Since the hypotheses are kept as general as possible and introduced only when they are needed, the results are not obstructed by the irrelevant details of concrete logical systems, and allows one to observe and exploit the causality relations between logical properties.

Institutions. The theory of institutions arose as a reaction to the explosion of logics used in computer science that took place about four decades ago. These systems typically shared a number of properties, which however had to be proved for each of them separately. Goguen and Burstall proposed in [13] an abstract framework in which results could be developed at a general level and specialized to particular systems avoiding the need of repeating the same arguments each time the logical context is updated or changed in any way. An institution is a meta-structure that puts together syntax, semantics, and the satisfaction relation between them, without specifying any particulars. Goguen and Burstall's goal was achieved at least in the area of formal specification, verification and development of software systems [31, 14]. An institutional framework is at the core of the mathematical foundations of almost all algebraic specification approaches to these. Three examples are CASL [1], CafeOBJ [9], and CITP [11, 12]. Institution theory is also friendly to formalised mathematics, perhaps because of its origins in computer science. The reader will have ample opportunities to verify that by looking at our proofs.

Our contribution. The main contribution of the present work is that we prove Lindström's theorem in a way that is independent of any concrete syntactic structure (syntax free) and independent of any concrete semantics (semantics free). It can be said that this completes Lindström's own work, which was syntax free (abstract logics), but not semantics free (usual first-order structures). It is also the first study of Lindström's theorem in the framework of institutions, in the spirit of abstract model theory as we see it. Here are a few direct consequences of our approach.

We achieve a modular, structured proof of the characterisation of first-order logics in terms of compactness and downward Löwenheim-Skolem property at a very general level of abstraction, which clearly reveals implications between logical properties and avoids inessential technical assumptions such as the Relativisation and Elimination properties of Ebbinghaus, Flum and Thomas [10], or the Relativisation and Quantifier properties of Chang, Keisler [3]. The results apply directly to a wide range of logical systems used in mathematical logic and computer science.

In particular, our results apply to many-sorted first-order logic (with equality) which, despite contrary appearances, is a non-trivial generalisation of the single-sorted version of first-order logic. For example, interpolation property holds in single-sorted first-order logic without any restrictions, but for the many-sorted case interpolation only holds under additional assumptions. The assumptions may be mild, yet finding a most general criterion ensuring that interpolation property holds when formulated in terms of pushouts, was put forward as an open problem by Tarlecki in [33]. The solution (only one of the pushout morphisms needs to be injective on sorts) was given in [23]. Another example is completeness, which is developed at an institution-independent level, but it is applicable to many-sorted first-order logics under the assumption that the carrier sets of the models consist of non-empty sets [29, 22, 20]. The absence of this assumption creates challenges for proving completeness that have no straightforward solutions. Therefore, it is worth mentioning that the institution-independent version of Lindström's Theorem developed in the present contribution is applicable to many-sorted first-order logic in which models may interpret some sorts as empty sets. Also, the classical proof of Lindström's Theorem is tailored to signatures which consists only of relation symbols and constants. This is no loss of generality, since functions can be simulated with relations and some additional axioms. While this argument is valid for first-order logic, it fails when dealing only with algebraic signatures, which is the case of the OBJ family of algebraic specification languages such as CafeOBJ [9] or Maude [5]. In addition, the ordering relation among sorts, a feature of both CafeOBJ and Maude, increases the complexity level of many-sorted first-order logic. Fortunately, the development of the results at the appropriated level of abstraction and the modularization principles that guided our study allowed us to cover also the case of order-sorted algebra [15, 25] with minimal effort.

Compactness. The conditions necessary for the development of ultraproduct method were made clear by Diaconescu [6] in an abstract framework provided by an institution satisfying some assumptions easy to verify in concrete examples of logical systems. Institution-independent compactness is obtained as an application of ultraproduct method. In the process, it was demonstrated that category-based multi-signature framework characteristic of institution theory leads to greater generality and structured proofs. Institution independent compactness is obtained as a consequence of completeness in [29], using a generalization of Henkin's method from classical model theory, and [22], using model-theoretic forcing.

Downward Löwenheim-Skolem Theorem. A method for proving downward Löwenheim-Skolem theorem within an arbitrary institution satisfying certain properties is based on forcing and it was developed in [16]. The results are applicable to logics in which the semantics is restricted to models with non-empty carrier sets and the syntax is restricted to countable languages. These restrictions were overcome in [18], which employs a novel technique for proving a more refined version of the theorem.

Fraïssé-Hintikka Theorem. This result, stating that elementary equivalence can be characterised in terms of finite Ehrenfeucht-Fraïssé games, was generalized to an institution independent setting by the present authors [21], using an argument not calling on quantifier rank or normal forms of sentences. The first part of the present article is largely a preparation for an application of this abstract version of Fraïssé-Hintikka Theorem to the abstract internal characterisation of elementary equivalence, carried out in Section 4. This characterisation, in turn, is the key to a modular approach to Lindström's theorem.

2 Institutions

We recall the notion of an *institution*, originally from [13].

Definition 1 (Institution)**.** *An institution* $\mathcal{I} = (\mathsf{Sig}^{\mathcal{I}}, \mathsf{Sen}^{\mathcal{I}}, \mathsf{Mod}^{\mathcal{I}}, \models^{\mathcal{I}})$ *consists of:*

1. *A category* $\mathsf{Sig}^{\mathcal{I}}$*, whose objects are called* signatures*.*

2. *A functor* $\mathsf{Sen}^{\mathcal{I}} : \mathsf{Sig}^{\mathcal{I}} \to \mathsf{Set}$*, providing for each signature* Σ *a set whose elements are called* (Σ-)sentences*.*

3. *A functor* $\mathsf{Mod}^{\mathcal{I}} : \mathsf{Sig}^{\mathcal{I}} \to \mathsf{Cat}^{\mathsf{op}}$*, providing for each signature* Σ *a category whose objects are called* (Σ-)models *and whose arrows are called* (Σ-)homomorphisms*.*

4. *A family of relations* $\models^{\mathcal{I}} = \{\models^{\mathcal{I}}_{\Sigma}\}_{\Sigma \in |\mathsf{Sig}^{\mathcal{I}}|}$*, where* $\models^{\mathcal{I}}_{\Sigma} \subseteq |\mathsf{Mod}^{\mathcal{I}}(\Sigma)| \times \mathsf{Sen}^{\mathcal{I}}(\Sigma)$ *is called* (Σ-)*satisfaction for all signatures* $\Sigma \in |\mathsf{Sig}^{\mathcal{I}}|$*, such that the following* satisfaction condition *holds:*

$$\mathfrak{A}' \models^{\mathcal{I}}_{\Sigma'} \mathsf{Sen}^{\mathcal{I}}(\chi)(e) \textit{ iff } \mathsf{Mod}^{\mathcal{I}}(\chi)(\mathfrak{A}') \models^{\mathcal{I}}_{\Sigma} e$$

for all $\chi : \Sigma \to \Sigma' \in \mathsf{Sig}^{\mathcal{I}}$*,* $\mathfrak{A}' \in |\mathsf{Mod}^{\mathcal{I}}(\Sigma')|$ *and* $e \in \mathsf{Sen}^{\mathcal{I}}(\Sigma)$*.*

Note that the notion of valuation does not exist in institutions. Instead, in institution theory all variables are treated as constants, and thus there are no open formulas. Quantification has to be defined accordingly – we will go into some details later, as the need arises.

In concrete examples, the category of signatures $\mathsf{Sig}^{\mathcal{I}}$ provides the vocabularies over which the sentences are built and the morphisms in $\mathsf{Sig}^{\mathcal{I}}$, called *signature morphisms*, represent a change of notation. Signature morphisms act covariantly on sentences, and contravariantly on models. More concretely, given a signature morphism $\chi : \Sigma \to \Sigma'$ the sentences over the signature Σ are mapped to the sentences over the signature Σ' by the function $\mathsf{Sen}^{\mathcal{I}}(\chi) : \mathsf{Sen}^{\mathcal{I}}(\Sigma) \to \mathsf{Sen}^{\mathcal{I}}(\Sigma')$. The Σ'-models are 'reduced' to the signature Σ by the functor $\mathsf{Mod}^{\mathcal{I}}(\chi) : \mathsf{Mod}^{\mathcal{I}}(\Sigma') \to \mathsf{Mod}^{\mathcal{I}}(\Sigma)$. We denote the *reduct* functor $\mathsf{Mod}^{\mathcal{I}}(\chi)$ by $\_\!\upharpoonright_{\chi}$ and the function $\mathsf{Sen}^{\mathcal{I}}(\chi)$ by χ. If $\mathfrak{A} = \mathfrak{A}'\!\upharpoonright_{\chi}$ we say that $\mathfrak{A}$ is the χ-*reduct* of $\mathfrak{A}'$, and $\mathfrak{A}'$ is a χ-*expansion* of $\mathfrak{A}$.

When there is no danger of confusion, we omit the superscript $\mathcal{I}$ from the notations of the institution components: for example, $\mathsf{Sig}^{\mathcal{I}}$ may be simply denoted by Sig. The notation surrounding the satisfaction relation is standard. Namely, for all signatures Σ, sets of Σ-sentences Γ and E, we have

1. For all Σ-models $\mathfrak{A}$, ($\mathfrak{A} \models E$) iff ($\mathfrak{A} \models e$ for all $e \in E$);

2. $\Gamma \models E$ iff for all Σ-models $\mathfrak{A}$, we have $\mathfrak{A} \models \Gamma$ implies $\mathfrak{A} \models E$;

3. $\Gamma \models\!\!| E$ iff $\Gamma \models E$ and $E \models \Gamma$.

Assumption 1. *Since institutional setting is very abstract, we will explicitly mention an assumption that in another context would be completely trivial: isomorphic models are elementarily equivalent.*

We do not know of any concrete institutions that do not satisfy the assumption above, but an artificial example may illustrate why the assumption is necessary at the institutional level of abstraction. It may happen that the set of sentences is rich enough to speak about non-structural properties of models: say, distinguish between models whose elements are blue, and models whose elements are green. Sentences are not assumed to have any particular internal structure, so we can add them by *fiat*. Since isomorphism only detects structural properties, blue and green models may be isomorphic but not satify the same sentences. Admittedly the example is pathological, but the definition of an institution does not preclude it.

2.1 Examples

We give a few typical examples of institutions in algebraic specification literature. We will give quite some details, especially in the first example, in order to illustrate a few common features of institutions, which are absent in other logical systems.

Example 2 (First-order logic (FOL))**.** *The first presentation of first-order logic as an institution is due to Goguen and Burstall [13].*

Signatures. *Signatures are of the form (S, F, P), where S is a set of sorts, $F = \{F_{ar\to s}\}_{(ar,s)\in S^*\times S}$ is a ($S^*\times S$-indexed) set of operation symbols, and $P = \{P_{ar}\}_{ar\in S^*}$ is a (S^*-indexed) set of relation symbols. If $ar = \varepsilon$ then an element of $F_{ar\to s}$ is called a* constant symbol*. Generally, ar ranges over arities, which are understood here as strings of sorts, so an arity gives the number of arguments together with their sorts. We overload the notation and let F and P also denote $\biguplus_{(ar,s)\in S^*\times S} F_{ar\to s}$ and $\biguplus_{ar\in S^*} P_{ar}$, respectively. Therefore, we may write $\sigma \in F_{ar\to s}$ or $(\sigma : ar \to s) \in F$; both have the same meaning, which is: σ is an operation symbol of type $ar \to s$. A first-order signature with no relation symbols is called algebraic signature.*

A signature morphism $\chi : \Sigma \to \Sigma'$, where $\Sigma = (S, F, P)$ and $\Sigma' = (S', F', P')$, is a triple $\chi = (\chi^{st}, \chi^{op}, \chi^{rl})$ of maps:

1. $\chi^{st} : S \to S'$,
2. $\chi^{op} = \{\chi^{op}_{ar\to s} : F_{ar\to s} \to F'_{\chi^{st}(ar)\to\chi^{st}(s)} \mid ar \in S^*, s \in S\}$,
3. $\chi^{rl} = \{\chi^{rl}_{ar} : P_{ar} \to P'_{\chi^{st}(ar)} \mid ar \in S^*\}$.

When there is no danger of confusion, we may let χ denote either of χ^{st}, $\chi^{op}_{ar\to s}$, χ^{rl}_{ar}.

Models. *Given a signature $\Sigma = (S, F, P)$, a Σ-model is a triple*

$$\mathfrak{A} = (\{\mathfrak{A}_s\}_{s\in S}, \{\sigma^{\mathfrak{A}}\}_{(ar,s)\in S^*\times S, \sigma\in F_{ar\to s}}, \{\pi^{\mathfrak{A}}\}_{ar\in S^*, \pi\in P_{ar}})$$

interpreting each sort s as a set $\mathfrak{A}_s$, each operation symbol $\sigma \in F_{ar\to s}$ as a function $\sigma^{\mathfrak{A}} : \mathfrak{A}_{ar} \to \mathfrak{A}_s$ (where $\mathfrak{A}_{ar}$ stands for $\mathfrak{A}_{s_1} \times \ldots \times \mathfrak{A}_{s_n}$ if $ar = s_1 \ldots s_n$), and each relation symbol $\pi \in P_{ar}$ as a relation $\pi^{\mathfrak{A}} \subseteq \mathfrak{A}_{ar}$. Morphisms between models are the usual Σ-homomorphisms, i.e., S-sorted functions that preserve the structure. The models over an algebraic signature are called algebras.

For any signature morphism $\chi : \Sigma \to \Sigma'$, where $\Sigma = (S, F, P)$ and $\Sigma' = (S', F', P')$, the model functor $\mathsf{Mod}(\chi) : \mathsf{Mod}(\Sigma') \to \mathsf{Mod}(\Sigma)$ is defined as follows:

1. *The reduct $\mathfrak{A}'\restriction_\chi$ of a Σ'-model $\mathfrak{A}'$ is a defined by $(\mathfrak{A}'\restriction_\chi)_s = \mathfrak{A}'_{\chi(s)}$ for each sort $s \in S$, and $x^{\mathfrak{A}'\restriction_\chi} = \chi(x)^{\mathfrak{A}'}$, for each operation symbol $x \in F$ or relation symbol $x \in P$. Note that, unlike the single-sorted case, the reduct functor modifies the universes of models. For the universe of $\mathfrak{A}'\restriction_\chi$ is $\{\mathfrak{A}'_{\chi(s)}\}_{s\in S}$, which means that the sorts outside the image of S are discarded. Otherwise, the notion of reduct is standard.*
2. *The reduct $h'\restriction_\chi$ of a homomorphism h' is defined by $(h'\restriction_\chi)_s = h'_{\chi(s)}$ for all sorts $s \in S$.*

One important example of Σ-model is the first-order Σ-structure of (ground) terms T_Σ that interprets each relation as the empty set. Algebraists will recognise it as the absolutely free algebra of terms.

Sentences. *We fix a countably infinite set of variable names $\{x_i \mid i \in \omega\}$. A first-order variable for a signature $\Sigma = (S, F, P)$ is a triple (x_i, s, Σ), where x_i is the name of the variable, and $s \in S$ is the sort of the variable. The last component Σ of (x_i, s, Σ) ensures that the variable (x_i, s, Σ) is different from any constant of the signature Σ. The set of Σ-sentences is given by the following grammar:*

$$e ::= t = t' \mid \pi(t_1, \ldots, t_n) \mid \neg e \mid \wedge E \mid \forall X \cdot e'$$

where (a) $t = t'$ is an equation with $t, t' \in T_{\Sigma,s}$ and $s \in S$, (b) $\pi(t_1, \ldots, t_n)$ is a relational atom with $\pi \in P_{s_1 \ldots s_n}$, $t_i \in T_{\Sigma,s_i}$ and $s_i \in S$ for all $i \in \{1, \ldots, n\}$, (c) E is a finite set of Σ-sentences, (d) X is a finite set of variables for Σ, and (e) e' is a $\Sigma[X]$-sentence, where $\Sigma[X] = (S, F[X], P)$ and $F[X]$ is obtained by adding the variables in X as constants to F. Let $\chi : \Sigma \to \Sigma'$ be a signature morphism. The translation of a variable (x_i, s, Σ) for Σ along χ is $(x_i, \chi(s), \Sigma')$. The function $\mathsf{Sen}^{\mathsf{FOL}}(\chi) : \mathsf{Sen}^{\mathsf{FOL}}(\Sigma) \to \mathsf{Sen}^{\mathsf{FOL}}(\Sigma')$ translates sentences symbolwise.

Satisfaction relation. *Satisfaction is the usual tarskian satisfaction based on the natural interpretations of ground terms t as elements $t^{\mathfrak{A}}$ in models $\mathfrak{A}$. For example, $\mathfrak{A} \models t_1 = t_2$ iff $t_1^{\mathfrak{A}} = t_2^{\mathfrak{A}}$.*

When there is no danger of confusion, we identify a first-order variable only by its name and sort. For example, we write $(x_i : s)$ instead of (x_i, s, Σ) when it is clear from the context that $(x_i : s)$ is a variable for Σ. In case of single-sorted signatures Σ, the second component s of a variable $(x : s)$ for Σ can be dropped as well. With this convention, for any inclusion of signatures $\iota : \Sigma \hookrightarrow \Sigma'$ the corresponding function $\mathsf{Sen}(\iota) : \mathsf{Sen}(\Sigma) \to \mathsf{Sen}(\Sigma')$ is also an inclusion.

Example 3 (Order-sorted algebra (OSA))**.** *There are several non-equivalent axiomatizations of OSA in the literature including the ones proposed by Goguen and Meseguer [15] and by Poigné [30]. The version we consider here originates in [25], and it enjoys better mathematical properties than the ones enumerated before. See [26] for a comparison.*

Signatures. *An order-sorted (algebraic) signature is a triple $\Sigma = (S, \leq, F)$ with $(S, \leq)$ a poset and (S, F) a many-sorted algebraic signature. The set $S/_{\equiv_\leq}$ of connected components of $(S, \leq)$ is the quotient of S under the equivalence relation $\equiv_\leq$ generated by $\leq$. The equivalence $\equiv_\leq$ can be extended to strings of elements from S in the usual way. We say that Σ is* sensible *if for any function symbols $(\sigma : w \to s), (\sigma : w' \to s') \in F$ we have $w \equiv_\leq w'$ implies $s \equiv_\leq s'$. Any term over a sensible order-sorted signature has a unique connected component. The notion of sensible signature is a minimal syntactic requirement to avoid excessive ambiguity [26]. A partial ordering $(S, \leq)$ is* filtered *if for all $s_1, s_2 \in S$, there is some $s \in S$ such that $s_1 \leq s$ and $s_2 \leq s$. A partial ordering $(S, \leq)$ is* locally filtered *if every connected component of it is filtered. An order-sorted signature $\Sigma = (S, \leq, F)$ is locally filtered if $(S, \leq)$ is locally filtered. One can obtain locally filtered signatures by adding top elements to each connected component. Hereafter, we assume that all order-sorted signatures are sensible and locally filtered.*

For the sake of simplicity, we let $[S]$ *denote* $S/_{\equiv_\leq}$, *and* $[s]$ *denote* $s/_{\equiv_\leq}$ *for all sorts* $s \in S$. *For any connected components* $[s_1], \ldots, [s_n], [s_{n+1}] \in [S]$, *we let* $[\sigma] \colon [s_1] \ldots [s_n] \to [s_{n+1}]$ *denote the set* $\{\sigma \colon s'_1 \ldots s'_n \to s'_{n+1} \mid s'_i \in [s_i] \text{ and } 1 \leq i \leq n+1\}$ *of 'subsort polymorphic' operators with name* σ *for those components. An order-sorted signature morphism* $\chi : (S, \leq, F) \to (S', \leq', F')$ *is an algebraic signature morphism* $\chi \colon (S, F) \to (S', F')$ *such that the function* $\chi : (S, \leq) \to (S', \leq')$ *is monotonic and* χ *maps the function symbols in each subsort polymorphic family* $[\sigma] : [s_1] \ldots [s_n] \to [s_{n+1}]$ *to a subset of the set* $[\chi(\sigma)] \colon [\chi(s_1)] \ldots [\chi(s_n)] \to [\chi(s_{n+1})]$.

