Interpolation in Logics with Constructors

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Abstract

We present a generic method for establishing the interpolation property by borrowing it across the logical systems from a base institution to prove it for its *constructor-based* variant. The framework used is that of the so-called *institution theory* invented by Goguen and Burstall which is a categorical-based formulation of the informal concept of *logical system* sufficiently abstract to capture many examples of logics used in computer science and mathematical logic, and expressive enough to elaborate our general results. We illustrate the applicability of the present work by instantiating the abstract results to constructor-based Horn clause logic and constructor-based Horn preorder algebra but applications are also expected for many other logical systems.

Keywords: algebraic specification, institution, interpolation, constructor-based, Horn logic.

1. Introduction

Constructor-based institutions are obtained from a base institution by enhancing the syntax with a sub-signature of constructor operators and restricting the semantics to reachable models, which consist of constructor-generated elements. The sentences and satisfaction condition are preserved from the base institution, while the signature morphisms are restricted such that the reducts of models that are reachable in the target signature are again reachable in the source signature. The sorts of constructors are called *constrained* and a sort that is not constrained is called *loose*. Several algebraic specification languages incorporate features to express reachability and

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to deal with constructors like, for instance, Larch [25] or CASL [2, 31].

By introducing constructor operators in the signatures we gain more expressivity for the specifications but some of the basic important properties of the institution are lost. In previous research [22, 23, 21], we studied conditions under which those properties hold for the constructor-based variants of Horn institutions [1]. For example in [22] and its extended version [23], we provided the proof rules for these logics and we obtained a ω-completeness [2] result, while in [21] we investigated the existence of pushouts of signature morphisms, amalgamation, and the existence of free models along the theory morphisms. In this paper we focus on conditions for the preservation of the interpolation property.

Craig interpolation [10], abbreviated CI, is classically stated as follows: for any semantic consequence $e_1 \models e_2$ in the *union language* $\mathcal{L}_1 \cup \mathcal{L}_2$, where e_i is a first-order sentence in the language $\mathcal{L}_i$, there exists a sentence e_0 in the *intersection language* $\mathcal{L}_1 \cap \mathcal{L}_2$, called the *interpolant* of e_1 and e_2, such that $e_1 \models e_0 \models e_2$. Following an approach originating in [39], we naturally generalize the inclusion square

$$\begin{array}{ccc} \mathcal{L}_2 & \hookrightarrow & \mathcal{L}_1 \cup \mathcal{L}_2 \\ \uparrow & & \uparrow \\ \mathcal{L}_1 \cap \mathcal{L}_2 & \hookrightarrow & \mathcal{L}_1 \end{array}$$

to a pushout of language translations (signature morphisms)

$$\begin{array}{ccc} \mathcal{L}_2 & \xrightarrow{\chi_2} & \mathcal{L} \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi_1} \\ \mathcal{L}_0 & \xrightarrow[\chi]{} & \mathcal{L}_1 \end{array}$$

and replace the sentences e_1, e_2, e_0 with *sets of sentences* E_1, E_2, E_0 to obtain the following form of CI: if $\varphi_1(E_1) \models \chi_2(E_2)$ then there exists a set E_0 of sentences in $\mathcal{L}_0$ such that $E_1 \models \chi(E_0)$ and $\varphi(E_0) \models E_2$. The papers

[1] Horn institutions are obtained from a base institution, for example the institution of first-order logic, by restricting the sentences to the so-called Horn sentences of the form $(\forall X) \bigwedge H \Rightarrow C$, where H is a set of atoms in the base institution, and C is an atom.

[2] Some proof rules contain infinite premises which can only be checked with induction schemes. As a consequence, the resulting entailment system is not compact.

[37] and [18] argue successfully that the formulation of CI in terms of sets of sentences is more appropriate than the traditional formulation of CI in terms of single sentences.

One of the reasons for the great interest in CI is the fact that it is the source of many other results. For structured specifications interpolation ensures a good compositional behavior of module semantics [3, 6, 18, 38]. Applications of CI deal with combining and decomposing theories and involve areas like structured theorem proving [1, 28], model checking [29], and automated reasoning [33, 34]. It has also received special attention within institution-independent model theory [12, 13, 24, 35, 15].