Models. *Let* $\Sigma = (S, \leq, F)$ *be an order-sorted signature. An order-sorted algebra* $\mathfrak{A}$ *over* Σ *is an* (S, F)*-algebra such that*

- $\mathfrak{A}_s \subseteq \mathfrak{A}_{s'}$ *if* $s \leq s'$, *and*
- $(\sigma \colon w \to s)^{\mathfrak{A}}(a) = (\sigma \colon w' \to s')^{\mathfrak{A}}(a)$ *for all function symbols* $(\sigma : w \to s) \in F$ *and* $(\sigma : w' \to s') \in ([\sigma] \colon [w] \to [s])$ *and any element* $a \in \mathfrak{A}_w \cap \mathfrak{A}_{w'}$.

According to [25], sensibility ensures the existence of the initial order-sorted algebra of terms T_Σ, *which is defined as follows:*

- $c \in T_{\Sigma,s}$ *for all constants* $c :\to s \in F$,
- $\sigma(t_1, \ldots, t_n) \in T_{\Sigma,s}$ *for all* $(\sigma : s_1 \ldots s_n \to s) \in F$, $i \in \{1, \ldots, n\}$ *and* $t_i \in T_{\Sigma,s_i}$,
- $t \in T_{\Sigma,s}$ *for all* $(s' \leq s) \in (S, \leq)$ *and* $t \in T_{\Sigma,s'}$, *and*
- *for each* $(\sigma : s_1 \ldots, s_n \to s) \in F$, *the function* $\sigma^{T_\Sigma} : T_{\Sigma,s_1} \times \cdots \times T_{\Sigma,s_n} \to T_{\Sigma,s}$ *is defined by* $\sigma^{T_\Sigma}(t_1, \ldots, t_n) = \sigma(t_1, \ldots, t_n)$ *for all* $i \in \{1, \ldots, n\}$ *and* $t_i \in T_{\Sigma,s_i}$.

The homomorphisms $h : \mathfrak{A} \to \mathfrak{B}$ *over* Σ *are many-sorted algebraic homomorphisms such that* $h_s(a) = h_{s'}(a)$ *for all sorts* $s, s' \in S$ *with* $[s] = [s']$ *and all elements* $a \in \mathfrak{A}_s \cap \mathfrak{A}_{s'}$. *Every isomorphism is clearly bijective. However, unless the underlying signature is locally filtered, it is easy to exhibit signatures where some bijective homomorphisms fail to be isomorphisms [14, Example 10.2.17]. Local filtration is also used in the construction of quotient order-sorted algebras.*

Sentences. *The set of atomic sentences over a signature consists of equations* $t = t'$, *where the sorts of both* t *and* t' *belong to the same connected component. The set of all sentences over a signature is constructed from equations by iteration of Boolean connectives and quantification.*

Satisfaction relation. *The satisfaction of equations is based on the natural interpretation of terms in order-sorted algebras. The satisfaction of all sentences is defined by induction on the structure of sentences as for first-order logic.*

Example 4 (Higher-order logic with Henkin semantics (HNK))**.** *Higher-order logic with Henkin semantics has been introduced and studied in [4] and [24]. In this article we consider a simplified version close to the "higher-order algebra" of [27] which does not consider* λ*-abstraction.*

For any set S *of sorts, let* $\overrightarrow{S}$ *be the set of* S*-types defined as the least set such that* $S \subseteq \overrightarrow{S}$ *and* $s_1 \to s_2 \in \overrightarrow{S}$ *when* $s_1, s_2 \in \overrightarrow{S}$. *A* HNK*-signature is a tuple* (S, F), *where* S *is a set of sorts and* F *is a family of sets of constants* $F = \{F_s\}_{s \in \overrightarrow{S}}$. *A signature* *morphism $\chi : (S, F) \to (S', F')$ consists of a function $\chi^{st} : S \to S'$ and a family of functions between operation symbols $\{\chi^{op}_s : F_s \to F'_{\chi^{type}(s)}\}_{s\in\overrightarrow{S}}$ where $\chi^{type} : \overrightarrow{S} \to \overrightarrow{S'}$ is the natural extension of χ^{st} to $\overrightarrow{S}$. For every signature (S, F), a (S, F)-model interprets each (a) sort $s \in S$ as a set, and (b) function symbol $\sigma \in F_s$ as an element of $\mathfrak{A}_s$, where for all types $s_1, s_2 \in \overrightarrow{S}$, $\mathfrak{A}_{s_1\to s_2} \subseteq [\mathfrak{A}_{s_1} \to \mathfrak{A}_{s_2}] = \{f \text{ function } \mid f : \mathfrak{A}_{s_1} \to \mathfrak{A}_{s_2}\}$. An (S, F)-model morphism $h : \mathfrak{A} \to \mathfrak{B}$ interprets each type $s \in \overrightarrow{S}$ as a function $h_s : \mathfrak{A}_s \to \mathfrak{B}_s$ such that $h(\sigma^{\mathfrak{A}}) = \sigma^{\mathfrak{B}}$, for all function symbols $\sigma \in F_s$, and the following diagram commutes*

$$\begin{tikzcd}
\mathfrak{A}_{s_1} \arrow[r, "f"] \arrow[d, "h_{s_1}"'] & \mathfrak{A}_{s_2} \arrow[d, "h_{s_2}"] \\
\mathfrak{B}_{s_1} \arrow[r, "h_{s_1\to s_2}(f)"'] & \mathfrak{B}_{s_2}
\end{tikzcd}$$

for all types $s_1, s_2 \in \overrightarrow{S}$ and functions $f \in \mathfrak{A}_{s_1\to s_2}$.

The set of terms T_Σ over a signature $\Sigma = (S, F)$ is defined inductively: (a) $\sigma \in T_{\Sigma,s}$ for all types $s \in \overrightarrow{S}$ and all function symbols $\sigma \in F_s$, and (b) $t_1t_2 \in T_{\Sigma,s_2}$ for all types $s_1 \to s_2 \in \overrightarrow{S}$ and all terms $t_1 \in T_{\Sigma,s_1\to s_2}$ and $t_2 \in T_{\Sigma,s_1}$. The set of atomic sentences over Σ consists of all equations of the form $t_1 = t_2$, where t_1 and t_2 are terms of the same type. Full sentences are constructed from equations by iteration of Boolean connectives and quantification over variables of any type.

The satisfaction of equations $t_1 = t_2$ is based on the natural interpretations of the terms t_1 and t_2 as functions $t_1^{\mathfrak{A}}$ and $t_2^{\mathfrak{A}}$ in higher-order models $\mathfrak{A}$. As for first-order logic, we have $\mathfrak{A} \models t_1 = t_2$ iff $t_1^{\mathfrak{A}} = t_2^{\mathfrak{A}}$. The satisfaction of all sentences is defined by induction on the structure of sentences as in the case of first-order logic.

Example 5 (HNK')**.** *This institution is obtained from* HNK *by restricting the types in a signature to types that have depths less than some cardinal number, finite or ω, fixed for the signature.*

Signatures. *The signatures are of the form (S, F, κ), where (S, F) is a higher-order signature and κ is a cardinal with $0 < \kappa \leq \omega$ such that all the depths of the types of the function symbols in F are strictly less than κ, that is, $F = \{F_s\}_{s\in\overrightarrow{S}_\kappa}$, where $\overrightarrow{S}_\kappa$ is the set of all types of depth strictly less than κ.*

The signature morphisms $\chi : (S, F, \kappa) \to (S', F', \kappa')$ consists of a higher-order signature morphism $\chi : (S, F) \to (S', F')$ such that $\kappa \leq \kappa'$.

Models. *The (S, F, κ)-models are (S, F)-models which interpret only the types in $\overrightarrow{S}_\kappa$. This means that a (S, F, κ)-model $\mathfrak{A}$ interprets (a) each sort $s \in S$ as a set $\mathfrak{A}_s$, (b) each type $s_1 \to s_2 \in \overrightarrow{S}_\kappa$ as $\mathfrak{A}_{s_1\to s_2} \subseteq [\mathfrak{A}_{s_1} \to \mathfrak{A}_{s_2}]$, and (c) each function symbol $\sigma \in F_s$, where $s \in \overrightarrow{S}_\kappa$, as an element of $\mathfrak{A}_s$.*

Sentences. *The (S, F, κ)-sentences consists of all sentences in* $\mathsf{Sen}^{\mathsf{HNK}}(S, F)$ *formed with terms of types that have the depth strictly less than κ.*

Satisfaction relation. *Satisfaction is the restriction of* $\models^{\mathsf{HNK}}$ *to* HNK' *sentences.*

Notice that HNK' is a generalisation of HNK, since working in an HNK signature (S, F) is exactly the same as working in an HNK' signature (S, F, ω).

Example 6 (Subinstitutions)**.** *Let* $\mathcal{I} = (\mathsf{Sig}, \mathsf{Sen}, \mathsf{Mod}, \models)$ *be an institution and* $\mathsf{Sig}' \subseteq \mathsf{Sig}$ *a subcategory of signature morphisms. We overload the notation and let*

- $\mathsf{Sen} : \mathsf{Sig}' \to \mathit{Set}$ *be the restriction of* $\mathsf{Sen} : \mathsf{Sig} \to \mathit{Set}$ *to* Sig'*,*
- $\mathsf{Mod} : \mathsf{Sig}' \to \mathit{Set}$ *be the restriction of* $\mathsf{Mod} : \mathsf{Sig} \to \mathit{Set}$ *to* Sig'*, and*
- $\models$ *denote also* $\{\models_\Sigma\}_{\Sigma \in |\mathsf{Sig}'|}$.

Then $\mathcal{I}' = (\mathsf{Sig}', \mathsf{Sen}, \mathsf{Mod}, \models)$ *is also an institution.*

Example 7 (Institution of presentations)**.** *In any institution* $\mathcal{I} = (\mathsf{Sig}, \mathsf{Sen}, \mathsf{Mod}, \models)$*, a presentation is a pair* (Σ, E) *consisting of a signature* $\Sigma \in |\mathsf{Sig}|$ *and a set of* Σ*-sentences* E*. A presentation morphism* $\varphi : (\Sigma, E) \to (\Sigma', E')$ *is a signature morphism* $\varphi : \Sigma \to \Sigma'$ *such that* $E' \models_{\Sigma'} \varphi(E)$*. Note that presentation morphisms are closed under composition. The institution of presentations over* $\mathcal{I}$*, denoted by* $\mathcal{I}^{pres}$ *is defined as follows:*

- Sig^{pres} *is the category of presentations over* $\mathcal{I}$*,*
- $\mathsf{Sen}^{pres}(\Sigma, E) = \mathsf{Sen}(\Sigma)$*,*
- $\mathsf{Mod}^{pres}(\Sigma, E)$ *is the full subcategory of* $\mathsf{Mod}(\Sigma)$ *of models satisfying* E*, and*
- $\mathfrak{A} \models^{pres}_{(\Sigma,E)} e$ *iff* $\mathfrak{A} \models_\Sigma e$*, for each* (Σ, E)*-model* $\mathfrak{A}$ *and* Σ*-sentence* e*.*

For the sake of simplicity, we drop the superscript *pres* from the following notations: Sen^{pres}, Mod^{pres} and $\models^{pres}$. A presentation (Σ, E) is typically an axiomatisation of a theory, with E the set of axioms. We give three examples below.

Example 8 (Subsystems of second-order arithmetic)**.** *A concrete example of presentation in* FOL *is* second-order arithmetic $\mathsf{Z}_2 = (\Sigma_{\mathsf{Z}_2}, \mathsf{T}_{\mathsf{Z}_2})$*, where*

- Σ_{Z_2} *is the first-order signature with sorts Nat and Set, function symbols* $0 :\to \mathit{Nat}$*,* $s\_ : \mathit{Nat} \to \mathit{Nat}$*,* $\_+\_ : \mathit{Nat}\ \mathit{Nat} \to \mathit{Nat}$ *and* $\_\times\_ : \mathit{Nat}\ \mathit{Nat} \to \mathit{Nat}$*, and membership relation* $\_\in\_ : \mathit{Nat}\ \mathit{Set}$*. The* number variables *are usually denoted by lower case letters* $x, y, \ldots$*, while* set variables *are usually denoted by upper case letters* $A, B, \ldots$*.*
- *The axioms* $\mathsf{T}_{\mathsf{Z}_2}$ *are the usual Peano axioms, together with*
  - *the* induction axiom
    $$\forall A : \mathit{Set} \cdot \big(0 \in A \wedge \forall x : \mathit{Nat} \cdot (x \in A \Rightarrow s\ x \in A) \Rightarrow \forall x : \mathit{Nat} \cdot x \in A\big)$$
  - *universal closures of the* comprehension scheme
    $$\exists A : \mathit{Set} \cdot \forall x : \mathit{Nat} \cdot (x \in A \Leftrightarrow \rho[x])$$
    for any sentence $\rho[x]$ *over* $\Sigma_{\mathsf{Z}_2}[x]$*. One important requirement is that* A *does not occur in* $\rho[x]$*, since if* A *could occur in* $\rho[x]$*, then the formula* $x \notin A$ *would produce an inconsistent comprehension axiom* $\exists A : \mathit{Set} \cdot \forall x : \mathit{Nat} \cdot (x \in A \Leftrightarrow x \notin A)$*.*

Subsystems of Z_2 *are obtained by restricting the formula* $\rho[x]$ *in certain ways. These systems are central to* reverse mathematics*. A comprehensive handbook of reverse mathematics is [32]. Clearly, any subsystem of* Z_2 *is a presentation over* FOL*.*

Example 9 (Relation Algebras)**.** *Another example of presentation in first-order logic is* $\mathsf{RA} = (\Sigma_{\mathsf{RA}}, \mathsf{T}_{\mathsf{RA}})$*, which consists of the axioms of relation algebras. The signature* Σ_{RA} *has one sort, say Rel (not of relations, however, but of algebra elements), no relation symbols, and the following function symbols:*

1. Boolean operations: *the constants* 0 *for bottom,* 1 *for top, the unary* $-$ *for the complement, and the binary* $\circ$ *and* $+$ *for lattice meet and join* [1]*, and*
2. operations of monoids with involution: *the constant id for the identity, the unary* $\breve{}$ *for converse (involution), and the binary* $⨟$ *for composition (multiplication). Composition* $⨟$ *binds stronger than meet* $\circ$ *and join* $+$*.*

The axioms T_{RA} *of relation algebras consist of*

A1) axioms defining Boolean algebras,

A2) axioms defining monoids with involution,

A3) $\forall a, b, c \cdot (a + b) ⨟ c = a ⨟ c + b ⨟ c$ *and* $\forall a, b, c \cdot a ⨟ (b + c) = a ⨟ b + a ⨟ c$*,*

A4) $\forall a, b \cdot (a + b)^{\smile} = a^{\smile} + b^{\smile}$*, and the* triangle laws:

A5) $\forall a, b, c \cdot\ a ⨟ b \circ c \neq 0 \ \Leftrightarrow\ a^{\smile} ⨟ c \circ b \neq 0 \ \Leftrightarrow\ b ⨟ c^{\smile} \circ a^{\smile} \neq 0.$

Example 10 (Representable Relation Algebras)**.** *Yet another example of presentation in first-order logic is given by the axioms of representable relation algebras. The signature* Σ_{RRA} *consists of (a) two sorts, Elt for elements, and Rel for relations.*[2] *(b) all the function symbols for Boolean algebras and monoids with involution from Example 9, and (c) a ternary relation* λ : *Rel Elt Elt denoting relations between elements. The axioms* $\mathsf{T}_{\mathsf{RRA}}$ *consist of (A1) — (A4) defined in Example 9, and*

R1) $\forall x : Elt, y : Elt, a : Rel, b : Rel \cdot \lambda(a + b, x, y) \Leftrightarrow \lambda(a, x, y) \vee \lambda(b, x, y)$*,*

R2) $\forall x : Elt, y : Elt, a : Rel \cdot \lambda(-a, x, y) \Leftrightarrow \lambda(1, x, y) \wedge \neg\lambda(a, x, y)$*,*

R3) $\forall x : Elt, y : Elt \cdot \lambda(id, x, y) \Leftrightarrow x = y$*,*

R4) $\forall x : Elt, y : Elt, a : Rel \cdot \lambda(a^{\smile}, x, y) \Leftrightarrow \lambda(a, y, x)$*,*

R5) $\forall x : Elt, y : Elt, a : Rel, b : Rel \cdot \lambda(a ⨟ b, x, y) \Leftrightarrow \exists z : Elt \cdot \lambda(a, x, z) \wedge \lambda(b, z, y)$*,*

R6) $\forall a : Rel \cdot a \neq 0 \Rightarrow \exists x : Elt, y : Elt \cdot \lambda(a, x, y)$*.*

A representable relation algebra is the reduct to the signature Σ_{RA} *of any* Σ_{RRA}*-model which satisfies* $\mathsf{T}_{\mathsf{RRA}}$*.*

[1] We chose rather unusual symbols for lattice operations to avoid confusion with $\wedge$ and $\vee$ as conjunction and disjunction.

[2] Note that here *Rel* are real relations: the elements of *Rel* of Example 9 can be thought of as names for elements of *Rel* of the present example, so the abstract relation algebras are represented as concrete algebras of binary relations.

Relation algebras of Example 9 are just an example of an equational theory in first-order logic. Representable relation algebras are a little more involved. Any representable relation algebra is isomorphic to a Boolean subalgebra of $\mathcal{P}(E)$, where E is an equivalence relation on some set U, with the non-Boolean operations intepreted by the natural operations on binary relations: composition, converse and identity. Not all relation algebras are representable, that is, $\mathsf{Mod}(\Sigma_{\mathsf{RRA}}, \mathsf{T}_{\mathsf{RRA}}) \upharpoonright_{\Sigma_{\mathsf{RA}}}$ is a proper subcategory of $\mathsf{Mod}(\Sigma_{\mathsf{RA}}, \mathsf{T}_{\mathsf{RA}})$. Monk [28] shows that there exists no finite axiomatisation of representable relation algebras over the signature Σ_{RA}. By Example 10, there exists a finite axiomatisation of representable relation algebras in an extended signature with two sorts; it follows, in particular, that representable relation algebras have a recursive equational axiomatisation. Incidentally, this also demonstrates the naturalness of the many-sorted approach.

2.2 Internal logic

The following institutional notions dealing with the semantics of Boolean connectives and quantifiers were defined in [33].