The interpolation property is very difficult to obtain, in general. In constructor-based institutions interpolation holds under certain extra conditions which are added on top of the hypothesis under which it holds for the base institution. In this paper we provide a general method of borrowing interpolation from a base institution for its constructor-based variant across institution morphisms. This result depends on *sufficient completeness* (see Section 4, Theorem 4.5). Intuitively, a presentation (Σ, Γ), where Γ is a set of formulas over the signature Σ, is sufficient complete if every term can be reduced to a term formed with constructors and operators of loose sorts using the equations in Γ.

In the literature the interpolation results for constructor-based institutions are rather negative. According to [32], CASL institution does not enjoy the interpolation property, in general, while in [4] the interpolants for constructor-based observational logic consist of infinitary sentences. As far as we know, the interpolation problem in logics with constructors is still open and therefore, the conditions under which we prove interpolation for constructor-based Horn institutions are new.

We assume that the reader is familiar with the basic notions of category theory. See [27] for the standard definitions of category, functor, pushout, etc., which are omitted here.

2. Institutions

The concept of institution formalizes the intuitive notion of logical system, and has been defined by Goguen and Burstall in the seminal paper [20].

Definition 2.1. An *institution* $\mathcal{I} = (\mathbb{S}ig^{\mathcal{I}}, \mathbb{S}en^{\mathcal{I}}, \mathbb{M}od^{\mathcal{I}}, \models^{\mathcal{I}})$ consists of

1. a category $\mathbb{S}ig^{\mathcal{I}}$, whose objects are called *signatures*,

2. a functor $\mathbb{S}en^{\mathcal{I}} : \mathbb{S}ig^{\mathcal{I}} \to \mathbb{S}et$, providing for each signature Σ a set whose elements are called *(Σ-)sentences*,

3. a functor $\mathbb{M}od^{\mathcal{I}} : \mathbb{S}ig^{\mathcal{I}} \to \mathbb{CAT}^{op}$, providing for each signature Σ a category whose objects are called *(Σ-)models* and whose arrows are called *(Σ-)morphisms*,

4. a relation $\models^{\mathcal{I}}_{\Sigma} \subseteq |\mathbb{M}od^{\mathcal{I}}(\Sigma)| \times \mathbb{S}en^{\mathcal{I}}(\Sigma)$ for each signature $\Sigma \in |\mathbb{S}ig^{\mathcal{I}}|$, called *($\Sigma$-)satisfaction*, such that for each morphism $\varphi : \Sigma \to \Sigma'$ in $\mathbb{S}ig^{\mathcal{I}}$, the following *satisfaction condition* holds:

$$M' \models^{\mathcal{I}}_{\Sigma'} \mathbb{S}en^{\mathcal{I}}(\varphi)(e) \text{ iff } \mathbb{M}od^{\mathcal{I}}(\varphi)(M') \models^{\mathcal{I}}_{\Sigma} e$$

for all $M' \in |\mathbb{M}od^{\mathcal{I}}(\Sigma')|$ and $e \in \mathbb{S}en^{\mathcal{I}}(\Sigma)$.

We denote the *reduct* functor $\mathbb{M}od^{\mathcal{I}}(\varphi)$ by $\_\!\upharpoonright_{\varphi}$ and the sentence translation $\mathbb{S}en^{\mathcal{I}}(\varphi)$ by $\varphi(\_)$. When $M = M'\!\upharpoonright_{\varphi}$ we say that M is the φ-reduct of M' and M' is a φ-expansion of M. When there is no danger of confusion, we omit the superscript from the notations of the institution components; for example $\mathbb{S}ig^{\mathcal{I}}$ may be simply denoted by $\mathbb{S}ig$.

Definition 2.2. For all institutions $\mathcal{I} = (\mathbb{S}ig^{\mathcal{I}}, \mathbb{S}en^{\mathcal{I}}, \mathbb{M}od^{\mathcal{I}}, \models^{\mathcal{I}})$ and any subcategory $\mathbb{S}ig' \subseteq \mathbb{S}ig^{\mathcal{I}}$ of signature morphisms, we define the restriction $\mathcal{I}' = (\mathbb{S}ig^{\mathcal{I}'}, \mathbb{S}en^{\mathcal{I}'}, \mathbb{M}od^{\mathcal{I}'}, \models^{\mathcal{I}'})$ of $\mathcal{I}$ to $\mathbb{S}ig'$ as follows:

- $\mathbb{S}ig^{\mathcal{I}'} = \mathbb{S}ig'$,
- $\mathbb{S}en^{\mathcal{I}'} : \mathbb{S}ig' \to \mathbb{S}et$ is the restriction of $\mathbb{S}en^{\mathcal{I}}$ to $\mathbb{S}ig'$,
- $\mathbb{M}od^{\mathcal{I}'} : \mathbb{S}ig' \to \mathbb{CAT}^{op}$ is the restriction of $\mathbb{M}od^{\mathcal{I}'}$ to $\mathbb{S}ig'$, and
- $\models^{\mathcal{I}'} = (\models^{\mathcal{I}}_{\Sigma})_{\Sigma \in |\mathbb{S}ig'|}$.

When there is no danger of confusion we overload the notation by letting

- $\mathbb{S}en$ to denote both $\mathbb{S}en^{\mathcal{I}}$ and its restriction $\mathbb{S}en^{\mathcal{I}'}$ to $\mathbb{S}ig'$,
- $\mathbb{M}od$ to denote both $\mathbb{M}od^{\mathcal{I}}$ and its restriction $\mathbb{M}od^{\mathcal{I}'}$ to $\mathbb{S}ig'$, and
- $\models$ to denote both $\models^{\mathcal{I}}$ and its restriction $\models^{\mathcal{I}'}$ to $\mathbb{S}ig'$.

Hence, $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ and $\mathcal{I}' = (\mathbb{S}ig', \mathbb{S}en, \mathbb{M}od, \models)$.

Example 2.1 (First-order logic (**FOL**) [20])**.** Signatures are first-order many-sorted signatures (with sort names, operation names and predicate names); sentences are the usual closed formulae of first-order logic built over atomic formulae given either as equalities or atomic predicate formulae; models are the usual first-order structures; satisfaction of a formula in a structure is defined in the standard way.

Let (S, F, P) be a first-order signature, where S is a set of sorts, $F = (F_{w\to s})_{(w,s)\in S^*\times S}$ is a family of operation symbols and $P = (P_w)_{w\in S^*}$ is a family of relation symbols. *A Horn sentence* for the signature (S, F, P) is a sentence of the form $(\forall X)(\bigwedge H) \Rightarrow C$, where X is a finite set of variables, H is a finite set of (relational or equational) atoms, $\bigwedge H$ is the conjunction of the formulas in H, and C is a (relational or equational) atom. In the tradition of logic programming Horn sentences are known as *Horn clauses*. The institution of Horn clause logic (**HCL**) has the same signatures and models as **FOL** but only Horn sentences as sentences.

Example 2.2 (Constructor-based first-order logic (**CFOL**))**.** The **CFOL** signatures are of the form (S, F, F^c, P), where (S, F, P) is a first-order signature, and $F^c \subseteq F$ is a distinguished subfamily of sets of operation symbols called *constructors*. The constructors determine the set of *constrained* sorts $S^c \subseteq S$: $s \in S^c$ iff there exists a constructor $\sigma \in F^c_{w\to s}$. We call the sorts in $S^l = S - S^c$ *loose*. The (S, F, F^c, P)-sentences are the usual *first-order sentences*.

The (S, F, F^c, P)-models are the usual first-order structures M with the carrier sets for the constrained sorts consisting of interpretations of terms formed with constructors and elements of loose sorts, i.e. there exists a set Y of variables of loose sorts, and a function $f : Y \to M$ such that for every constrained sort $s \in S^c$ the function $f^\#_s : (T_{(S,F^c)}(Y))_s \to M_s$ is a surjection, where $T_{(S,F^c)}(Y)$ is the term algebra over the set Y of variables, and $f^\# : T_{(S,F^c)}(Y) \to M$ is the unique extension of $f : Y \to M$ to a (S, F^c)-morphism.

A signature morphism $\varphi : (S, F, F^c, P) \to (S', F', F'^c, P')$ in **CFOL** is a first-order signature morphism $\varphi : (S, F, P) \to (S', F', P')$ such that the constructors are preserved along the signature morphisms (i.e. if $\sigma \in F^c$ then $\varphi(\sigma) \in F'^c$) and no "new" constructors are introduced for "old" constrained sorts (i.e. if $s \in S^c$ and $\sigma' \in F'^c_{w'\to\varphi(s)}$ then there exists $\sigma \in F^c_{w\to s}$ such that $\varphi(\sigma) = \sigma'$). Variants of **CFOL** were studied in [5] and [4].