Definition 11 (Internal logic)**.** *Given a signature Σ in an institution, a Σ-sentence γ is a semantic*

1. negation *of a Σ-sentence e if for each Σ-model $\mathfrak{A}$,*

$$\mathfrak{A} \models \gamma \textit{ iff } \mathfrak{A} \not\models e,$$

2. conjunction *of a (finite) set of Σ-sentences E if for each Σ-model $\mathfrak{A}$,*

$$\mathfrak{A} \models \gamma \textit{ iff } \mathfrak{A} \models e \textit{ for all } e \in E,$$

3. universal χ-quantification *of a Σ'-sentence e', where $\chi : \Sigma \to \Sigma'$, if for each Σ-model $\mathfrak{A}$,*

$$\mathfrak{A} \models \gamma \textit{ iff } \mathfrak{A}' \models_{\Sigma'} e' \textit{ for all } \chi\textit{-expansions } \mathfrak{A}' \textit{ of } \mathfrak{A}.$$

Distinguished negation is usually denoted by $\neg\_$, distinguished conjunction by $\wedge\_$, and distinguished universal χ-quantification by $\forall \chi \cdot \_$. □

As we have already mentioned, quantifiers need a special treatment. Intuitively, the sentence $\forall x \cdot e[x]$ should hold in $\mathfrak{A}$ if and only if the open formula $e[x]$ is satisfied by $\mathfrak{A}$ on all valuations v into $\mathfrak{A}$. This is equivalent to saying that for all expansions $(\mathfrak{A}, a)$ of $\mathfrak{A}$, we have that $(\mathfrak{A}, a)$ satisfies $e[x/a]$; and it is the way quantification is rendered in institutions. Namely, let $\chi : \Sigma \to \Sigma[x]$ be a signature morphism that adds the variable x as a new constant to Σ. The sentence $\forall x \cdot e'$ is an abbreviation of $\forall \chi \cdot e'$ and the third clause in Definition 11 ensures that we consider all χ-expansions of $\mathfrak{A}$. Thus, the classical notion of valuation is incorporated into expansions. Variables belong to an extended language: one can think of them as names coming from some external pool and used as needed to name special constants. Incidentally, this approach avoids the usual caveats about accidental binding of free variables.

Sentence building operators (Boolean connectives and quantifiers, but possibly also infinitary connectives, modalities and suchlike) are part of the metalanguage and they are used to construct sentences which belong to the internal language of individual institutions using (an appropriate extension of) the universal semantics presented above.

Here we will only consider classical finitary Boolean connectives and quantifiers. We also use the classical definitions of $\_ \vee \_$, $\exists \chi \cdot \_$, etc. For example, $\exists \chi \cdot e' := \neg \forall \chi \cdot \neg e'$ and $\top := \wedge \emptyset$. For standard logical operators we adopt the following convention about their binding strength: $\neg$ binds stronger than $\wedge$, which binds stronger than $\vee$, which binds stronger than $\Rightarrow$, which binds stronger than quantifiers; quantifiers $\exists$ and $\forall$ have the same binding strength.

An institution $\mathcal{I}$ is said to be semantically closed under negation (conjunction, universal quantification, etc.) if every sentence in $\mathcal{I}$ has a semantic negation (conjunction, universal quantification, etc.) according to Definition 11.

2.3 Compactness and downward Löwenheim-Skolem property

Here we recall two fundamental model theoretic results on which the proof of Lindström's theorem rests.

Definition 12 (Compactness). *An institution* $\mathcal{I} = (\mathsf{Sig}, \mathsf{Sen}, \mathsf{Mod}, \models)$ *is* compact *if for all signatures* $\Sigma \in |\mathsf{Sig}|$ *and all sets of* Σ*-sentences* Γ *we have* $\Gamma \models_\Sigma \gamma$ *iff* $\Gamma_f \models_\Sigma \gamma$ *for some finite subset* $\Gamma_f \subseteq \Gamma$.

Theorem 13. FOL, OSA *and* HNK' *are compact.*

Institutional proof of compactness of FOL can be found, for example, in [7]. Same abstract results are applicable to OSA too. For compactness of HNK' one can easily adapt the arguments used for HNK in [7].

The *downward Löwenheim-Skolem (DLS) property* of FOL is the property that each satisfiable and countable set of sentences has a countable model.[3] In order to state this definition for an arbitrary institution $\mathcal{I} = (\mathsf{Sig}, \mathsf{Sen}, \mathsf{Mod}, \models)$, we will consider a subfunctor $\mathsf{Mod}_c : \mathsf{Sig} \to \mathsf{Cat}^{\mathsf{op}}$ of the model functor $\mathsf{Mod} : \mathsf{Sig} \to \mathsf{Cat}^{\mathsf{op}}$.

Definition 14 (Downward Löwenheim-Skolem property). *Let* $\mathcal{I} = (\mathsf{Sig}, \mathsf{Sen}, \mathsf{Mod}, \models)$ *be an institution equipped with a model subfunctor* $\mathsf{Mod}_c : \mathsf{Sig} \to \mathsf{Cat}^{\mathsf{op}}$ *of* Mod. $\mathcal{I}$ *has the* DLS property (via Mod_c) *if for all signatures* $\Sigma \in |\mathsf{Sig}|$ *and all satisfiable and countable sets of* Σ*-sentences* Γ*, there exists a model* $\mathfrak{A} \in |\mathsf{Mod}_c(\Sigma)|$ *such that* $\mathfrak{A} \models \Gamma$.

Technically, our definition is parameterised by the functor Mod_c. However, we do not make it explicit, since whenever an institution $\mathcal{I}$ is given, the functor Mod_c is implicitly fixed. In the examples of institutions introduced in Section 2.1, Mod_c selects countable models among all models: to be precise, Mod_c is the model functor which maps signatures to classes of models with countable carrier sets.

Theorem 15 (Downward Löwenheim-Skolem Theorem). FOL, OSA *and* HNK' *have the DLS property.*

[3] In many-sorted setting, we say that a model is countable if the carrier set of each sort is countable.

An institutional proof for FOL is given in [18], which is applicable to OSA too. For HNK' one can again adapt the arguments used in [18] for HNK.

To compare the expressivity of two institutions that share the same model functor, we extend the terminology used in [10] to institutional model theory.

Definition 16 (Expressivity)**.** *Consider two institutions $\mathcal{I} = (\mathsf{Sig}, \mathsf{Sen}, \mathsf{Mod}, \models)$ and $\mathcal{J} = (\mathsf{Sig}, \mathsf{Sen}^{\mathcal{J}}, \mathsf{Mod}, \models^{\mathcal{J}})$ that share the category of signatures and the model functor.*

- *$\mathcal{J}$ is* at least as strong as *$\mathcal{I}$, in symbols $\mathcal{I} \lesssim \mathcal{J}$, if for each signature $\Sigma \in |\mathsf{Sig}|$ and any sentence $\rho \in \mathsf{Sen}^{\mathcal{I}}(\Sigma)$ there exists a sentence $e_\rho \in \mathsf{Sen}^{\mathcal{J}}(\Sigma)$ such that $\mathfrak{A} \models^{\mathcal{I}} \rho$ iff $\mathfrak{A} \models^{\mathcal{J}} e_\rho$ for all Σ-models $\mathfrak{A}$.*
- *$\mathcal{J}$ has the same expressive power as $\mathcal{I}$, in symbols $\mathcal{I} \sim \mathcal{J}$, iff $\mathcal{I} \lesssim \mathcal{J}$ and $\mathcal{J} \lesssim \mathcal{I}$.*

To lighten the notation a little, we adopt the following convention: assuming that $\mathcal{I} \lesssim \mathcal{J}$ as in Definition 16, for all $\Sigma \in |\mathsf{Sig}|$, $\rho \in \mathsf{Sen}^{\mathcal{I}}(\Sigma)$ and $\gamma \in \mathsf{Sen}^{\mathcal{J}}(\Sigma)$, we will write $\rho \models \gamma$ to mean $e_\rho \models^{\mathcal{J}} \gamma$. Now we are ready to define Lindström property.

Definition 17 (Lindström property)**.** *Let $\mathcal{I} = (\mathsf{Sig}, \mathsf{Sen}, \mathsf{Mod}, \models)$ be an institution equipped with a model subfunctor $\mathsf{Mod}_c : \mathsf{Sig} \to \mathsf{Cat}^{\mathsf{op}}$ of Mod such that*

(a) $\mathcal{I}$ is semantically closed under Boolean connectives,

(b) $\mathcal{I}$ compact, and

(c) $\mathcal{I}$ has the DLS property of Definition 14.

$\mathcal{I}$ has the Lindström property *if for all institutions $\mathcal{J} = (\mathsf{Sig}, \mathsf{Sen}^{\mathcal{J}}, \mathsf{Mod}, \models^{\mathcal{J}})$ such that $\mathcal{I} \lesssim \mathcal{J}$ and $\mathcal{J}$ has the properties (a)–(c), we have $\mathcal{I} \sim \mathcal{J}$.*

The following lemma will be useful later on to prove our results.

Lemma 18. *Let $\mathcal{I} = (\mathsf{Sig}, \mathsf{Sen}, \mathsf{Mod}, \models)$ and $\mathcal{J} = (\mathsf{Sig}, \mathsf{Sen}^{\mathcal{J}}, \mathsf{Mod}, \models^{\mathcal{J}})$ be two institutions such that $\mathcal{I} \lesssim \mathcal{J}$ and $\mathcal{J}$ is compact. For all $\Sigma \in |\mathsf{Sig}|$, $T \subseteq \mathsf{Sen}^{\mathcal{I}}(\Sigma)$ and $\gamma \in \mathsf{Sen}^{\mathcal{J}}(\Sigma)$ such that $T \models \gamma$, there exists a finite set $T_f \subseteq T$ such that $T_f \models \gamma$.*

3 Object-level description of isomorphic models

Here we give an object-level characterisation of isomorphic models, in disjoint languages. Intuitively, we want to be able to take two isomorphic models and construct their disjoint union in an extended signature that remembers which ingredients (sorts, functions, relations) come from which model.

Definition 19 (Isomorphism structure)**.** *An institution $\mathcal{I} = (\mathsf{Sig}, \mathsf{Sen}, \mathsf{Mod}, \models)$ is equipped with an* isomorphism structure *if there exist*

- *a subcategory of signature morphisms $\mathsf{Sig}^{\mathtt{ISO}} \subseteq \mathsf{Sig}$,*
- *a functor $\mathtt{ISO} : \mathsf{Sig}^{\mathtt{ISO}} \to \mathsf{Sig}$,*
- *two natural transformations $\mathtt{iso}_1, \mathtt{iso}_2 : 1_{\mathsf{Sig}^{\mathtt{ISO}}} \Rightarrow \mathtt{ISO}$,*
- *a set of $\mathtt{ISO}(\Sigma)$-sentences S_Σ for each signature $\Sigma \in |\mathsf{Sig}^{\mathtt{ISO}}|$,*

$$
\begin{tikzcd}
\Sigma' \arrow[r, "\mathtt{iso}_i(\Sigma')"] & \mathtt{ISO}(\Sigma') & (\mathtt{ISO}(\Sigma'), \mathsf{S}_{\Sigma'}) \\
\Sigma \arrow[u, "\chi"] \arrow[r, "\mathtt{iso}_i(\Sigma)"'] & \mathtt{ISO}(\Sigma) \arrow[u, "\mathtt{ISO}(\chi)"'] & (\mathtt{ISO}(\Sigma), \mathsf{S}_{\Sigma}) \arrow[u, "\mathtt{ISO}(\chi)"']
\end{tikzcd}
$$

Figure 1: Isomorphism

satisfying the following properties:

1. *For all* $\Sigma \in |\mathsf{Sig}^{\mathtt{ISO}}|$ *and all* Σ*-models* $\mathfrak{A}$ *and* $\mathfrak{B}$ *the following are equivalent:*
   (a) $\mathfrak{A}$ *and* $\mathfrak{B}$ *are isomorphic,*
   (b) *there exists* $\mathfrak{D} \in |\mathsf{Mod}(\mathtt{ISO}(\Sigma), \mathsf{S}_\Sigma)|$ *s.t.* $\mathfrak{D}\restriction_{\mathtt{iso}_1(\Sigma)} = \mathfrak{A}$ *and* $\mathfrak{D}\restriction_{\mathtt{iso}_2(\Sigma)} = \mathfrak{B}$.
2. *For all* $\chi : \Sigma \to \Sigma' \in \mathsf{Sig}^{\mathtt{ISO}}$,
   (a) *the diagram shown to the left in Figure 1 has the following property: for all* Σ'*-models* $\mathfrak{A}$ *and* $\mathfrak{B}$*, and all* $\mathtt{ISO}(\Sigma)$*-models* $\mathfrak{C}$ *such that* $\mathfrak{C}\restriction_{\mathtt{iso}_1(\Sigma)} = \mathfrak{A}\restriction_\chi$ *and* $\mathfrak{C}\restriction_{\mathtt{iso}_2(\Sigma)} = \mathfrak{B}\restriction_\chi$*, there exists an* $\mathtt{ISO}(\Sigma')$*-model* $\mathfrak{D}$ *such that* $\mathfrak{D}\restriction_{\mathtt{iso}_1(\Sigma')} = \mathfrak{A}$, $\mathfrak{D}\restriction_{\mathtt{iso}_2(\Sigma')} = \mathfrak{B}$ *and* $\mathfrak{D}\restriction_{\mathtt{ISO}(\chi)} = \mathfrak{C}$*, and*
   (b) *the diagram shown to the right in Figure 1 is a presentation morphism.*

In the examples that follow, $\mathsf{Sig}^{\mathtt{ISO}}$ is the broad subcategory of all signature morphisms injective on sorts, and the functor $\mathtt{ISO}$ maps each signature Σ to a signature $\mathtt{ISO}(\Sigma)$ over which there exists a set of sentences S_Σ that describes at an object level the relation of isomorphism between Σ-models.

3.1 Isomorphic first-order models

Of course, FOL is expressive enough to support an object-level description of isomorphic first-order models. We will show that FOL has an isomorphism structure as described in Definition 19.

Definition 20 (Disjoint union of first-order signatures)**.** *For any first-order signature* Σ*, the disjoint union* $\Sigma \uplus \Sigma$ *is defined as follows:*

> *Let* Σ^1 *and* Σ^2 *be two copies of* Σ *obtained by adding a superscript to the symbols in* Σ*; if* x *is a sort, function or relation symbol in* Σ *then* x^1 *is a sort, function or relation symbol in* Σ^1 *and* x^2 *is a sort, function or relation symbol in* Σ^2.

Then $\Sigma^1 \cup \Sigma^2 = \Sigma \uplus \Sigma$. There are many equivalent ways to define a disjoint union of two signatures. Any of them can be used in the following developments.

Proposition 21 (Object-level characterisation of isomorphic first-order models)**.**
FOL *is equipped with the following isomorphism structure:*

- $\mathsf{Sig}^{\mathtt{ISO}} \subseteq \mathsf{Sig}^{\mathsf{FOL}}$ *is the broad subcategory of all first-order signature morphisms injective on sorts.*

- *For each $\Sigma = (S, F, P)$, the signature* $\mathtt{ISO}(\Sigma)$ *is obtained from $\Sigma^1 \cup \Sigma^2$ described in Definition 20 by adding a new relation symbol $h_s : s^1 s^2$ for each sort $s \in S$, in symbols,* $\mathtt{ISO}(\Sigma) \coloneqq (S^1 \cup S^2, F^1 \cup F^2 \cup \{h_s : s^1 s^2 \mid \text{ for all } s \in S\})$.

  The morphism $\mathtt{iso}_i : \Sigma \to \mathtt{ISO}(\Sigma)$ maps each symbol x from Σ to x^i, for all $i \in \{1, 2\}$. The set S_Σ, which says that $h \coloneqq \{h_s\}_{s \in S}$ is an isomorphism, consists of the following sentences:

  A1) h is an injective function: for all $s \in S$,

  (a) $\forall x : s^1 \cdot \exists y : s^2 \cdot h_s(x, y)$, *and*

  (b) $\forall x_1 : s^1, y_1 : s^2, x_2 : s^1, y_2 : s^2 \cdot h_s(x_1, y_1) \wedge h_s(x_2, y_2) \Rightarrow (x_1 = x_2 \Leftrightarrow y_1 = y_2)$.

  A2) h is surjective: $\forall y : s^2 \cdot \exists x : s^1 \cdot h_s(x, y)$, *for all $s \in S$.*

  A3) h preserves the structure:

  (a) *for all* $(\sigma \colon s_1 \dots s_n \to s) \in F$,
  $$\forall x_1 : s_1^1, y_1 : s_1^2, \dots, x_n : s_n^1, y_n : s_n^2 \cdot \bigwedge_{i=1}^n h_{s_i}(x_i, y_i) \Rightarrow h_s(\sigma^1(x_1, \dots, x_n), \sigma^2(y_1, \dots, y_n)).$$

  (b) *for all* $(\pi \colon s_1 \dots s_n) \in P$,
  $$\forall x_1 : s_1^1, y_1 : s_1^2, \dots, x_n : s_n^1, y_n : s_n^2 \cdot \bigwedge_{i=1}^n h_{s_i}(x_i, y_i) \Rightarrow (\pi^1(x_1, \dots, x_n) \Leftrightarrow \pi^2(y_1, \dots, y_n)).$$

- *Let $\chi \colon \Sigma \to \Sigma'$ be a signature morphism injective on sorts, where $\Sigma = (S, F, P)$ and $\Sigma' = (S', F', P')$. Then* $\mathtt{ISO}(\chi) \colon \mathtt{ISO}(\Sigma) \to \mathtt{ISO}(\Sigma')$ *maps*

  – *x^i to $\chi(x)^i$, where x is any sort, function or relation symbol in Σ and $i \in \{1, 2\}$,*

  – *h_s to $h_{\chi(s)}$, for all sorts $s \in S$.*

Proof. For each signature morphism $\chi \colon \Sigma \to \Sigma' \in \mathsf{Sig}^{\mathtt{ISO}}$ and any $i \in \{1, 2\}$, we have $\mathtt{iso}_i(\Sigma')(\chi(x)) = \chi(x)^i = \mathtt{ISO}(\chi)(x^i) = \mathtt{ISO}(\chi)(\mathtt{iso}_i(\Sigma)(x))$ for all sorts, function or relation symbols x in Σ. Therefore, the following diagram is commutative.

$$\begin{tikzcd}
\Sigma' \arrow[r, "\mathtt{iso}_i(\Sigma')"] & \mathtt{ISO}(\Sigma') \\
\Sigma \arrow[u, "\chi"] \arrow[r, "\mathtt{iso}_i(\Sigma)"'] & \mathtt{ISO}(\Sigma) \arrow[u, "\mathtt{ISO}(\chi)"']
\end{tikzcd}$$

It follows that $\mathtt{iso}_i : 1_{\mathsf{Sig}^{\mathtt{ISO}}} \Rightarrow \mathtt{ISO}$ is a natural transformation, for all $i \in \{1, 2\}$.

1. It is straightforward to show that two first-order models $\mathfrak{A}$ and $\mathfrak{B}$ over the same signature Σ are isomorphic iff there exists $\mathfrak{C} \in |\mathsf{Mod}(\mathtt{ISO}(\Sigma), \mathsf{S}_\Sigma)|$ such that $\mathfrak{C}\!\upharpoonright_{\mathtt{iso}_1(\Sigma)} = \mathfrak{A}$ and $\mathfrak{C}\!\upharpoonright_{\mathtt{iso}_2(\Sigma)} = \mathfrak{B}$.