The institution of constructor-based Horn clause logic (**CHCL**) is the restriction of **CFOL** to Horn sentences.

Example 2.3 (Preorder algebra (**POA**) [17, 16])**.** The **POA** signatures are just the ordinary algebraic signatures. The **POA** models are *preordered algebras* which are interpretations of the signatures into the category of preorders $\mathbb{P}re$ rather than the category of sets $\mathbb{S}et$. This means that each sort gets interpreted as a preorder, and each operation as a preorder functor, which means a preorder-preserving (i.e. monotonic) function. A *preordered algebra morphism* is just a family of preorder functors (preorder-preserving functions) which is also an algebra morphism.

The sentences have two kinds of atoms: equations and *preorder atoms*. A preorder atom $t \leq t'$ is satisfied by a preorder algebra M when the interpretations of the terms are in the preorder relation of the carrier, i.e. $M_t \leq M_{t'}$. Full sentences are constructed from equational and preorder atoms by using Boolean connectives and first-order quantification.

Horn preorder algebra (**HPOA**) is the restriction of **POA** to Horn sentences. Their constructor-based variants **CPOA** and **CHPOA** are obtained similarly to the first-order case.

Assumption 2.1. Throughout this paper, for all institutions above, we assume that the signature morphisms allow mappings of non-constructor constants to ground terms. This makes it possible to treat first-order substitutions [3] in the comma category [4] of signature morphisms.

2.1. Presentations

Let $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution. A presentation (Σ, E) consists of a signature Σ and a set E of Σ-sentences. A presentation morphism $\varphi : (\Sigma, E) \to (\Sigma', E')$ is a signature morphism $\varphi : \Sigma \to \Sigma'$ such that $E' \models \varphi(E)$. The presentation morphisms form a category denoted $\mathbb{S}ig^{pres}$ with the composition inherited from the category of signatures. The model functor $\mathbb{M}od$ can be extended from the category of signatures $\mathbb{S}ig$ to the category of presentations $\mathbb{S}ig^{pres}$, by mapping a presentation (Σ, E) to the full

[3] First-order substitutions are the substitutions of terms for variables. Second-order substitutions are the substitutions of formulas for predicates.

[4] Given a category $\mathcal{C}$ and an object $A \in |\mathcal{C}|$, the comma category $A/\mathcal{C}$ has arrows $A \xrightarrow{f} B \in \mathcal{C}$ as objects, and $h \in \mathcal{C}(B, B')$ with $f; h = f'$ as arrows.

subcategory [5] $\mathbb{M}od(\Sigma, E)$ of $\mathbb{M}od(\Sigma)$ consisting of models that satisfy E. The correctness of the definition of $\mathbb{M}od : \mathbb{S}ig^{pres} \to \mathbb{CAT}^{op}$ is guaranteed by the satisfaction condition of the base institution. This leads to the *institution of presentations* $\mathcal{I}^{pres} = (\mathbb{S}ig^{pres}, \mathbb{S}en, \mathbb{M}od, \models)$ over the base institution $\mathcal{I}$, where the notations $\mathbb{S}en$ and $\models$ are overloaded such that

- for all $(\Sigma, E) \xrightarrow{\varphi} (\Sigma', E') \in \mathbb{S}ig^{pres}$ we have $\mathbb{S}en((\Sigma, E) \xrightarrow{\varphi} (\Sigma', E')) = \mathbb{S}en(\Sigma \xrightarrow{\varphi} \Sigma')$, and
- for all $M \in \mathbb{M}od(\Sigma, E)$ and $\rho \in \mathbb{S}en(\Sigma, E)$ we have $M \models_{(\Sigma,E)} \rho$ iff $M \models_{\Sigma} \rho$.

If $\mathcal{T} \subseteq \mathbb{S}ig$ then we denote by $\mathcal{T}^{pres}$ the subcategory of presentation morphisms $(\Sigma, E) \xrightarrow{\varphi} (\Sigma', E')$ such that $\Sigma \xrightarrow{\varphi} \Sigma' \in \mathcal{T}$.

2.2. Internal Logic

Let $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution and $\Sigma \in |\mathbb{S}ig|$ a signature.

- For all $E \subseteq \mathbb{S}en(\Sigma)$, $E^* = \{M \in \mathbb{M}od(\Sigma) \mid M \models e \text{ for each } e \in E\}$.
- For all $\mathcal{M} \subseteq \mathbb{M}od(\Sigma)$, $\mathcal{M}^* = \{e \in \mathbb{S}en(\Sigma) \mid M \models e \text{ for each } M \in \mathcal{M}\}$.

If E and E' are sets of sentences of the same signature, then

- $E^* \subseteq E'^*$ is denoted by $E \models E'$, and
- E and E' are semantically equivalent, denoted $E \models\!\!\mid E'$, when $E \models E'$ and $E' \models E$.

Definition 2.3. An institution $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ has

- *conjunctions* when for every signature $\Sigma \in |\mathbb{S}ig|$ and each Σ-sentences e_1 and e_2 there exists a Σ-sentence e such that $e^* = e_1^* \cap e_2^*$, usually denoted by $e_1 \wedge e_2$;
- *implications* when for each e_1 and e_2 as above there exists e such that $e^* = (\mathbb{M}od(\Sigma) - e_1^*) \cup e_2^*$, usually denoted by $e_1 \Rightarrow e_2$;

[5] $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$ if $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ such that $\mathcal{C}(A, B) = \mathcal{C}'(A, B)$ for all objects $A, B \in |\mathcal{C}'|$.

- *universal $\mathcal{D}$-quantifications* for a subcategory $\mathcal{D} \subseteq \mathbb{S}ig$ of signature morphisms when for all $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$ and each $e' \in \mathbb{S}en(\Sigma')$ there exists $e \in \mathbb{S}en(\Sigma)$ such that

  $$e^* = \{M \in \mathbb{M}od(\Sigma) \mid \text{for all } \varphi\text{-expansions } M' \text{ of } M \text{ we have } M' \in e'^*\}$$

  here denoted by $(\forall\chi)e'$.

2.3. Pushouts

Let $\varphi : (S, F, F^c, P) \to (S', F', F'^c, P')$ be a signature morphism in **CFOL**. We say that φ^{op} is *injective* if for all arities $w \in S^*$ and sorts $s \in S$, $\varphi^{op}_{w\to s}$ is injective. The same applies to φ^{ct}, the constructor symbols component, and φ^{rl}, the relation symbols component. φ^{op} is encapsulated means that no "new" operation symbol, i.e. outside the image of φ, is allowed to have the sort in the image of φ. More precise, if $\sigma' \in F'_{w'\to s'}$ then for all $s \in S$ such that $s' = \varphi^{st}(s)$ there exists $\sigma \in F_{w\to s}$ such that $\varphi^{op}(\sigma) = \sigma'$. The same applies to φ^{ct}.

Definition 2.4 ($(xyzt)$-signature morphisms)**.** A **CFOL** signature morphism $\varphi : (S, F, F^c, P) \to (S', F', F'^c, P')$ is a $(xyzt)$-morphism, with $x, t \in \{i, *\}$ and $y, z \in \{i, e, *\}$, where i stands for "injective", e for "encapsulated", and $*$ for "all", when

(1) it does not map constants to terms,

(2) the sort component $\varphi^{st} : S \to S'$ has the property x,

(3) the operation component $\varphi^{op} = (\varphi^{op}_{w\to s} : F_{w\to s} \to F'_{\varphi^{st}(w)\to\varphi^{st}(s)})_{(w,s)\in S^*\times S}$ has the property y,

(4) the constructor component $\varphi^{ct} = (\varphi^{ct}_{w\to s} : F^c_{w\to s} \to F'^c_{\varphi^{st}(w)\to\varphi^{st}(s)})_{(w,s)\in S^*\times S}$ has the property z, and

(5) the relation component $\varphi^{rl} = (\varphi^{rl}_w : P_w \to P'_{\varphi^{st}(w)})_{w\in S^*}$ has the property t.

This notational convention can be extended to other institutions too, such as **CPOA** or **CFOL**^{pres}. In case of **CPOA**, because there are no relation symbols, the last component is missing. In case of **CFOL**^{pres}, a presentation morphism $(\Sigma, E) \xrightarrow{\varphi} (\Sigma', E')$ is

(i) a $(xyzt)$-morphism if $E' = \varphi(E)$ and $\Sigma \xrightarrow{\varphi} \Sigma'$ is a $(xyzt)$-morphism;

(ii) a $(xyzt)^{pres}$-morphism if $\Sigma \overset{\varphi}{\to} \Sigma'$ is a $(xyzt)$-morphism.