2. For all signature morphisms $\chi : \Sigma \to \Sigma' \in \mathsf{Sig}^{\mathtt{ISO}}$, since $\mathtt{ISO}(\chi)(\mathsf{S}_\Sigma) \subseteq \mathsf{S}_{\Sigma'}$, $\mathtt{ISO}(\chi) : (\mathtt{ISO}(\Sigma), \mathsf{S}_\Sigma) \to (\mathtt{ISO}(\Sigma'), \mathsf{S}_{\Sigma'})$ is a presentation morphism.

   Now, let $\mathfrak{A}, \mathfrak{B} \in |\mathsf{Mod}(\Sigma')|$ and $\mathfrak{C} \in |\mathsf{Mod}(\mathtt{ISO}(\Sigma))|$ such that $\mathfrak{A}\!\upharpoonright_\chi = \mathfrak{C}\!\upharpoonright_{\mathtt{iso}_1(\Sigma)}$ and $\mathfrak{B}\!\upharpoonright_\chi = \mathfrak{C}\!\upharpoonright_{\mathtt{iso}_2(\Sigma)}$. Let $\mathfrak{D}$ be the $\mathtt{ISO}(\Sigma')$-structure defined as follows:

(a) $\mathfrak{D}$ interprets each symbol in $(\Sigma')^1$ as $\mathfrak{A}$ and each symbol in $(\Sigma')^2$ as $\mathfrak{B}$.

(b) For all sorts $s' \in \chi(S)$, $\mathfrak{D}$ interprets $h_{s'}$ as $h_s^{\mathfrak{C}}$, where $s = \chi^{-1}(s')$. Since χ is injective on sorts, $h_{s'}^{\mathfrak{D}}$ is well-defined for all sorts $s' \in \chi(S)$.

(c) For all sorts $s' \in S' \setminus \chi(S)$, $h_{s'}^{\mathfrak{D}}$ is the empty set.

By (a), we get $\mathfrak{D}\restriction_{\mathtt{iso}_1(\Sigma')} = \mathfrak{A}$ and $\mathfrak{D}\restriction_{\mathtt{iso}_2(\Sigma')} = \mathfrak{B}$.

(i) Since $\mathfrak{A}\restriction_{\chi} = \mathfrak{C}\restriction_{\mathtt{iso}_1(\Sigma)}$, $\mathfrak{C}$ interprets each symbol in Σ^1 as $\mathfrak{A}\restriction_{\chi}$.

(ii) Since $\mathfrak{B}\restriction_{\chi} = \mathfrak{C}\restriction_{\mathtt{iso}_2(\Sigma)}$, $\mathfrak{C}$ interprets each symbol in Σ^2 as $\mathfrak{B}\restriction_{\chi}$.

(iii) By (b), $h_s^{\mathfrak{C}} = h_{\chi(s)}^{\mathfrak{D}}$ for all sorts $s \in S$.

By (i), (ii) and (iii), we get $\mathfrak{D}\restriction_{\mathtt{ISO}(\chi)} = \mathfrak{C}$. □

The injectivity on sorts required for $\mathsf{Sig}^{\mathtt{ISO}}$ is necessary for satisfying the condition (2a) of Definition 19. Also, it is useful to specify h_s as a relation rather than function. The reason is again the modularization of isomorphism structures expressed by Definition 19 (2a).

3.2 Isomorphic order-sorted algebras

We show that OSA is equipped with an isomorphism structure. We use superscripts to define the disjoint union of order-sorted signatures.

Proposition 22 (Object-level characterisation of isomorphic order-sorted algebras)**.** *OSA is equipped with the following isomorphic structure:*

1. *$\mathsf{Sig}^{\mathtt{ISO}} \subseteq \mathsf{Sig}^{\mathsf{OSA}}$ is the broad subcategory of all order-sorted signature morphisms injective on sorts.*

2. *For each OSA signature $\Sigma = (S, \leq, F)$,*

$$\mathtt{ISO}(\Sigma) \coloneqq \quad (S^1 \cup S^2 \cup \{Bool\}, F^1 \cup F^2 \cup \{true :\to Bool\} \ \cup \{h_s : s^1\ s^2 \to Bool \mid s \in S\}\ ).$$

The signature morphism $\mathtt{iso}_i \colon \Sigma \to \mathtt{ISO}(\Sigma)$ maps each sort or function symbol x from Σ to x^i, where $i \in \{1, 2\}$. The set S_{Σ}, which says that $h \coloneqq \{h_s\}_{s \in S}$ is an isomorphism, consists of the following sentences:

A1) h is an injective function:

(a) for all sorts $s \in S$,
$\forall x : s^1 \cdot \exists y : s^2 \cdot h_s(x, y) = true$.

(b) for all $s_1, s_2 \in S$ such that $s_1 \equiv_{\leq} s_2$,
$$\forall x_1 : s_1^1, y_1 : s_1^2, x_2 : s_2^1, y_2 : s_2^2 \cdot h_{s_1}(x_1, y_1) = true \wedge h_{s_2}(x_2, y_2) = true \Rightarrow (x_1 = x_2 \Leftrightarrow y_1 = y_2).$$

A2) h is surjective: $\forall y : s^2 \cdot \exists x : s^1 \cdot h_s(x, y) = true$, for all sorts $s \in S$.

A3) h preserves the structure: for all $(\sigma : s_1 \dots s_n \to s) \in F$,

$$\forall x_1 : s_1^1, y_1 : s_1^2, \dots, x_n : s_n^1, y_n : s_n^2 \cdot \bigwedge_{i=1}^n h_{s_i}(x_i, y_i) = \mathit{true} \Rightarrow (h_s(\sigma^1(x_1, \dots, x_n), \sigma^2(y_1, \dots, y_n)) = \mathit{true}.$$

Proposition 22 is a straightforward generalization of Proposition 21 to the order-sorted case.

3.3 Isomorphic higher-order models

Here we show that HNK' is equipped with an isomorphism structure. The disjoint union of higher-order signatures is defined as for first-order signatures, using superscripts.

Proposition 23 (Object-level characterisation of isomorphic higher-order models)**.** HNK' *is equipped with the following isomorphism structure:*

- $\mathsf{Sig}^{\mathtt{ISO}} \subseteq \mathsf{Sig}^{\mathsf{HNK}'}$ *is the broad subcategory of all signature morphisms injective on sorts.*
- *For each* HNK' *signature* $\Sigma = (S, F, \kappa)$,

  $$\mathtt{ISO}(\Sigma) \coloneqq \quad (S^1 \cup S^2 \cup \{\mathit{Bool}\}, F^1 \cup F^2 \cup \{\mathit{true} : \mathit{Bool}\} \ \cup \{h_s \colon s^1 \to s^2 \to \mathit{Bool} \mid s \in \overrightarrow{S}_\kappa\}, 2 + \kappa \ ).$$

  The signature morphism $\mathtt{iso}_i \colon \Sigma \to \mathtt{ISO}(\Sigma)$ *maps each sort or function symbol* x *from* Σ *to* x^i, *where* $i \in \{1, 2\}$. *The set* S_Σ, *which says that* $h \coloneqq \{h_s\}_{s \in \overrightarrow{S}_\kappa}$ *is an isomorphism, consists of the following sentences:*

  A1) h is an injective function: for all types $s \in \overrightarrow{S}_\kappa$,

  (a) $\forall x : s^1 \cdot \exists y : s^2 \cdot h_s\ x\ y = \mathit{true}$, *and*

  (b) $\forall x_1 : s^1, x_2 : s^1, y_1 : s^2, y_2 : s^2 \cdot h_s\ x_1\ y_1 = \mathit{true} \wedge h_s\ x_2\ y_2 = \mathit{true} \Rightarrow (x_1 = x_2 \Leftrightarrow y_1 = y_2)$.

  A2) h is surjective: $\forall y : s^2 \cdot \exists x : s^1 \cdot h_s\ x\ y = \mathit{true}$, *for all types* $s \in \overrightarrow{S}_\kappa$.

  A3) h preserves the structure:

  (a) for all $\sigma : s \in F$, $h_s\ \sigma^1\ \sigma^2 = \mathit{true}$.

  (b) for all types $s_1 \to s_2 \in \overrightarrow{S}_\kappa$,

  $$\forall f : s_1^1 \to s_2^1, g : s_1^2 \to s_2^2, x : s^1, y : s_1^2 \cdot h_{s_1 \to s_2} f\ g = \mathit{true}\ \wedge h_{s_1} x\ y = \mathit{true} \Rightarrow h_{s_2}(f\ x)(g\ y) = \mathit{true}.$$

- *Let* $\chi \colon \Sigma \to \Sigma'$ *be a signature morphism, where* $\Sigma = (S, F, \kappa)$ *and* $\Sigma' = (S', F', \kappa')$. *The signature morphism* $\mathtt{ISO}(\chi) \colon \mathtt{ISO}(\Sigma) \to \mathtt{ISO}(\Sigma')$ *maps*
  - x^i *to* $\chi(x)^i$, *where* x *is any sort or function symbol in* Σ *and* $i \in \{1, 2\}$,
  - h_s *to* $h_{\chi(s)}$, *for all types* $s \in \overrightarrow{S}_\kappa$, *and*
  - *it is the identity on the rest of the symbols.*

The proof of Proposition 23 is conceptually the same as the proof of Proposition 21.

$$\begin{tikzcd} \Sigma' \arrow[r, "\mathtt{ele}_i(\Sigma')"] & \mathtt{ELE}(\Sigma') & (\mathtt{ELE}(\Sigma'), \mathsf{E}_{\Sigma'}) \\ \Sigma \arrow[u, "\chi"] \arrow[r, "\mathtt{ele}_i(\Sigma)"'] & \mathtt{ELE}(\Sigma) \arrow[u, "\mathtt{ELE}(\chi)"'] & (\mathtt{ELE}(\Sigma), \mathsf{E}_{\Sigma}) \arrow[u, "\mathtt{ELE}(\chi)"'] \end{tikzcd}$$

Figure 2: Elementary equivalence

4 Object-level description of elementary equivalence

Our next goal is to obtain an object-level characterisation of elementary equivalence. In analogy to the previous section, we will define two crucial properties and then show that three benchmark examples of logics, formalised as institutions, have them. To this end we will apply the institution independent version of Fraïssé-Hintikka Theorem from [21]. This section is a main stepping stone in our proof of Lindström's theorem.

Definition 24 (Elementary equivalence structure)**.** *An institution $\mathcal{I}$ of the form* $(\mathsf{Sig}, \mathsf{Sen}, \mathsf{Mod}, \models)$ *is equipped with an* elementary equivalence structure *if there exist*

- *two subcategories of signature morphisms* $\mathsf{Sig}^{\mathtt{FIN}} \subseteq \mathsf{Sig}^{\mathtt{ELE}} \subseteq \mathsf{Sig}$*,*
- *a functor* $\mathtt{ELE} : \mathsf{Sig}^{\mathtt{ELE}} \to \mathsf{Sig}$*,*
- *two natural transformations* $\mathtt{ele}_1, \mathtt{ele}_2 : 1_{\mathsf{Sig}^{\mathtt{ELE}}} \Rightarrow \mathtt{ELE}$*,*
- *a set of sentences* E_Σ *over* $\mathtt{ELE}(\Sigma)$*, for each signature* $\Sigma \in |\mathsf{Sig}^{\mathtt{FIN}}|$*,*

satisfying the following properties:

1. *For all* $\Sigma \in |\mathsf{Sig}^{\mathtt{FIN}}|$ *and all* Σ*-models* $\mathfrak{A}$ *and* $\mathfrak{B}$ *the following are equivalent:*
   (a) $\mathfrak{A}$ *and* $\mathfrak{B}$ *are elementarily equivalent,*
   (b) *there exists* $\mathfrak{C} \in |\mathsf{Mod}(\mathtt{ELE}(\Sigma), \mathsf{E}_\Sigma)|$ *s.t.* $\mathfrak{C}\upharpoonright_{\mathtt{ele}_1(\Sigma)} = \mathfrak{A}$ *and* $\mathfrak{C}\upharpoonright_{\mathtt{ele}_2(\Sigma)} = \mathfrak{B}$*.*
2. *For all signature morphisms* $\chi \colon \Sigma \to \Sigma' \in \mathsf{Sig}^{\mathtt{ELE}}$*, the diagram shown to the left in Figure 2 has the following property:*
   for all Σ'*-models* $\mathfrak{A}$ *and* $\mathfrak{B}$*, and all* $\mathtt{ELE}(\Sigma)$*-models* $\mathfrak{C}$ *such that* $\mathfrak{C}\upharpoonright_{\mathtt{ele}_1(\Sigma)} = \mathfrak{A}\upharpoonright_\chi$ *and* $\mathfrak{C}\upharpoonright_{\mathtt{ele}_2(\Sigma)} = \mathfrak{B}\upharpoonright_\chi$*, there exists an* $\mathtt{ELE}(\Sigma')$*-model* $\mathfrak{D}$ *such that* $\mathfrak{D}\upharpoonright_{\mathtt{ele}_1(\Sigma')} = \mathfrak{A}$*,* $\mathfrak{D}\upharpoonright_{\mathtt{ele}_2(\Sigma')} = \mathfrak{B}$ *and* $\mathfrak{D}\upharpoonright_{\mathtt{ELE}(\chi)} = \mathfrak{C}$*.*
3. *For all signature morphisms* $\chi : \Sigma \to \Sigma' \in \mathsf{Sig}^{\mathtt{FIN}}$*, the diagram shown to the right in Figure 2 is a presentation morphism.*

In the examples of institutions considered here, $\mathsf{Sig}^{\mathtt{ELE}}$ is the broad subcategory of all signature morphisms injective on sorts and $\mathsf{Sig}^{\mathtt{FIN}}$ is the subcategory of signature morphisms injective on sorts whose objects consist of a finite number of symbols. Then $\mathtt{ELE}$ maps each signature Σ to a signature $\mathtt{ELE}(\Sigma)$ such that if Σ is finite there exists a set of sentences E_Σ over $\mathtt{ELE}(\Sigma)$ which describes the relation of finite isomorphism between Σ-models. By an application of Fraïssé-Hintikka Theorem, E_Σ describes

$$
\begin{tikzcd}
\mathtt{ELE}(\Sigma') \arrow[r, "\mathtt{bf}(\Sigma')"] & \mathtt{BF}(\Sigma') & (\mathtt{ELE}(\Sigma'), \mathsf{E}_{\Sigma'}) \arrow[r, "\mathtt{bf}(\Sigma')"] & (\mathtt{BF}(\Sigma'), \mathsf{B}_{\Sigma'}) \\
\mathtt{ELE}(\Sigma) \arrow[u, "\mathtt{ELE}(\chi)"'] \arrow[r, "\mathtt{bf}(\Sigma)"'] & \mathtt{BF}(\Sigma) \arrow[u, "\mathtt{BF}(\chi)"'] & (\mathtt{ELE}(\Sigma), \mathsf{E}_{\Sigma}) \arrow[u, "\mathtt{ELE}(\chi)"'] \arrow[r, "\mathtt{bf}(\Sigma)"'] & (\mathtt{BF}(\Sigma), \mathsf{B}_{\Sigma}) \arrow[u, "\mathtt{BF}(\chi)"']
\end{tikzcd}
$$

Figure 3: Back-and-forth equivalence

elementary equivalence between Σ-models. The description involves counting the number of times partial isomorphisms can be extended, so $\mathtt{ELE}(\Sigma)$ will typically have a sort *Nat* of natural numbers.

We will also need a stronger structure describing back-and-forth equivalence. Elementary equivalence is too weak to force isomorphism, but two countable models are back-and-forth equivalent iff they are isomorphic. Definition 25 provides an institution-independent infrastructure that enables an object-level description of back-and-forth equivalence.

Definition 25 (Back-and-forth equivalence structure)**.** *An institution $\mathcal{I}$ of the form* $(\mathsf{Sig}, \mathsf{Sen}, \mathsf{Mod}, \models)$ *is equipped with a* back-and-forth equivalence structure *if it is equipped with an elementary equivalence structure and, in addition, there exist:*

- *a model functor* $\mathsf{Mod}_c \colon \mathsf{Sig} \to \mathsf{Cat}^{\mathsf{op}}$*, which is a subfunctor of* Mod*,*
- *a functor* $\mathtt{BF} \colon \mathsf{Sig}^{\mathtt{ELE}} \to \mathsf{Sig}$*,*
- *a natural transformation* $\mathtt{bf} \colon \mathtt{ELE} \Rightarrow \mathtt{BF}$*,*
- *a countable set of* $\mathtt{BF}(\Sigma)$*-sentences* B_Σ*, for all signatures* $\Sigma \in |\mathsf{Sig}^{\mathtt{FIN}}|$*,*

satisfying the following properties:

1. *For any signature* $\Sigma \in |\mathsf{Sig}^{\mathtt{FIN}}|$*,*
   (a) *for all models* $\mathfrak{A} \in |\mathsf{Mod}(\mathtt{ELE}(\Sigma), \mathsf{E}_\Sigma)|$ *and all finite subsets* $\mathsf{T} \subseteq \mathsf{B}_\Sigma$ *there exists a* $\mathtt{bf}(\Sigma)$*-expansion* $\mathfrak{B}$ *of* $\mathfrak{A}$ *such that* $\mathfrak{B} \models \mathsf{T}$*, and*
   (b) *for all* $\mathfrak{A} \in |\mathsf{Mod}_c(\mathtt{BF}(\Sigma), \mathsf{B}_\Sigma)|$ *we have* $\mathfrak{A}\restriction_{\mathtt{ele}_1(\Sigma);\mathtt{bf}(\Sigma)} \cong \mathfrak{A}\restriction_{\mathtt{ele}_2(\Sigma);\mathtt{bf}(\Sigma)}$.
2. *For all signature morphisms* $\chi \colon \Sigma \to \Sigma' \in \mathsf{Sig}^{\mathtt{ELE}}$*, the diagram shown to the left in Figure 3 has the following property: for all* $\mathtt{ELE}(\Sigma')$*-models* $\mathfrak{A}$*, and all* $\mathtt{BF}(\Sigma)$*-models* $\mathfrak{B}$ *such that* $\mathfrak{B}\restriction_{\mathtt{bf}(\Sigma)} = \mathfrak{A}\restriction_{\mathtt{ELE}(\chi)}$*, there exists a* $\mathtt{BF}(\Sigma')$*-model* $\mathfrak{C}$ *such that* $\mathfrak{C}\restriction_{\mathtt{bf}(\Sigma')} = \mathfrak{A}$*,* $\mathfrak{C}\restriction_{\mathtt{BF}(\chi)} = \mathfrak{A}$*.*
3. *For all signature morphisms* $\chi \colon \Sigma \to \Sigma' \in \mathsf{Sig}^{\mathtt{FIN}}$*, the diagram shown to the right in Figure 3 is a commutative square of presentation morphisms.*

In our examples, the signature $\mathtt{BF}(\Sigma)$ is obtained from $\mathtt{ELE}(\Sigma)$ by adding a new constant c of sort *Nat*. The set B_Σ is obtained from E_Σ by adding a countably infinite set of sentences which collectively say that c is a nonstandard number, which has an infinite number of predecessors. Note that (1a) above says that B_Σ is finitely satisfiable, so compactness (if it holds) will yield a $\mathtt{BF}(\Sigma)$-model of B_Σ. Since the elements of sort *Nat* count the number of times partial isomorphisms can be extended, in any model $\mathfrak{C}$ of B_Σ, a certain partial isomorphism between Σ-models living inside $\mathfrak{C}$ will be extendable infinitely many times, giving back-and-forth equivalence. Then (1b) says that for a *countable* $\mathfrak{C}$, these Σ-models inside $\mathfrak{C}$ (i.e. reducts of $\mathfrak{C}$) will be isomorphic.