Proposition 2.1. *[21]* **CFOL** *has* $((i{*}e{*}),({*}{*}{*}{*}))$*-pushouts* [6]. *Moreover, if* $\{\Sigma_2 \overset{\chi}{\leftarrow} \Sigma_0 \overset{\varphi}{\to} \Sigma_1, \Sigma_1 \overset{\chi_1}{\to} \Sigma \overset{\varphi_2}{\leftarrow} \Sigma_2\}$ *is a pushout of signature morphisms,* φ *is a* $(i{*}e{*})$*-morphism and* χ *is a* $({*}{*}{*}{*})$*-morphism then* φ_2 *is a* $(i{*}e{*})$*-morphism and* χ_1 *is a* $({*}{*}{*}{*})$*-morphism.*

Sketch of proof. Let $\Sigma_2 \overset{\chi}{\leftarrow} \Sigma_0 \overset{\varphi}{\to} \Sigma_1$ be a span of **CFOL** signature morphisms such that $\Sigma_i = (S_i, F_i, F_i^c, P_i)$ for all $i \in \{0,1,2\}$, φ is a $(i*e*)$-morphism, and χ is a $({*}{*}{*}{*})$-morphism. Let $\{(S_2,F_2,P_2) \overset{\chi}{\leftarrow} (S_0,F_0,P_0) \overset{\varphi}{\to} (S_1,F_1,P_1), (S_1,F_1,P_1) \overset{\chi_1}{\to} (S,F,P) \overset{\varphi_2}{\leftarrow} (S_2,F_2,P_2)\}$ be a pushout of **FOL** signature morphisms. We define $F^c = \chi_1(F_1^c) \cup \varphi_2(F_2^c)$ and $\Sigma = (S,F,F^c,P)$. It follows that $\{\Sigma_2 \overset{\chi}{\leftarrow} \Sigma_0 \overset{\varphi}{\to} \Sigma_1, \Sigma_1 \overset{\chi_1}{\to} \Sigma \overset{\varphi_2}{\leftarrow} \Sigma_2\}$ is a pushout of **CFOL** signature morphisms such that φ_2 is a $(i*e*)$-morphism and χ_1 is a $({*}{*}{*}{*})$-morphism. □

Proposition 2.2. *[4] In* **CFOL***, the category of* $(**e*)$*-morphisms has pushouts.*

Sketch of proof. The pushout construction in **FOL** is lifted to **CFOL** as in the proof of Proposition 2.1. □

The pushouts of **CPOA** signature morphisms are constructed as in **CFOL** but without considering the relation symbols.

Proposition 2.3. *[20] In any institution* $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ *the forgetful functor* $F : \mathbb{S}ig^{pres} \to \mathbb{S}ig$*, defined by* $F((\Sigma,E) \overset{\varphi}{\to} (\Sigma',E')) = \Sigma \overset{\varphi}{\to} \Sigma'$ *for all presentation morphisms* $(\Sigma,E) \overset{\varphi}{\to} (\Sigma',E') \in \mathbb{S}ig^{pres}$*, lifts pushouts* [7].

Sketch of proof. If $\{\Sigma_2 \overset{\chi}{\leftarrow} \Sigma_0 \overset{\varphi}{\to} \Sigma_1, \Sigma_1 \overset{\chi_1}{\to} \Sigma \overset{\varphi_2}{\leftarrow} \Sigma_2\}$ is a pushout of signature morphisms then the following square of presentation morphisms is a pushout

[6] A category C has $(\mathcal{L},\mathcal{R})$-pushouts, where $\mathcal{L} \subseteq \mathcal{C}$ and $\mathcal{R} \subseteq \mathcal{C}$, if for each span of morphisms $A_2 \overset{v}{\leftarrow} A_0 \overset{u}{\to} A_1$ such that $u \in \mathcal{L}$ and $v \in \mathcal{R}$ there exists a pushout $\{A_2 \overset{v}{\leftarrow} A_0 \overset{u}{\to} A_1, A_1 \overset{v_1}{\to} A \overset{u_2}{\leftarrow} A_2\}$.