4.1 Elementarily equivalent first-order models

We are now ready to give an object level characterisation of elementarily equivalent first-order models.

Definition 26 (Partial isomorphism)**.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two first-order models over a signature $\Sigma = (S, F, P)$. A partial isomorphism $p\colon \mathfrak{A} \nrightarrow \mathfrak{B}$ is an injective partial many-sorted function $\{p_s\colon \mathfrak{A}_s \nrightarrow \mathfrak{B}_s\}_{s\in S}$ that is*

1. *a homomorphism, in the following sense:*
   (a) *for all $\sigma\colon s_1 \dots s_n \to s_{n+1} \in F$ and $a_1 \in dom(p)_{s_1}, \dots, a_{n+1} \in dom(p)_{s_{n+1}}$, $\sigma^{\mathfrak{A}}(a_1, \dots, a_n) = a_{n+1}$ iff $\sigma^{\mathfrak{B}}(p_{s_1}(a_1), \dots, p_{s_n}(a_n)) = p_{s_{n+1}}(a_{n+1})$;*
   (b) *for all $\pi\colon s_1 \dots s_n \in P$ and $a_1 \in dom(p)_{s_1}, \dots, a_n \in dom(p)_{s_n}$, we have $(a_1, \dots, a_n) \in \pi^{\mathfrak{A}}$ iff $(p_{s_1}(a_1), \dots, p_{s_n}(a_n)) \in \pi^{\mathfrak{B}}$; and*
2. *the domain of p includes the interpretation of all constants, which means that $\{c^{\mathfrak{A}} \mid c\colon\to s \in F\} \subseteq dom(p)$.*

Our definition of partial isomorphism is slightly different from [10] since it requires that the interpretation of all constants to be included in the domain.

Example 27 (Unnested isomorphism)**.** *An* unnested isomorphism *is a partial isomorphism $p\colon \mathfrak{A} \nrightarrow \mathfrak{B}$ such that $dom(p) = \{c^{\mathfrak{A}} \mid c\colon\to s \in F\}$.*

By Fraïssé-Hintikka Theorem, elementary equivalence between two first-order models amounts to the existence of extensions of certain partial isomorphisms. We will now define tools needed for an object-level description of these extensions in first-order logic. The definition of *finitely isomorphic* first-order models over single-sorted signatures, which can be found, for example, in [10], can be easily adapted to many-sorted first-order models with possibly empty domains.

Definition 28 (Finitely isomorphic first-order models)**.** *Two first-order models $\mathfrak{A}$ and $\mathfrak{B}$ over the same signature $\Sigma = (S, F, P)$ are* finitely isomorphic*, in symbols, $\mathfrak{A} \cong_f \mathfrak{B}$, if there exists a family $\mathsf{F} = \{\mathsf{F}_k\}_{k\in\omega}$ of non-empty sets of partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ such that for all $k \in \omega$ and all $p \in \mathsf{F}_{k+1}$ the following two properties hold:*

1. *(Forth property) For all $s \in S$ and all $a \in \mathfrak{A}_s$ there exists $q \in \mathsf{F}_k$ such that $dom(p) \cup \{a\} \subseteq dom(q)$ and $q(x) = p(x)$ for all $x \in dom(p)$.*
2. *(Back property) For all $s \in S$ and all $b \in \mathfrak{B}_s$, there exists $q \in \mathsf{F}_k$ such that $range(p) \cup \{b\} \subseteq range(q)$ and $q(x) = p(x)$ for all $x \in dom(p)$.*

Intuitively, the back and forth properties can be expressed as follows: a partial isomorphism in F_{k+1} can be extended $k+1$ times; the corresponding extensions lie in F_k, F_{k-1}, ..., F_1, and F_0, respectively. If $\{\mathsf{F}_k\}_{k\in\omega}$ has the the back and forth properties, we write $\{\mathsf{F}_k\}_{k\in\omega} : \mathfrak{A} \cong_f \mathfrak{B}$. Next, we recall Fraïssé-Hintikka Theorem in many-sorted first-order logic. See [21] for details, including the connection to Ehrenfeucht-Fraïssé games in an institutional setting.

Theorem 29 (Fraïssé-Hintikka Theorem in FOL)**.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two first-order models over a signature Σ with a finite number of function and relation symbols. Then the following are equivalent:*

1. *$\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent.*
2. *$\mathfrak{A}$ and $\mathfrak{B}$ are finitely isomorphic.*

Based on Theorem 29, we give an object-level characterisation of elementary equivalence in first-order logic.

Proposition 30 (Object-level characterisation of elementarily equivalence in FOL)**.** FOL *is equipped with the following elementary equivalence structure:*

- $\mathsf{Sig}^{\mathtt{ELE}} \subseteq \mathsf{Sig}^{\mathsf{FOL}}$ *is the broad subcategory of all signature morphisms injective on sorts.*
- $\mathsf{Sig}^{\mathtt{FIN}} \subseteq \mathsf{Sig}^{\mathsf{FOL}}$ *is the subcategory of signature morphisms injective on sorts whose objects are finite first-order signatures.*
- *For each $\Sigma \in |\mathsf{Sig}^{\mathsf{FOL}}|$, the signature $\mathtt{ELE}(\Sigma)$ is obtained from $\Sigma^1 \cup \Sigma^2$ described in Definition 20 by adding the following new symbols:*
  - *a sort Nat, together with a binary relation $\_<\_ : Nat\ Nat$ and a predecessor function $pre\_ : Nat \to Nat$,*
  - *a sort Par for the names of partial functions, together with a binary relation $\mathsf{F} : Nat\ Par$ for finitely extendable partial isomorphisms, and*
  - *a ternary relation symbol $app_s : Par\ s^1 s^2$ for each sort $s \in S$, for graphs of partial functions.*

  In symbols, $\mathtt{ELE}(\Sigma) \coloneqq (\ S^1 \cup S^2 \cup \{Nat, Par\}, F^1 \cup F^2 \cup \{pre\_ : Nat \to Nat\}, P^1 \cup P^2 \cup \{\_<\_ : Nat\ Nat, \mathsf{F} : Nat\ Par\} \cup \{app_s : Par\ s^1\ s^2 \mid s \in S\})$.
- *For each finite first-order signature Σ, the set E_Σ, which says that two first-order models over Σ are finitely isomorphic, consists of the following sentences:*

  A1) p is a partial injective function: for each sort $s \in S$,
  $$\forall p : Par, x_1 : s^1, x_2 : s^1, y_1 : s^2, y_2 : s^2 \cdot app_s(p, x_1, y_1) \wedge app_s(p, x_2, y_2) \Rightarrow (x_1 = x_2 \Leftrightarrow y_1 = y_2).$$
  A2) p preserves and reflects functions: for each $(\sigma : s_1 \dots s_n \to s_{n+1}) \in F$,
  $$\forall p : Par, x_1 : s_1^1, y_1 : s_1^2, \dots, x_{n+1} : s_{n+1}^1, y_{n+1} : s_{n+1}^2 \cdot \bigwedge_{i=1}^{n+1} app_{s_i}(p, x_i, y_i) \Rightarrow (\sigma^1(x_1, \dots, x_n) = x_{n+1} \Leftrightarrow \sigma^2(y_1, \dots, y_n) = y_{n+1}).$$

A3) p preserves and reflects relations: for each $(\pi\colon s_1 \dots s_n \in P)$,

$$\forall p : Par, x_1 : s_1^1, y_1 : s_1^2, \dots, x_n : s_n^1, y_n : s_n^2 \cdot \bigwedge_{i=1}^{n} app_{s_i}(p, x_i, y_i) \Rightarrow (\pi^1(x_1, \dots, x_n) \Leftrightarrow \pi^2(y_1, \dots, y_n)).$$

A4) $dom(p)$ includes the interpretation of all constants: for each $c :\to s \in F$, $\forall p : Par \cdot app_s(p, c^1, c^2)$.

A5) $<$ *is irreflexive:* $\forall k : Nat \cdot \neg(k < k)$.
$<$ *is transitive:* $\forall k_1 : Nat, k_2 : Nat, k_3 : Nat \cdot k_1 < k_2 \wedge k_2 < k_3 \Rightarrow k_1 < k_3$.
$<$ *is a totally defined:* $\forall k_1 : Nat, k_2 : Nat \cdot k_1 < k_2 \vee k_1 = k_2 \vee k_2 < k_1$.
Every element except the first (if it exists) has a predecessor:

$$\forall k_1 : Nat \cdot (\exists k_2 : Nat \cdot k_2 < k_1) \Rightarrow (pre\ k_1 < k_1) \wedge \neg \exists k_2 : Nat \cdot (pre\ k_1 < k_2 \wedge k_2 < k_1).$$

Every element has a successor: $\forall k_1 : Nat \cdot \exists k_2 : Nat \cdot k_1 = pre\ k_2$.

A6) p has the back and forth properties:
$\forall k : Nat, p : Par \cdot \mathsf{forth}(k, p) \wedge \mathsf{back}(k, p)$, *where*

- $\mathsf{forth}(k, p)$ *is the formula*

$$(pre\ k < k) \wedge \mathsf{F}(k, p) \Rightarrow \bigwedge_{s \in S} \forall x_s : s^1 \cdot \exists q_s : Par, y_s : s^2 \cdot (q_s \text{ extends } p) \wedge app_s(q_s, x_s, y_s) \wedge \mathsf{F}(pre\ k, q_s),$$

- $\mathsf{back}(k, p)$ *is the formula*

$$(pre\ k < k) \wedge \mathsf{F}(k, p) \Rightarrow \bigwedge_{s \in S} \forall y_s : s^2 \cdot \exists q_s : Par, x_s : s^1 \cdot (q_s \text{ extends } p) \wedge app_s(q_s, x_s, y_s) \wedge \mathsf{F}(pre\ k, q_s),$$

- g `extends` f *is the formula*

$$\bigwedge_{s \in S} \forall x_s : s^1, y_s : s^2 \cdot app_s(f, x_s, y_s) \Rightarrow app_s(g, x_s, y_s).$$

A7) For all natural numbers k, the set of partial isomorphisms that can be extended k times is not empty: $\forall k : Nat \cdot \exists p : Par \cdot \mathsf{F}(k, p)$.

- *Let* $\chi\colon \Sigma \to \Sigma'$ *be a first-order signature morphism injective on sorts. The signature morphism* $\mathtt{ELE}(\chi)\colon \mathtt{ELE}(\Sigma) \to \mathtt{ELE}(\Sigma')$ *maps*
  - x^i *to* $\chi(x)^i$, *where x is any sort, function or relation symbol in* Σ *and* $i \in \{1, 2\}$,
  - $app_s\colon Par\ s^1 s^2$ *to* $app_{\chi(s)}\colon Par\ \chi(s)^1 \chi(s)^2$ *for all sorts s in* Σ, *and*
  - *it is the identity on the rest of the symbols.*

Proof. Let $\Sigma = (S, F, P)$ be a first-order signature. Since S is finite, the sentence at (A6) is well-defined. Since F and P are finite, by Fraïssé-Hintikka Theorem, finitely isomorphic relation yield to elementary equivalence. The set E_Σ of first-order sentences over $\mathtt{ELE}(\Sigma)$ defined above is finite, as a consequence of the finiteness of Σ.

1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two models over a finite signature $\Sigma = (S, F, P)$. We show that $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent iff there exists $\mathfrak{C} \in |\mathsf{Mod}^{\mathsf{FOL}}(\mathtt{ELE}(\Sigma), \mathsf{E}_\Sigma)|$ such that $\mathfrak{C}\!\upharpoonright_{\mathtt{ele}_1(\Sigma)} = \mathfrak{A}$ and $\mathfrak{C}\!\upharpoonright_{\mathtt{ele}_2(\Sigma)} = \mathfrak{B}$.

"⇒" Assume that $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent. By Theorem 29, we have $\{\mathsf{F}_k\}_{k\in\omega} : \mathfrak{A} \cong_f \mathfrak{B}$ for some family $\{\mathsf{F}_k\}_{k\in\omega}$ of non-empty sets of partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$. Let $\mathfrak{C}$ be the first-order model over $\mathtt{ELE}(\Sigma)$ which interprets

(a) all symbols in Σ^1 as $\mathfrak{A}$, all symbols in Σ^2 as $\mathfrak{B}$,

(b) the sort Nat as ω,

(c) the sort Par as the set of all partial isomorphisms $p\colon \mathfrak{A} \nrightarrow \mathfrak{B}$,

(d) $\mathit{app}_s^{\mathfrak{C}} \coloneqq \{(p,a,b) \mid p\colon \mathfrak{A} \nrightarrow \mathfrak{B}, a \in \mathfrak{A}_s, b \in \mathfrak{B}_s \text{ s.t. } p(a) = b\}$ for all $s \in S$,

(e) $\mathsf{F}^{\mathfrak{C}} \coloneqq \{(k,p) \mid k \in \omega \text{ and } p \in \mathsf{F}_k\}$.

It is straightforward to check that $\mathfrak{C} \models \mathsf{E}_\Sigma$.

"⇐" Let $\mathfrak{C} \in |\mathsf{Mod}^{\mathsf{FOL}}(\mathtt{ELE}(\Sigma), \mathsf{E}_\Sigma)|$ such that $\mathfrak{C}\upharpoonright_{\mathtt{ele}_1(\Sigma)} = \mathfrak{A}$ and $\mathfrak{C}\upharpoonright_{\mathtt{ele}_2(\Sigma)} = \mathfrak{B}$. By (A5), $(\mathfrak{C}_{Nat}, <^{\mathfrak{C}})$ is a discrete linear ordering with no top element. Therefore, there exists an infinite sequence

$$c_0 <^{\mathfrak{C}} c_1 <^{\mathfrak{C}} c_2 <^{\mathfrak{C}} \dots$$

such that $\mathit{pre}^{\mathfrak{C}} c_{k+1} = c_k$ for all $k \in \omega$. By (A1)–(A4), each $c_p \in \mathfrak{C}_{Par}$ can be regarded as a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ in the following way: $c_p\colon \mathfrak{A} \nrightarrow \mathfrak{B}$ such that $c_p(a) = b$ iff $(c_p, a, b) \in \mathit{app}_s^{\mathfrak{C}}$ for all $s \in S$, $a \in \mathfrak{A}_s$ and $b \in \mathfrak{B}_s$. For all $k \in \omega$, let $\mathsf{F}_k \coloneqq \{c_p \in \mathfrak{C}_{Par} \mid \mathsf{F}^{\mathfrak{C}}(c_k, c_p)\}$, which is not empty due to (A7). By (A6), the family $\{\mathsf{F}_k\}_{k\in\omega}$ has the back and forth properties required by Definition 28. Therefore, $\{\mathsf{F}_k\}_{k\in\omega} : \mathfrak{A} \cong_f \mathfrak{B}$. By Theorem 29, $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent.

2. For any first-order signature morphism injective on sorts $\chi\colon \Sigma \to \Sigma'$, we have $\mathtt{iso}_i(\Sigma')(\chi(x)) = \chi(x)^i = \mathtt{ELE}(\chi)(x^i) = \mathtt{ELE}(\chi)(\mathtt{ele}_i(\Sigma)(x))$ for all sorts, function or relation symbols x from Σ. Therefore, the following diagram is commutative.

$$\begin{tikzcd}
\Sigma' \arrow[r, "\mathtt{ele}_i(\Sigma')"] & \mathtt{ELE}(\Sigma') \\
\Sigma \arrow[u, "\chi"] \arrow[r, "\mathtt{ele}_i(\Sigma)"'] & \mathtt{ELE}(\Sigma) \arrow[u, "\mathtt{ELE}(\chi)"']
\end{tikzcd}$$

Let $\mathfrak{A}, \mathfrak{B} \in |\mathsf{Mod}(\Sigma')|$ and $\mathfrak{C} \in |\mathsf{Mod}(\mathtt{ELE}(\Sigma))|$ such that $\mathfrak{A}\upharpoonright_\chi = \mathfrak{C}\upharpoonright_{\mathtt{ele}_1(\Sigma)}$ and $\mathfrak{B}\upharpoonright_\chi = \mathfrak{C}\upharpoonright_{\mathtt{ele}_2(\Sigma)}$. Let $\mathfrak{D}$ be the $\mathtt{ELE}(\Sigma')$-structure, which interprets

(a) each symbol in $(\Sigma')^1$ as $\mathfrak{A}$ and each symbol in $(\Sigma')^2$ as $\mathfrak{B}$,

(b) the sort Nat as $\mathfrak{C}_{Nat}$, the relation symbol $<$ as $<^{\mathfrak{C}}$, and the function symbol pre as $\mathit{pre}^{\mathfrak{C}}$,

(c) the sort Par as $\mathfrak{C}_{Par}$, and the relation symbol F as $\mathsf{F}^{\mathfrak{C}}$,

(d) the relation symbol $\mathit{app}_{s'}$ as $\mathit{app}^{\mathfrak{C}}_{\chi^{-1}(s')}$ for all sorts $s' \in \chi(S)$, and

(e) the relation symbol $\mathit{app}_{s'}$ as the empty set for all sorts $s' \in S' \setminus \chi(S)$.

Since χ is injective on sorts, $app^{\mathfrak{D}}_{s'}$ is well-defined for all $s' \in \chi(S)$.

It is easy to show that $\mathfrak{D}\upharpoonright_{\mathtt{ele}_1(\Sigma')} = \mathfrak{A}$, $\mathfrak{D}\upharpoonright_{\mathtt{ele}_2(\Sigma')} = \mathfrak{B}$ and $\mathfrak{D}\upharpoonright_{\mathtt{ELE}(\chi)} = \mathfrak{C}$. □

Now we strengthen the elementary equivalence structure to a back-and-forth equivalence structure.

Definition 31 (Partially isomorphic first-order models)**.** *Two models $\mathfrak{A}$ and $\mathfrak{B}$ over the same signature $\Sigma = (S, F, P)$ are* partially isomorphic *or* back-and-forth equivalent *if there exists a non-empty set F of partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ such that for all partial isomorphism $p \in \mathsf{F}$ the following two properties hold:*

1. *(Forth property) For all $s \in S$ and all $a \in \mathfrak{A}_s$ there exists $q \in F$ such that $dom(p) \cup \{a\} \subseteq dom(q)$ and $p(x) = q(x)$ for all $x \in dom(p)$.*
2. *(Back property) For all $s \in S$ and all $b \in \mathfrak{B}_s$ there exists $q \in F$ such that $range(p) \cup \{b\} \subseteq range(q)$ and $p(x) = q(x)$ for all $x \in dom(p)$.*

The back and forth properties from Definition 31 amount to the existence of a partial isomorphism which has an infinite number of extensions. The following lemma says that back-and-forth equivalence can be turn into isomorphism provided that the universes of the underlying models are countable.

Lemma 32. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two first-order models over the same signature Σ. If $\biguplus_{s\in S} \mathfrak{A}_s$ and $\biguplus_{s\in S} \mathfrak{B}_s$ are countable then $\mathfrak{A} \cong_p \mathfrak{B}$ iff $\mathfrak{A} \cong \mathfrak{B}$.*

The proof of Lemma 32 is a straightforward generalization of [10, Lemma 1.5 (d)] to the many-sorted case. The result relies on the fact that the elements of both $\mathfrak{A}$ and $\mathfrak{B}$ are countable. Without the countability assumption only the backward implication holds.

Proposition 33. FOL *is equipped with a back-and-forth equivalence structure, which extends the elementary equivalence structure described in Proposition 30 as follows:*

- $\mathsf{Mod}_c \colon \mathsf{Sig}^{\mathsf{FOL}} \to \mathsf{Cat}^{\mathsf{op}}$ *is the subfunctor of* $\mathsf{Mod}^{\mathsf{FOL}}$ *which maps each signature Σ to the full subcategory of Σ-models with countable carrier sets.*
- *The signature $\mathtt{BF}(\Sigma)$ extends $\mathtt{ELE}(\Sigma)$ with a new constant $c :\to Nat$, and the signature morphism $\mathtt{bf}(\Sigma) \colon \mathtt{ELE}(\Sigma) \hookrightarrow \mathtt{BF}(\Sigma)$ is an inclusion.*
- *For each finite first-order signature Σ, the set B_Σ extends $\mathtt{bf}(\Sigma)(\mathsf{E}_\Sigma)$ with a countably infinite set $\{pre^{n+1}c < pre^n c \mid n \in \omega\}$, which says that c denotes a nonstandard number which has an infinite number of predecessors.*

Proof. Let $\Sigma = (S, F, P)$ be a first-order signature.

- Let $\chi \colon \Sigma \to \Sigma'$ be a signature morphism injective on sorts. Let $\mathfrak{A}$ be a $\mathtt{ELE}(\Sigma')$-model and $\mathfrak{B}$ a $\mathtt{BF}(\Sigma)$-model such that $\mathfrak{A}\upharpoonright_{\mathtt{ELE}(\chi)} = \mathfrak{B}\upharpoonright_{\mathtt{bf}(\Sigma)}$. We define $\mathfrak{C}$ as the $\mathtt{bf}(\Sigma')$-expansion of $\mathfrak{A}$, which interprets c as $\mathfrak{B}$, i.e. $c^{\mathfrak{C}} = c^{\mathfrak{B}}$. Obviously, $\mathfrak{C}\upharpoonright_{\mathtt{bf}(\Sigma')} = \mathfrak{A}$ and $\mathfrak{C}\upharpoonright_{\mathtt{BF}(\chi)} = \mathfrak{B}$.

- Let $\mathfrak{A} \in |\mathsf{Mod}^{\mathsf{FOL}}(\mathtt{ELE}(\Sigma), \mathsf{E}_\Sigma)|$, and let $\{pre^{n+1}c < pre^n c \mid n < k\}$ be a finite subset of $\{pre^{n+1}c < pre^n c \mid n \in \omega\}$, where $k \in \omega$. Since $(\mathfrak{A}_{Nat}, <^{\mathfrak{A}})$ has no upper bound, there exists an element $a \in \mathfrak{A}_{Nat}$ with at least k predecessors. Let $\mathfrak{B}$ be the $\mathtt{bf}(\Sigma)$-expansion of $\mathfrak{A}$ which interprets c as a. It is straightforward to show that $\mathfrak{B} \models \{pre^{n+1}c < pre^n c \mid n < k\}$.

- Assume that Σ is finite and let $\mathfrak{D} \in |\mathsf{Mod}_c(\mathsf{BF}(\Sigma), \mathsf{B}_\Sigma)|$ be a countable model. We define the models $\mathfrak{A} = \mathfrak{D}\upharpoonright_{\mathtt{ele}_1(\Sigma);\mathtt{BF}(\Sigma)}$ and $\mathfrak{B} = \mathfrak{D}\upharpoonright_{\mathtt{ele}_2(\Sigma);\mathtt{BF}(\Sigma)}$. We show that $\mathfrak{A} \cong \mathfrak{B}$:

1	$\mathfrak{D} \models \exists p : Par \cdot \mathsf{F}(d_c, p)$, where d_c is the interpretation of c into $\mathfrak{D}$	since $\mathfrak{D} \models \forall k : Nat \cdot \exists p : Par \cdot \mathsf{F}(k, p)$
2	$\mathfrak{D} \models \mathsf{F}(d_c, d_p)$ for some $d_p \in \mathfrak{D}_{Par}$	by the definition of satisfaction
3	d_p can be regarded as a partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, that is, $d_p \colon \mathfrak{A} \nrightarrow \mathfrak{B}$ such that $d_p(a) = b$ iff $(d_p, a, b) \in app_s^{\mathfrak{D}}$ for all $s \in S$, $a \in \mathfrak{A}_s$ and $b \in \mathfrak{B}_s$	by E_Σ described in Proposition 30
4	$d_p : \mathfrak{A} \nrightarrow \mathfrak{B}$ can be extended infinitely many times	since $\mathfrak{D} \models \mathsf{F}(d_c, d_p)$ and d_c has an infinite number of predecessors
5	$\mathfrak{A}$ and $\mathfrak{B}$ are countable	since $\mathfrak{D}$ is countable
6	$\mathfrak{A} \cong \mathfrak{B}$	by Lemma 32 from 4 and 5

$\square$

4.2 Elementarily equivalent order-sorted algebras

The definitions of partial isomorphism, and of finitely isomorphic and partially isomorphic models can be replicated for order-sorted algebras based on the definition of order-sorted homomorphism. One can straightforwardly check that the institution-independent version of Fraïssé-Hintikka Theorem from [21] is applicable to OSA too.

Theorem 34 (Fraïssé-Hintikka Theorem in OSA). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two order-sorted algebras over an order-sorted signature Σ with a finite number of function symbols. Then the following are equivalent:*

1. *$\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent.*
2. *$\mathfrak{A}$ and $\mathfrak{B}$ are finitely isomorphic.*

Proposition 35 (Object-level characterisation of elementary equivalence in OSA). OSA *is equipped with the following elementary equivalence structure:*

- $\mathsf{Sig}^{\mathtt{ELE}}$ *is the subcategory of order-sorted signature morphisms injective on sorts.*

- $\mathsf{Sig}^{\mathtt{FIN}}$ *is the subcategory of order-sorted signature morphisms injective on sorts whose objects are finite order-sorted signatures.*

- *For each finite signature $\Sigma = (S, \leq, F)$,*

$$\begin{aligned}\mathtt{ELE}(\Sigma) \coloneqq \;\; (\; & S^1 \cup S^2 \cup \{Bool, Nat, Par\}, \leq^1 \cup \leq^2, F^1 \cup F^2 \;\cup \\ & \{true :\to Bool, pre\_ : Nat \to Nat, \_<\_ : Nat\ Nat \to Bool\} \;\cup \\ & \{app_s : Par\ s^1\ s^2 \to Bool \mid s \in S\} \cup \{\mathsf{F} : Nat\ Par \to Bool\} \;).\end{aligned}$$

  For each finite signature $\Sigma = (S, \leq, F)$, the set E_Σ, which says that two order-sorted algebras are finitely isomorphic, consists of the following sentences:

A1) p is a partial injective function: for all sorts $s_1, s_2 \in S$ *such that* $s_1 \equiv_{\leq} s_2$,

$$\forall p : Par, x_1 : s_1^1, y_1 : s_1^2, x_2 : s_2^1, y_2 : s_2^2 \cdot (app_{s_1}(p, x_1, y_1) = true) \ \wedge \\ (app_{s_2}(p, x_2, y_2) = true) \Rightarrow (x_1 = x_2 \Leftrightarrow y_1 = y_2).$$

A2) p preserves and reflects functions: for each $(\sigma\colon s_1 \dots s_n \to s_{n+1}) \in F$,

$$\forall p : Par, x_1 : s_1^1, y_1 : s_1^2, \dots, x_{n+1} : s_{n+1}^1, y_{n+1} : s_{n+1}^2 \cdot \bigwedge_{i=1}^{n+1} app_{s_i}(p, x_i, y_i) = true \\ \Rightarrow (\sigma^1(x_1, \dots, x_n) = x_{n+1} \Leftrightarrow \sigma^2(y_1, \dots, y_n) = y_{n+1}).$$

A3) dom(p) includes the interpretation of all constants: for each $(c :\to s) \in F$,

$$\forall p : Par \cdot app_s(p, c^1, c^2) = true.$$

A4) $<$ *is irreflexive:* $\forall k : Nat \cdot \neg(k < k = true)$.

$<$ *is transitive:*

$$\forall k_1 : Nat, k_2 : Nat, k_3 : Nat \cdot (k_1 < k_2 = true) \wedge (k_2 < k_3 = true) \Rightarrow \\ (k_1 < k_3 = true).$$

$<$ *is totally defined:*

$$\forall k_1 : Nat, k_2 : Nat \cdot (k_1 < k_2 = true) \vee (k_1 = k_2) \vee (k_2 < k_1 = true).$$

Every element except the first (if it exists) has a predecessor:

$$\forall k_1 : Nat \cdot (\exists k_2 : Nat \cdot k_2 < k_1 = true) \Rightarrow ((pre\ k_1) < k_1 = true) \ \wedge \\ \neg \exists k_2 : Nat \cdot ((pre\ k_1) < k_2 = true \ \wedge k_2 < k_1 = true).$$

Every element has a successor: $\forall k_1 : Nat \cdot \exists k_2 : Nat \cdot (pre\ k_2 = k_1)$.

A5) p has the back and forth properties:

$\forall k : Nat, p : Par \cdot \mathsf{forth}(k, p) \wedge \mathsf{back}(k, p)$, *where*

- $\mathsf{forth}(k, p)$ *is the formula*

$$(pre\ k) < k = true \wedge \mathsf{F}(k, p) = true \Rightarrow \\ \bigwedge_{s \in S} \forall x_s : s^1 \cdot \exists q_s : Par, y_s : s^2 \cdot (q_s \ \mathsf{extends}\ p) \ \wedge \\ app_s(q_s, x_s, y_s) = true \ \wedge \ \mathsf{F}((pre\ k), q_s) = true,$$

- $\mathsf{back}(k, p)$ *is the formula*

$$(pre\ k) < k = true \wedge \mathsf{F}(k, p) = true \Rightarrow \\ \bigwedge_{s \in S} \forall y_s : s^2 \cdot \exists q_s : Par, x_s : s^1 \cdot (q_s \ \mathsf{extends}\ p) \ \wedge \\ app_s(q_s, x_s, y_s) = true \wedge \mathsf{F}((pre\ k), q_s) = true,$$

- g `extends` f *is the formula*

$$\bigwedge_{s \in S} \forall x_s : s^1, y_s : s^2 \cdot app_s(f, x_s, y_s) = true \Rightarrow app_s(g, x_s, y_s) = true.$$

A6) For all natural numbers k, the set of partial isomorphisms that can be extended k times is not empty: $\forall k : Nat \cdot \exists p : Par \cdot \mathsf{F}(k, p) = true$.

- *For each order-sorted signature morphism* $\chi\colon \Sigma \to \Sigma'$, *the signature morphism* $\mathtt{ELE}(\chi)\colon \mathtt{ELE}(\Sigma) \to \mathtt{ELE}(\Sigma')$ *maps*
  - x^i *to* $\chi(x)^i$, *where x is any sort or function symbol in* Σ *and* $i \in \{1, 2\}$,
  - $(app_s : Par\ s^1\ s^2 \to Bool)$ *to* $(app_{\chi(s)} : Par\ \chi(s)^1\ \chi(s)^2 \to Bool)$ *for all sorts s in* Σ, *and*

– *it is the identity on the rest of the symbols.*

Since Σ is finite, S is finite and the sentence at (A5) is well-defined. By Theorem 34, finitely isomorphic relation yield to elementary equivalence provided that the underlying signature has a finite number of function symbols. Proposition 35 is a straightforward generalization of Proposition 30 to order-sorted case.

Proposition 36. *The elementary equivalence structure from Proposition 35 can be extended to a back-and-forth equivalence structure as follows:*

- $\mathsf{Mod}_c \colon \mathsf{Sig}^{\mathsf{OSA}} \to \mathsf{Cat}^{\mathsf{op}}$ *is the subfunctor of* $\mathsf{Mod}^{\mathsf{OSA}}$*, which maps each signature to the category of order-sorted algebras with countable carrier sets.*
- *The signature* $\mathtt{BF}(\Sigma)$ *is obtained from* $\mathtt{ELE}(\Sigma)$ *by adding a new constant* $c :\to Nat$*, and* $\mathtt{bf}(\Sigma) \colon \mathtt{ELE}(\Sigma) \hookrightarrow \mathtt{BF}(\Sigma)$ *is an inclusion.*
- *For each finite signature* Σ*, the set* B_Σ *extends* $\mathtt{bf}(\Sigma)(\mathsf{E}_\Sigma)$ *with a countably infinite set* $\{(pre^{n+1}c) < (pre^{n}c) = true \mid n \in \omega\}$*.*

Proposition 36 is a straightforward generalization of Proposition 33 to the order-sorted case.

4.3 Elementarily equivalent higher-order models

The definitions of partial isomorphism, and of finitely isomorphic and partially isomorphic models can be straightforwardly replicated for HNK' using types instead of sorts. Fraïssé-Hintikka Theorem holds in HNK' too.

Theorem 37 (Fraïssé-Hintikka Theorem in HNK')**.** *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be two higher-order models over a signature* Σ *with a finite number of function symbols. Then the following are equivalent:*

1. $\mathfrak{A}$ *and* $\mathfrak{B}$ *are elementarily equivalent.*
2. $\mathfrak{A}$ *and* $\mathfrak{B}$ *are finitely isomorphic.*

Theorem 37 allows us to give an object-level characterisation of elementarily equivalent higher-order models.

Proposition 38 (Object-level characterisation of elementary equivalence in HNK')**.** HNK' *is equipped with the following elementary equivalence structure:*

- $\mathsf{Sig}^{\mathtt{ELE}} \subseteq \mathsf{Sig}^{\mathsf{HNK}'}$ *is the subcategory of signature morphisms injective on sorts.*
- $\mathsf{Sig}^{\mathtt{FIN}} \subseteq \mathsf{Sig}^{\mathsf{HNK}'}$ *is the subcategory of signature morphisms injective on sorts whose objects are finite* HNK' *signatures.*
- *For each finite signature* $\Sigma = (S, F, \kappa)$*,*

$$\begin{aligned}\mathtt{ELE}(\Sigma) \coloneqq \quad (\ S^1 \cup S^2 \cup \{Bool, Nat, Par\},\ F^1 \cup F^2 \cup \{true : Bool\} \ \cup \\ \{pre : Nat \to Nat, <: Nat \to Nat \to Bool\} \ \cup \\ \{app_s : Par \to s^1 \to s^2 \to Bool \mid s \in S\} \ \cup \\ \{\mathsf{F} : Nat \to Par \to Bool\} \quad).\end{aligned}$$

For each finite signature $\Sigma = (S, F, \kappa)$*, the set* E_Σ*, which says that two higher-order models are finitely isomorphic, consists of the following sentences:*

A1) *p is a partial injective function: for each type* $s \in \overrightarrow{S}_\kappa$,

$$\forall p : Par, x_1 : s^1, x_2 : s^1, y_1 : s^2, y_2 : s^2 \cdot (app_s\ p\ x_1\ y_1 = true)\ \wedge$$
$$(app_s\ p\ x_2\ y_2 = true) \Rightarrow (x_1 = x_2 \Leftrightarrow y_1 = y_2).$$

A2) *p is compatible with functions: for all types* $s_1 \to s_2 \in \overrightarrow{S}_\kappa$,

$$\forall p : Par, f : s_1^1 \to s_2^1, g : s_1^2 \to s_2^2, x_1 : s_1^1, y_1 : s_1^2, x_2 : s_2^1, y_2 : s_2^2 \cdot$$
$$(app_{s_1 \to s_2}\ p\ f\ g = true)\ \wedge\ (app_{s_1}\ p\ x_1\ y_1 = true)\ \wedge$$
$$(app_{s_2}\ p\ x_2\ y_2 = true) \Rightarrow (f\ x_1 = x_2 \Leftrightarrow g\ y_1 = y_2).$$

A3) *dom*(p) *includes the interpretation of all functions symbols:*

$\forall p : Par \cdot app_s\ p\ \sigma^1\ \sigma^2 = true$, *for all* $(\sigma : s) \in F$.

A4) $<$ *is irreflexive:* $\forall k : Nat \cdot \neg(<\ k\ k = true)$.

$<$ *is transitive:*

$$\forall k_1 : Nat, k_2 : Nat, k_3 : Nat \cdot (<\ k_1\ k_2 = true) \wedge (<\ k_2\ k_3 = true)$$
$$\Rightarrow (<\ k_1\ k_3 = true).$$

$<$ *is a totally defined:*

$$\forall k_1 : Nat, k_2 : Nat \cdot (<\ k_1\ k_2 = true) \vee (k_1 = k_2) \vee (<\ k_2\ k_1 = true).$$

Every element except the first (if it exists) has a predecessor:

$$\forall k_1 : Nat \cdot (\exists k_2 : Nat \cdot\ <\ k_2\ k_1 = true) \Rightarrow (<\ (pre\ k_1)\ k_1 = true)\ \wedge$$
$$\neg\exists k_2 : Nat \cdot (<\ (pre\ k_1)\ k_2 = true \wedge <\ k_2\ k_1 = true).$$

Every element has a successor: $\forall k_1 : Nat \cdot \exists k_2 : Nat \cdot (pre\ k_2 = k_1)$.

A5) *p has the back and forth properties:*

$\forall k : Nat, p : Par \cdot \mathsf{forth}(k, p) \wedge \mathsf{back}(k, p)$, *where*

- $\mathsf{forth}(k, p)$ *is the formula*

$$<\ (pre\ k)\ k = true \wedge \mathsf{F}\ k\ p = true \Rightarrow$$
$$\bigwedge_{s \in \overrightarrow{S}_\kappa} \forall x_s : s^1 \cdot \exists q_s : Par, y_s : s^2 \cdot (q_s\ \mathsf{extends}\ p)\ \wedge$$
$$app_s\ q_s\ x_s\ y_s = true\ \wedge\ \mathsf{F}\ (pre\ k)\ q_s = true,$$

- $\mathsf{back}(k, p)$ *is the formula*

$$<\ (pre\ k)\ k = true \wedge \mathsf{F}\ k\ p = true \Rightarrow$$
$$\bigwedge_{s \in \overrightarrow{S}_\kappa} \forall y_s : s^2 \cdot \exists q_s : Par, x_s : s^1 \cdot (q_s\ \mathsf{extends}\ p)\ \wedge$$
$$app_s\ q_s\ x_s\ y_s = true \wedge \mathsf{F}\ (pre\ k)\ q_s = true,$$

- g `extends` f *is the formula*

$$\bigwedge_{s \in \overrightarrow{S}_\kappa} \forall x_s : s^1, y_s : s^2 \cdot (app_s f\ x_s\ y_s = true \Rightarrow app_s g\ x_s\ y_s = true).$$

A6) *For all natural numbers k, the set of partial isomorphisms that can be extended k times is not empty:* $\forall k : Nat \cdot \exists p : Par \cdot \mathsf{F}\ k\ p = true$.

- *For each signature morphism* $\chi : \Sigma \to \Sigma'$ *in* HNK', *the signature morphism* $\mathtt{ELE}(\chi) : \mathtt{ELE}(\Sigma) \to \mathtt{ELE}(\Sigma')$ *maps*
  - x^i *to* $\chi(x)^i$, *where x is any sort or function symbol in* Σ *and* $i \in \{1, 2\}$,
  - $(app_s\colon Par \to s^1 \to s^2 \to Bool)$ *to* $(app_{\chi(s)}\colon Par \to \chi(s)^1 \to \chi(s)^2 \to Bool)$ *for all types s in* Σ, *and*

– *it is the identity on the rest of the symbols.*

Since Σ is finite, $\overrightarrow{S}_\kappa$ is finite and the sentence at (A5) is well-defined. By Theorem 37, finitely isomorphic relation yield to elementary equivalence provided that the underlying signature has a finite number of function symbols. The proof of Proposition 38 is conceptually the same as the proof of Proposition 30.

Proposition 39. *The elementary equivalence structure from Proposition 38 can be extended to a back-and-forth equivalence structure as follows:*

- $\mathsf{Mod}_c\colon \mathsf{Sig}^{\mathsf{HNK}'} \to \mathsf{Cat}^{\mathsf{op}}$ *is the subfunctor of* $\mathsf{Mod}^{\mathsf{HNK}'}$*, which maps each* HNK' *signature to the category of higher-order models with countable carrier sets.*
- *The signature* $\mathtt{BF}(\Sigma)$ *is obtained from* $\mathtt{ELE}(\Sigma)$ *by adding a new constant* $c : Nat$*, and* $\mathtt{bf}(\Sigma)\colon \mathtt{ELE}(\Sigma) \hookrightarrow \mathtt{BF}(\Sigma)$ *is an inclusion.*
- *For each finite* HNK' *signature* Σ*, the set* B_Σ *extends* $\mathtt{bf}(\Sigma)(\mathsf{E}_\Sigma)$ *with a countably infinite set* $\{< (pre^{n+1}c)(pre^n c) = true \mid n \in \omega\}$.

The proof of Proposition 39 is conceptually the same as the proof of Proposition 33.

5 Lindström's theorem

We prove an institution independent version of Lindström's theorem, not relying on any particular form of syntax or semantics. In this sense our proof is both syntax and semantics free.

Theorem 40. *Let* $\mathcal{I} = (\mathsf{Sig}, \mathsf{Sen}, \mathsf{Mod}, \models)$ *and* $\mathcal{J} = (\mathsf{Sig}, \mathsf{Sen}^{\mathcal{J}}, \mathsf{Mod}, \models^{\mathcal{J}})$ *be two institutions such that* $\mathcal{I} \lesssim \mathcal{J}$*. In addition, we assume that* $\mathcal{I}$ *is semantically closed under Boolean connectives, and* $\mathcal{J}$ *is compact. If* $\mathfrak{A} \equiv^{\mathcal{I}} \mathfrak{B}$ *implies* $\mathfrak{A} \equiv^{\mathcal{J}} \mathfrak{B}$ *for all signatures* $\Sigma \in |\mathsf{Sig}|$ *and all* Σ*-models* $\mathfrak{A}$ *and* $\mathfrak{B}$*, then* $\mathcal{I} \sim \mathcal{J}$*.*

Proof. The proof is performed in three steps:

S1) Assume that $\mathfrak{A} \models^{\mathcal{J}} \gamma$, where $\Sigma \in |\mathsf{Sig}|$, $\mathfrak{A} \in |\mathsf{Mod}(\Sigma)|$ and $\gamma \in \mathsf{Sen}^{\mathcal{J}}(\Sigma)$. We show that there exists $e_{(\mathfrak{A},\gamma)} \in \mathsf{Sen}^{\mathcal{I}}(\Sigma)$ such that $\mathfrak{A} \models^{\mathcal{I}} e_{(\mathfrak{A},\gamma)}$ and $e_{(\mathfrak{A},\gamma)} \models \gamma$:

1	$\mathsf{Th}^{\mathcal{I}}(\mathfrak{A}) \models \gamma$	by the following proof steps
1.1	assume $\mathfrak{B} \models \mathsf{Th}^{\mathcal{I}}(\mathfrak{A})$	
1.2	$\mathfrak{A} \equiv^{\mathcal{I}} \mathfrak{B}$	as $\mathcal{I}$ is semantically closed under negations and $\mathsf{Th}^{\mathcal{I}}(\mathfrak{A})$ is a complete theory
1.3	$\mathfrak{A} \equiv^{\mathcal{J}} \mathfrak{B}$	since $\equiv^{\mathcal{I}} \subseteq \equiv^{\mathcal{J}}$
1.4	$\mathfrak{B} \models^{\mathcal{J}} \gamma$	since $\mathfrak{A} \models^{\mathcal{J}} \gamma$
2	$E \models \gamma$ for some $E \subseteq \mathsf{Th}^{\mathcal{I}}(\mathfrak{A})$ finite	by Lemma 18
3	$e_{\mathfrak{A},\gamma} \mathrel{\models\!\mid} \bigwedge E$ for some $e_{(\mathfrak{A},\gamma)} \in \mathsf{Sen}^{\mathcal{I}}(\Sigma)$	since $\mathcal{I}$ is semantically closed under conjunctions
4	$\mathsf{Th}^{\mathcal{I}}(\mathfrak{A}) \models e_{(\mathfrak{A},\gamma)}$ and $e_{(\mathfrak{A},\gamma)} \models \gamma$	from 2 and 3

S2) $|\mathsf{Mod}(\Sigma,\gamma)| = \bigcup_{\mathfrak{D}\in|\mathsf{Mod}(\Sigma,\gamma)|} |\mathsf{Mod}(\Sigma, e_{(\mathfrak{D},\gamma)})|$ for all signatures Σ and all sentences $\gamma \in \mathsf{Sen}^{\mathcal{J}}(\Sigma)$:

(a) If $\mathfrak{A} \in |\mathsf{Mod}(\Sigma,\gamma)|$ then $\mathfrak{A} \models^{\mathcal{I}} e_{(\mathfrak{A},\gamma)}$, which means $\mathfrak{A} \in |\mathsf{Mod}(\Sigma, e_{(\mathfrak{A},\gamma)})|$, and we get $\mathfrak{A} \in \bigcup_{\mathfrak{D}\in|\mathsf{Mod}(\Sigma,\gamma)|} |\mathsf{Mod}(\Sigma, e_{(\mathfrak{D},\gamma)})|$.

(b) If $\mathfrak{A} \in \bigcup_{\mathfrak{D}\in|\mathsf{Mod}(\Sigma,\gamma)|} |\mathsf{Mod}(\Sigma, e_{(\mathfrak{D},\gamma)})|$ then $\mathfrak{A} \in |\mathsf{Mod}(\Sigma, e_{(\mathfrak{D},\gamma)})|$ for some Σ-model $\mathfrak{D}$ such that $\mathfrak{D} \models^{\mathcal{J}} \gamma$. We have $\mathfrak{A} \models^{\mathcal{I}} e_{(\mathfrak{D},\gamma)}$, and since $e_{(\mathfrak{D},\gamma)} \models \gamma$, we get $\mathfrak{A} \models \gamma$, which means $\mathfrak{A} \in |\mathsf{Mod}(\Sigma,\gamma)|$.

S3) For all $\Sigma \in |\mathsf{Sig}|$ and all $\gamma \in \mathsf{Sen}^{\mathcal{J}}(\Sigma)$, there exist $n \in \omega$ and $\mathfrak{D}_1,\dots,\mathfrak{D}_n \in |\mathsf{Mod}(\Sigma,\gamma)|$ such that $|\mathsf{Mod}(\Sigma,\gamma)| = \bigcup_{i=1}^{n} |\mathsf{Mod}(\Sigma, e_{(\mathfrak{D}_i,\gamma)})|$.[4] Suppose towards a contradiction that $|\mathsf{Mod}(\Sigma,\gamma)| \supsetneq \bigcup_{i=1}^{n} |\mathsf{Mod}(\Sigma, e_{(\mathfrak{D}_i,\gamma)})|$ for all $n \in \omega$ and $\mathfrak{D}_1,\dots,\mathfrak{D}_n \in |\mathsf{Mod}(\Sigma,\gamma)|$. Then:

1	$\{\gamma\} \cup \{\neg e_{(\mathfrak{D}_1,\gamma)},\dots,\neg e_{(\mathfrak{D}_n,\gamma)}\}$ is satisfiable, for all $n \in \omega$ and all $\mathfrak{D}_1,\dots,\mathfrak{D}_n \in |\mathsf{Mod}(\Sigma,\gamma)|$	since $|\mathsf{Mod}(\Sigma,\gamma)| \supsetneq \bigcup_{i=1}^{n} |\mathsf{Mod}(\Sigma, e_{(\mathfrak{D}_i,\gamma)})|$
2	$\{\gamma\} \cup \{\neg e_{(\mathfrak{D},\gamma)} \mid \mathfrak{D} \in |\mathsf{Mod}(\Sigma,\gamma)|\}$ is satisfiable	since $\mathcal{I} \lesssim \mathcal{J}$ and $\mathcal{J}$ is compact
3	there exists a Σ-model $\mathfrak{A}$ such that $\mathfrak{A} \models^{\mathcal{J}} \gamma$ and $\mathfrak{A} \models^{\mathcal{I}} \neg e_{(\mathfrak{D},\gamma)}$ for all $\mathfrak{D} \in |\mathsf{Mod}(\Sigma,\gamma)|$	by the definition of satisfiability
4	$\mathfrak{A} \models^{\mathcal{I}} e_{(\mathfrak{B},\gamma)}$ for some $\mathfrak{B} \in |\mathsf{Mod}(\Sigma,\gamma)|$	by S2, since $\mathfrak{A} \models^{\mathcal{J}} \gamma$
5	$\mathfrak{A} \models^{\mathcal{I}} \neg e_{(\mathfrak{B},\gamma)}$	from 3
6	contradiction	from 4 and 5

It follows that $|\mathsf{Mod}(\Sigma,\gamma)| = \bigcup_{i=1}^{n} |\mathsf{Mod}(\Sigma, e_{(\mathfrak{D}_i,\gamma)})|$ for some $\mathfrak{D}_1,\dots,\mathfrak{D}_n \in |\mathsf{Mod}(\Sigma,\gamma)|$. Since $|\mathsf{Mod}(\Sigma, \bigvee_{i=1}^{n} e_{(\mathfrak{D}_i,\gamma)})| = \bigcup_{i=1}^{n} |\mathsf{Mod}(\Sigma, e_{(\mathfrak{D}_i,\gamma)})|$, we have that $\gamma \models\!\mid \bigvee_{i=1}^{n} e_{(\mathfrak{D}_i,\gamma)}$. □

The following result provides minimal conditions for the meaning of a sentence to depend on a finite number of symbols.

Theorem 41. *Let $\mathcal{I} = (\mathsf{Sig}, \mathsf{Sen}, \mathsf{Mod}, \models)$ and $\mathcal{J} = (\mathsf{Sig}, \mathsf{Sen}^{\mathcal{J}}, \mathsf{Mod}, \models^{\mathcal{J}})$ be two institutions such that $\mathcal{I} \lesssim \mathcal{J}$. Assume moreover that*

1. *$\mathcal{I}$ is equipped with (a) an isomorphism structure as described in Definition 19 such that $|\mathsf{Sig}^{\mathtt{ISO}}| = |\mathsf{Sig}|$, and (b) a back-and-forth equivalence structure as described in Definition 25 such that $\mathsf{Sig}^{\mathtt{ELE}} = \mathsf{Sig}^{\mathtt{ISO}}$ and for all signatures $\Sigma \in |\mathsf{Sig}|$ and all finite subsets $\mathsf{T} \subseteq \mathsf{S}_\Sigma$ there exists $\chi : \Sigma' \to \Sigma \in \mathsf{Sig}^{\mathtt{ISO}}$ such that $\Sigma' \in |\mathsf{Sig}^{\mathtt{FIN}}|$ and $\mathtt{ISO}(\chi)(\mathsf{S}_{\Sigma'}) \models \mathsf{T}$;*

2. *$\mathcal{J}$ has the following properties: (a) semantic closure under Boolean connectives, (b) compactness, and (c) DLS property via Mod_c.*[5]

For all signatures $\Sigma \in |\mathsf{Sig}|$ and all sentences $\gamma \in \mathsf{Sen}^{\mathcal{J}}(\Sigma)$ there exists a signature $\Sigma' \in |\mathsf{Sig}^{\mathtt{FIN}}|$ and a signature morphism $\chi : \Sigma' \to \Sigma \in \mathsf{Sig}^{\mathtt{ISO}}$ such that

$$\textit{if } \mathfrak{A}\upharpoonright_\chi \equiv^{\mathcal{I}} \mathfrak{B}\upharpoonright_\chi \textit{ then } (\mathfrak{A} \models^{\mathcal{J}} \gamma \textit{ iff } \mathfrak{B} \models^{\mathcal{J}} \gamma)$$

[4] For example, if γ is not satisfiable then take $n = 0$, which means that $\gamma \models\!\mid \bot$.

[5] The parameter Mod_c is given by the back-and-forth equivalence $\mathtt{BF}$.

for all Σ-models $\mathfrak{A}$ and $\mathfrak{B}$.

Proof. We perform the proof in two steps.

S1) Firstly, we prove that for all signatures $\Sigma \in |\mathsf{Sig}|$ and all sentences $\gamma \in \mathsf{Sen}^{\mathcal{J}}(\Sigma)$, there exist $\Sigma' \in |\mathsf{Sig}^{\mathtt{FIN}}|$ and $\chi : \Sigma' \to \Sigma \in \mathsf{Sig}^{\mathtt{ISO}}$ such that

$$\text{if } \mathfrak{A}\upharpoonright_{\chi} \cong \mathfrak{B}\upharpoonright_{\chi} \text{ then } (\mathfrak{A} \models^{\mathcal{J}} \gamma \text{ iff } \mathfrak{B} \models^{\mathcal{J}} \gamma)$$

for all Σ-models $\mathfrak{A}$ and $\mathfrak{B}$. Let $\Sigma \in |\mathsf{Sig}|$ and $\gamma \in \mathsf{Sen}^{\mathcal{J}}(\Sigma)$. Since $\mathcal{J}$ is semantically closed under Boolean connectives, there exists an $\mathtt{ISO}(\Sigma)$-sentence ρ semantically equivalent to $\mathtt{iso}_1(\Sigma)(\gamma) \Leftrightarrow \mathtt{iso}_2(\Sigma)(\gamma)$. We show that $\mathsf{S}_{\Sigma} \models \rho$:

1	let $\mathfrak{C}$ be an $\mathtt{ISO}(\Sigma)$-model such that $\mathfrak{C} \models^{\mathcal{I}} \mathsf{S}_{\Sigma}$	
2	$\mathfrak{A} \cong \mathfrak{B}$, where $\mathfrak{A} = \mathfrak{C}\upharpoonright_{\mathtt{iso}_1(\Sigma)}$ and $\mathfrak{B} = \mathfrak{C}\upharpoonright_{\mathtt{iso}_2(\Sigma)}$	by Definition 19(1), since $\mathfrak{C} \models^{\mathcal{I}} \mathsf{S}_{\Sigma}$
3	$\mathfrak{A} \models^{\mathcal{J}} \gamma$ iff $\mathfrak{B} \models^{\mathcal{J}} \gamma$	by Assumption 1, since $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic
4	$\mathfrak{C} \models^{\mathcal{J}} \mathtt{iso}_1(\Sigma)(\gamma)$ iff $\mathfrak{C} \models^{\mathcal{J}} \mathtt{iso}_2(\Sigma)(\gamma)$	by the satisfaction condition
5	$\mathfrak{C} \models^{\mathcal{J}} \rho$	as ρ is semantically equivalent to $\mathtt{iso}_1(\Sigma)(\gamma) \Leftrightarrow \mathtt{iso}_2(\Sigma)(\gamma)$

By Lemma 18, $\mathsf{T} \models \rho$ for some $\mathsf{T} \subseteq \mathsf{S}_{\Sigma}$ finite. By the properties of $\mathcal{I}$, there exist $\Sigma' \in |\mathsf{Sig}^{\mathtt{FIN}}|$ and $\chi \colon \Sigma' \to \Sigma \in \mathsf{Sig}^{\mathtt{ISO}}$ such that $\mathtt{ISO}(\chi)(\mathsf{S}_{\Sigma'}) \models \mathsf{T}$. It follows that $\mathtt{ISO}(\chi)(\mathsf{S}_{\Sigma'}) \models \rho$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two Σ-models such that $\mathfrak{A}\upharpoonright_{\chi} \cong \mathfrak{B}\upharpoonright_{\chi}$. We show that $\mathfrak{A} \models^{\mathcal{J}} \gamma$ iff $\mathfrak{B} \models^{\mathcal{J}} \gamma$:

1	$\mathfrak{C}\upharpoonright_{\mathtt{iso}_1(\Sigma')} = \mathfrak{A}\upharpoonright_{\chi}$ and $\mathfrak{C}\upharpoonright_{\mathtt{iso}_2(\Sigma')} = \mathfrak{B}\upharpoonright_{\chi}$ for some $\mathfrak{C} \in |\mathsf{Mod}(\mathtt{ISO}(\Sigma'), \mathsf{S}_{\Sigma'})|$	since $\mathfrak{A}\upharpoonright_{\chi}$ and $\mathfrak{B}\upharpoonright_{\chi}$ are isomorphic
2	let $\mathfrak{D}$ be an $\mathtt{ISO}(\chi)$-expansion of $\mathfrak{C}$ such that $\mathfrak{D}\upharpoonright_{\mathtt{iso}_1(\Sigma)} = \mathfrak{A}$ and $\mathfrak{D}\upharpoonright_{\mathtt{iso}_2(\Sigma)} = \mathfrak{B}$	by Definition 19(2a)
3	$\mathfrak{D} \models^{\mathcal{J}} \rho$	since $\mathfrak{D} \models^{\mathcal{I}} \mathtt{ISO}(\chi)(\mathsf{S}_{\Sigma'})$ and $\mathtt{ISO}(\chi)(\mathsf{S}_{\Sigma'}) \models \rho$
4	$\mathfrak{D} \models^{\mathcal{J}} \mathtt{iso}_1(\Sigma)(\gamma)$ iff $\mathfrak{D} \models^{\mathcal{J}} \mathtt{iso}_2(\Sigma)(\gamma)$	since ρ is semantically equivalent to $\mathtt{iso}_1(\Sigma)(\gamma) \Leftrightarrow \mathtt{iso}_2(\Sigma)(\gamma)$
5	$\mathfrak{A} \models^{\mathcal{J}} \gamma$ iff $\mathfrak{B} \models^{\mathcal{J}} \gamma$	by the satisfaction condition, since $\mathfrak{D}\upharpoonright_{\mathtt{iso}_1(\Sigma)} = \mathfrak{A}$ and $\mathfrak{D}\upharpoonright_{\mathtt{iso}_2(\Sigma)} = \mathfrak{B}$

$$\begin{tikzcd}
\mathtt{ISO}(\Sigma) & \Sigma \arrow[l, "\mathtt{iso}_i(\Sigma)"'] \arrow[r, "\mathtt{ele}_i(\Sigma)"] & \mathtt{ELE}(\Sigma) \arrow[r, "\mathtt{bf}(\Sigma)"] & \mathtt{BF}(\Sigma) \\
\mathtt{ISO}(\Sigma') \arrow[u, dotted, "\mathtt{ISO}(\chi)"] & \Sigma' \arrow[u, dotted, "\chi"'] \arrow[l, "\mathtt{iso}_i(\Sigma')"] \arrow[r, "\mathtt{ele}_i(\Sigma')"'] & \mathtt{ELE}(\Sigma') \arrow[u, dotted, "\mathtt{ELE}(\chi)"'] \arrow[r, "\mathtt{bf}(\Sigma')"'] & \mathtt{BF}(\Sigma') \arrow[u, dotted, "\mathtt{BF}(\chi)"']
\end{tikzcd}$$

S2) Let $\Sigma \in |\mathsf{Sig}|$ and $\gamma \in \mathsf{Sen}^{\mathcal{J}}(\Sigma)$. By (S1), there exists $\chi \colon \Sigma' \to \Sigma \in \mathsf{Sig}^{\mathtt{ISO}}$ with $\Sigma' \in |\mathsf{Sig}^{\mathtt{FIN}}|$ such that the meaning of γ depends only on Σ'. Suppose towards a contradiction that there exist some Σ-models $\mathfrak{A}$ and $\mathfrak{B}$ such that

$$\mathfrak{A}\upharpoonright_{\Sigma'} \equiv^{\mathcal{I}} \mathfrak{B}\upharpoonright_{\Sigma'},\ \mathfrak{A}\models^{\mathcal{J}}\gamma \text{ and } \mathfrak{B}\models^{\mathcal{J}}\neg\gamma \tag{1}$$

We show that $\mathtt{BF}(\chi)(\mathsf{T})\cup\{(\mathtt{ele}_1(\Sigma);\mathtt{bf}(\Sigma))(\gamma),(\mathtt{ele}_2(\Sigma);\mathtt{bf}(\Sigma))(\neg\gamma)\}$ is satisfiable, for any finite subset $\mathsf{T}\subseteq\mathsf{B}_{\Sigma'}$:

1	there exists $\mathfrak{C}\in|\mathsf{Mod}(\mathtt{ELE}(\Sigma'),\mathsf{E}_{\Sigma'})|$ such that $\mathfrak{A}\upharpoonright_{\chi}=\mathfrak{C}\upharpoonright_{\mathtt{ele}_1(\Sigma')}$ and $\mathfrak{B}\upharpoonright_{\chi}=\mathfrak{C}\upharpoonright_{\mathtt{ele}_2(\Sigma')}$	since $\mathfrak{A}\upharpoonright_{\chi}\equiv^{\mathcal{I}}\mathfrak{B}\upharpoonright_{\chi}$
2	there exists an $\mathtt{ELE}(\chi)$-expansion $\mathfrak{D}$ of $\mathfrak{C}$ such that $\mathfrak{D}\upharpoonright_{\mathtt{ele}_1(\Sigma)}=\mathfrak{A}$ and $\mathfrak{D}\upharpoonright_{\mathtt{ele}_2(\Sigma)}=\mathfrak{B}$	by Definition 24(2), since $\mathfrak{A}\upharpoonright_{\chi}=\mathfrak{C}\upharpoonright_{\mathtt{ele}_1(\Sigma')}$ and $\mathfrak{B}\upharpoonright_{\chi}=\mathfrak{C}\upharpoonright_{\mathtt{ele}_2(\Sigma')}$
3	$\mathfrak{D}\models^{\mathcal{I}}\mathtt{ELE}(\chi)(\mathsf{E}_{\Sigma'})$, $\mathfrak{D}\models^{\mathcal{J}}\mathtt{ele}_1(\Sigma)(\gamma)$ and $\mathfrak{D}\models^{\mathcal{J}}\mathtt{ele}_2(\Sigma)(\neg\gamma)$	by the satisfaction condition, as $\mathfrak{A}\models^{\mathcal{J}}\gamma$, $\mathfrak{B}\models^{\mathcal{J}}\neg\gamma$ and $\mathfrak{C}\models^{\mathcal{I}}\mathsf{E}_{\Sigma'}$
4	$\mathfrak{M}\models^{\mathcal{I}}\mathsf{T}$ for some $\mathtt{bf}(\Sigma')$-expansion $\mathfrak{M}$ of $\mathfrak{C}$	by Definition 25(1a), since $\mathfrak{C}\models^{\mathcal{I}}\mathsf{E}_{\Sigma'}$ and $\mathsf{T}\subseteq\mathsf{B}_{\Sigma'}$ is finite
5	there exists a $\mathtt{BF}(\chi)$-expansion $\mathfrak{N}$ of $\mathfrak{M}$ such that $\mathfrak{N}\upharpoonright_{\mathtt{bf}(\Sigma)}=\mathfrak{D}$	by Definition 25(2), since $\mathfrak{D}\upharpoonright_{\mathtt{ELE}(\chi)}=\mathfrak{C}$ and $\mathfrak{M}\upharpoonright_{\mathtt{bf}(\Sigma')}=\mathfrak{C}$
6	$\mathfrak{N}\models^{\mathcal{I}}\mathtt{BF}(\chi)(\mathsf{T})$, $\mathfrak{N}\models^{\mathcal{J}}(\mathtt{ele}_1(\Sigma);\mathtt{bf}(\Sigma))(\gamma)$ and $\mathfrak{N}\models^{\mathcal{J}}(\mathtt{ele}_2(\Sigma);\mathtt{bf}(\Sigma))(\neg\gamma)$	by the satisfaction condition, since $\mathfrak{N}\upharpoonright_{\mathtt{BF}(\chi)}\models^{\mathcal{I}}\mathsf{T}$, $\mathfrak{N}\upharpoonright_{\mathtt{bf}(\Sigma)}\models^{\mathcal{J}}\mathtt{ele}_1(\Sigma)(\gamma)$, $\mathfrak{N}\upharpoonright_{\mathtt{bf}(\Sigma)}\models^{\mathcal{J}}\mathtt{ele}_2(\Sigma)(\neg\gamma)$

We show that assumption (1) made above is false:

1	$\{(\mathtt{ele}_1(\Sigma);\mathtt{bf}(\Sigma))(\gamma),(\mathtt{ele}_2(\Sigma);\mathtt{bf}(\Sigma))(\neg\gamma)\}\ \cup\ \mathtt{BF}(\chi)(\mathsf{B}_{\Sigma'})$ is satisfiable	since every finite subset is satisfiable, $\mathcal{I}\lesssim\mathcal{J}$ and $\mathcal{J}$ is compact
2	$\mathfrak{C}\models^{\mathcal{J}}\{(\mathtt{ele}_1(\Sigma);\mathtt{bf}(\Sigma))(\gamma),(\mathtt{ele}_2(\Sigma);\mathtt{bf}(\Sigma))(\neg\gamma)\}$ and $\mathfrak{C}\models^{\mathcal{I}}\mathtt{BF}(\chi)(\mathsf{B}_{\Sigma'})$ for some $\mathfrak{C}\in|\mathsf{Mod}_c(\mathtt{BF}(\Sigma))|$	since $\mathcal{I}\lesssim\mathcal{J}$ and $\mathcal{J}$ has the DLS property
3	$\mathfrak{A}'\upharpoonright_{\chi}\cong\mathfrak{B}'\upharpoonright_{\chi}$, where $\mathfrak{A}'=\mathfrak{C}\upharpoonright_{\mathtt{ele}_1(\Sigma);\mathtt{bf}(\Sigma)}$ and $\mathfrak{B}'=\mathfrak{C}\upharpoonright_{\mathtt{ele}_2(\Sigma);\mathtt{bf}(\Sigma)}$	by Definition 25(1b), since $\mathfrak{C}\upharpoonright_{\mathtt{BF}(\chi)}\models^{\mathcal{I}}\mathsf{B}_{\Sigma'}$
4	$\mathfrak{A}'\models^{\mathcal{J}}\gamma$ iff $\mathfrak{B}'\models^{\mathcal{J}}\gamma$	by (S1)
5	$\mathfrak{A}'\models^{\mathcal{J}}\gamma$ and $\mathfrak{B}'\models^{\mathcal{J}}\neg\gamma$	from 2, by the satisfaction condition
6	contradiction	from 4 and 5

Hence, ($\mathfrak{A}\models\gamma$ iff $\mathfrak{B}\models\gamma$) whenever $\mathfrak{A}\upharpoonright_{\chi}\equiv^{\mathcal{I}}\mathfrak{B}\upharpoonright_{\chi}$. □

The first part of the proof shows, using the isomorphism structure, that there exists a finite signature Σ' such that γ cannot distinguish between the models $\mathfrak{A}$ and $\mathfrak{B}$ if they are isomorphic over Σ'. The second part of the proof shows, using the back-and-forth equivalence structure, that isomorphism can be weakened to elementary equivalence and γ still cannot distinguish between the models $\mathfrak{A}$ and $\mathfrak{B}$. So, γ depends only on the properties that can be expressed in a finite signature.

Lindström's theorem is a direct consequence of Theorem 40 and Theorem 41.

Theorem 42 (Lindström's theorem)**.** *Let* $\mathcal{I}=(\mathsf{Sig},\mathsf{Sen},\mathsf{Mod},\models)$ *be an institution equipped with*

1. *an isomorphism structure such that* $|\mathsf{Sig}^{\mathtt{ISO}}|=|\mathsf{Sig}|$,

2. *a back-and-forth equivalence structure such that* $\mathsf{Sig}^{\mathtt{ELE}} = \mathsf{Sig}^{\mathtt{ISO}}$ *and for all signatures* $\Sigma \in |\mathsf{Sig}|$ *and all finite subsets* $\mathsf{T} \subseteq \mathsf{S}_\Sigma$ *there exists a signature morphism* $\chi \colon \Sigma' \to \Sigma \in \mathsf{Sig}^{\mathtt{ISO}}$ *such that* $\Sigma' \in |\mathsf{Sig}^{\mathtt{FIN}}|$ *and* $\mathtt{ISO}(\chi)(\mathsf{S}_{\Sigma'}) \models \mathsf{T}$.

If $\mathcal{I}$ is (a) semantically closed under Boolean connectives, (b) compact and (c) it has the DLS property, then $\mathcal{I}$ has the Lindström property.

Proof. By Theorem 40, it suffices to prove that $\mathfrak{A} \equiv^{\mathcal{I}} \mathfrak{B}$ implies $\mathfrak{A} \equiv^{\mathcal{J}} \mathfrak{B}$ for all signatures Σ, and all Σ-models $\mathfrak{A}$ and $\mathfrak{B}$. Assume that $\mathfrak{A} \equiv^{\mathcal{I}} \mathfrak{B}$. Let γ be an arbitrary but fixed Σ-sentence in $\mathcal{J}$. By Theorem 41, there exist $\Sigma' \in |\mathsf{Sig}^{\mathtt{FIN}}|$ and $\chi \colon \Sigma' \to \Sigma \in \mathsf{Sig}^{\mathtt{ISO}}$ such that if $\mathfrak{A}\upharpoonright_\chi \equiv^{\mathcal{I}} \mathfrak{B}\upharpoonright_\chi$ then ($\mathfrak{A} \models^{\mathcal{J}} \gamma$ iff $\mathfrak{B} \models^{\mathcal{J}} \gamma$). Since $\mathfrak{A} \equiv^{\mathcal{I}} \mathfrak{B}$, we have $\mathfrak{A}\upharpoonright_\chi \equiv^{\mathcal{I}} \mathfrak{B}\upharpoonright_\chi$. It follows that $\mathfrak{A} \models^{\mathcal{J}} \gamma$ iff $\mathfrak{B} \models^{\mathcal{J}} \gamma$. Since γ was arbitrarily chosen, $\mathfrak{A} \equiv^{\mathcal{J}} \mathfrak{B}$. Since Σ was arbitrarily chosen, $\mathcal{I} \sim \mathcal{J}$. Hence, $\mathcal{I}$ has the Lindström property. □

Theorem 42 is applicable to many-sorted first-order logic, partial algebra and higher-order logic with Henkin semantics as presented in Example 5.

Corollary 43. FOL*,* OSA *and* HNK' *have the Lindström property.*

Proof. We prove the result only for FOL, as the other cases are similar.

By Theorem 15, FOL has the DLS property. By Theorem 13, FOL is compact. Therefore, it makes sense to consider an extension of FOL that preserves these properties.

By Proposition 21, FOL is equipped with an isomorphism structure described in Proposition 21 such that $|\mathsf{Sig}^{\mathtt{ISO}}| = |\mathsf{Sig}^{\mathsf{FOL}}|$. By Proposition 33, FOL is equipped with a back-and-forth equivalence structure obtained by extending the elementary equivalence structure described in Proposition 30 such that $\mathsf{Sig}^{\mathtt{ISO}} = \mathsf{Sig}^{\mathtt{ELE}}$.

For all first-order signatures Σ and all finite subsets $\mathsf{T} \subseteq \mathsf{S}_\Sigma$, there exists a finite subsignature $\Sigma' \subseteq \Sigma$ which contains all symbols of sorts, functions and relations that occur in T. Notice that $\mathsf{T} \subseteq \mathtt{ISO}(\chi)(\mathsf{S}_{\Sigma'})$, where $\chi \colon \Sigma' \hookrightarrow \Sigma$ is an inclusion. It follows that $\mathtt{ISO}(\chi)(\mathsf{S}_{\Sigma'}) \models \mathsf{T}$.

By Theorem 42, FOL has the Lindström property. □

6 Local Lindström's theorem

The framework of institutions is essentially a multi-signature environment. Signature morphisms are ubiquitous: apart from making the framework easily adaptable to various concrete instantiations, they are also crucial internally for quantification. In the classical environment however, one typically works with a fixed theory in a finite language. The next definition is geared to this kind of setup. For a single finite signature Σ, we define a minimal institution wrapping Σ up in enough logic to enable stating and proving Lindström's theorem for Σ, as it were, locally.

Definition 44. *Consider an institution* $\mathcal{I} = (\mathsf{Sig}, \mathsf{Sen}, \mathsf{Mod}, \models)$ *equipped with a back-and-forth equivalence structure as described in Definition 25, and a set of sentences* T *over a signature* $\Sigma \in |\mathsf{Sig}^{\mathtt{FIN}}|$*. Let* $\mathcal{I}_{(\Sigma,\mathsf{T})} = (\mathsf{Sig}_{(\Sigma,\mathsf{T})}, \mathsf{Sen}^{pres}, \mathsf{Mod}^{pres}, \models^{pres})$ *be the restriction of* $\mathcal{I}^{pres}$ *to a subcategory* $\mathsf{Sig}_{(\Sigma,\mathsf{T})}$ *as described in Example 6, where* $\mathsf{Sig}_{(\Sigma,\mathsf{T})}$ *is defined as follows:*

1. *Its objects are* (Σ, T)*,* $(\mathtt{ELE}(\Sigma), \mathsf{ET})$ *and* $(\mathtt{BF}(\Sigma), \mathsf{BT})$*, where*
   (a) $\mathsf{ET} = \mathtt{ele}_1(\Sigma)(\mathsf{T}) \cup \mathtt{ele}_2(\Sigma)(\mathsf{T})$ *and*
   (b) $\mathsf{BT} = (\mathtt{ele}_1(\Sigma); \mathtt{bf}(\Sigma))(\mathsf{T}) \cup (\mathtt{ele}_2(\Sigma); \mathtt{bf}(\Sigma))(\mathsf{T})$.
2. *Its arrows are the identities plus the following presentation morphisms:*
   (a) $\mathtt{ele}_i(\Sigma) : (\Sigma, \mathsf{T}) \to (\mathtt{ELE}(\Sigma), \mathsf{ET})$*, for all* $i \in \{1, 2\}$*, and*
   (b) $\mathtt{bf}(\Sigma) : (\mathtt{ELE}(\Sigma), \mathsf{ET}) \to (\mathtt{BF}(\Sigma), \mathsf{BT})$.

Theorem 45 (Local Lindström's theorem)**.** *Let* $\mathcal{I} = (\mathsf{Sig}, \mathsf{Sen}, \mathsf{Mod}, \models)$ *be an institution equipped with a back-and-forth equivalence structure as described in Definition 25. Let* (Σ, T) *be a presentation such that* $\Sigma \in |\mathsf{Sig}^{\mathtt{FIN}}|$ *and* T *is countable.*

> *If* $\mathcal{I}_{(\Sigma,\mathsf{T})}$ *is (a) semantically closed under Boolean connectives, (b) compact and (c) it has the DLS property, then for all institutions* $\mathcal{J}$ *such that* $\mathcal{I}_{(\Sigma,\mathsf{T})} \lesssim \mathcal{J}$ *and* $\mathcal{J}$ *has the properties (a)–(c), we have that for each sentence in* $\mathsf{Sen}^{\mathcal{J}}(\Sigma)$ *there exists a sentence in* $\mathsf{Sen}^{\mathcal{I}}(\Sigma)$ *satisfied by the same class of models.*

Proof. The proof follows the lines of the proof of Theorem 42. It relies on an analogue of Theorem 41, simplified by the assumption that the signature of γ is from $\mathsf{Sig}^{\mathtt{FIN}}$, so the isomorphism property is not required. Moreover, the condition $|\mathsf{Sig}^{\mathtt{ELE}}| = |\mathsf{Sig}_{(\Sigma,\mathsf{T})}|$ is not required either, and indeed, in general it does not hold, as in our case $|\mathsf{Sig}^{\mathtt{ELE}}| = (\Sigma, \mathsf{T})$. We leave the details to the reader. □

Corollary 46. *Let* (Σ, T) *be a first-order presentation such that* Σ *consists of a finite number of symbols.* $\mathsf{FOL}_{(\Sigma,\mathsf{T})}$ *has the local Lindström property, that is:*

> *For all institutions* $\mathcal{J}$ *such that* $\mathsf{FOL}_{(\Sigma,\mathsf{T})} \lesssim \mathcal{J}$ *and* $\mathcal{J}$ *is (a) semantically closed under Boolean connectives, (b) compact, and (c) it has the DLS property, we have that for each sentence in* $\mathsf{Sen}^{\mathcal{J}}(\Sigma)$ *there exists a first-order* Σ*-sentence satisfied by the same class of models.*

In particular, $\mathsf{FOL}_{\mathsf{RA}}$*,* $\mathsf{FOL}_{\mathsf{RRA}}$ *and* $\mathsf{FOL}_{\mathsf{Z}_2}$ *have the local Lindström property.*

In order to define $\mathcal{J}$, the language is extended not only over the signature Σ, but also over the signatures $\mathtt{ELE}(\Sigma)$ and $\mathtt{BF}(\Sigma)$. This is a simplification because it suffices to extend the language over three vocabularies and not the entire class. For example, if one wants to extend the language of Representable Relation Algebra with the Kleene operator then only three signatures need to be considered. It is problematic to give an extension for all first-order signatures because of the translation of sentences along signature morphisms.

7 Conclusions

We proved Lindström's theorem in the framework of institutions, which is both syntax and semantics free. The result immediately applies to many-sorted first-order logic, order-sorted algebra and a version of higher-order logic with Henkin semantics. It is also applicable to other logical systems formalised as institutions such as partial algebra [1], preorder algebra [9], membership algebra [25], and other combinations of these logics, which underlie algebraic specification languages such as CafeOBJ [9], Maude [5], or CASL [1].

We believe that applications to these systems are especially valuable, as the systems lack the austerity of the single-sorted first-order setting, and technical details make clear-cut applications of first-order results rather difficult. Expanding on this theme, in future we plan to cast Lindström's theorem in the framework of stratified institutions [8, 19, 17] and apply the result to hybrid logics.

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