[7] A functor $F : \mathcal{C} \to \mathcal{C}'$ lifts pushouts if for any pushout $\{F(A_2) \overset{F(v)}{\leftarrow} F(A_0) \overset{F(u)}{\to} F(A_1), F(A_1) \overset{v_1'}{\to} B \overset{u_2'}{\leftarrow} F(A_2)\}$ in $\mathcal{C}'$ there exists a pushout $\{A_2 \overset{v}{\leftarrow} A_0 \overset{u}{\to} A_1, A_1 \overset{v_1}{\to} A \overset{u_2}{\leftarrow} A_2\}$ in $\mathcal{C}$ such that $F(u_2) = u_2'$ and $F(v_1) = v_1'$.

in $\mathcal{I}^{pres}$.

$$\begin{array}{ccc}
(\Sigma_2, E_2) & \xrightarrow{\varphi_2} & (\Sigma, \chi_1(E_1) \cup \varphi_2(E_2)) \\
\chi \uparrow & & \uparrow \chi_1 \\
(\Sigma_0, E_0) & \xrightarrow[\varphi]{} & (\Sigma_1, E_1)
\end{array}$$

□

2.4. Basic Sets of Sentences

A set of sentences $B \subseteq Sen(\Sigma)$ is *basic* [11] if there exists a Σ-model M_B such that, for all Σ-models M, $M \models B$ iff there exists a morphism $M_B \to M$. We say that M_B is a *basic model* of B. If in addition the morphisms $M_B \to M$ is unique then the set B is called *epi basic*.

Lemma 2.4. *Any set of atoms in* **FOL** *and* **POA** *is epi basic.*

Proof. Let B be a set of atomic (S, F, P)-sentences in **FOL**. The basic model M_B it is the initial model of B and it is constructed as follows: on the quotient $(T_{(S,F)})_{\equiv_B}$ of the term model $T_{(S,F)}$ by the congruence generated by the equational atoms of B, we interpret each relation symbol $\pi \in P$ by $(M_B)_\pi = \{(t_1/_{\equiv_B}, \ldots, t_n/_{\equiv_B}) \mid \pi(t_1, \ldots, t_n) \in B\}$. By defining an appropriate notion of congruence for **POA**-models compatible with the preorder (see [16] or [9]) one may obtain the same result for **POA**. □

The proof of Lemma 2.4 is well known, and it can be found, for example, in [11] or [14], but since we want to make use of the construction of the basic model, we include it for the convenience of the reader.

2.5. Reachable Models

As implied by the choice of signature morphisms (non-constructor constants can be mapped to ground terms) we are going to treat substitutions as morphisms in the comma category of signatures. Consider two signature morphisms $\Sigma \xrightarrow{\chi_1} \Sigma_1$ and $\Sigma \xrightarrow{\chi_2} \Sigma_2$ of an institution. A signature morphisms $\Sigma_1 \xrightarrow{\theta} \Sigma_2$ such that $\chi_1; \theta = \chi_2$ is called a Σ-*substitution* between χ_1 and χ_2.

In **FOL**, consider $\Sigma \overset{\chi_1}{\hookrightarrow} \Sigma(X_1)$ and $\Sigma \overset{\chi_2}{\hookrightarrow} \Sigma(X_2)$ two inclusion signature morphisms, where $\Sigma = (S, F, P)$, X_i is a set of constant symbols disjoint from the constants of F, and $\Sigma(X_i) = (S, F \cup X_i, P)$. A substitution between χ_1 and χ_2 in **FOL** can be represented by a function $\theta : X_1 \to T_{(S,F)}(X_2)$. One can easily notice that θ can be extended to a first-order signature morphism between $\Sigma(X_1)$ and $\Sigma(X_2)$ which is the identity on Σ and maps every variable in X_1 to a term in $T_{(S,F)}(X_2)$, according to θ.

Definition 2.5. Let $\mathcal{D}$ be a broad subcategory [8] of signature morphisms of an institution $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$. We say that a Σ-model M is $\mathcal{D}$-reachable if for each span of signature morphisms $\Sigma_1 \overset{\chi_1}{\leftarrow} \Sigma_0 \overset{\chi}{\rightarrow} \Sigma$ in $\mathcal{D}$, each χ_1-expansion M_1 of $M\upharpoonright_\chi$ determines a substitution $\theta_{M_1} : \chi_1 \to \chi$ such that $M\upharpoonright_{\theta_{M_1}} = M_1$.

In concrete examples of institutions, $\mathcal{D}$-reachable models correspond to models with elements that are interpretations of ground terms.

Proposition 2.5. *[23] In* **FOL** *and* **POA** *assume that $\mathcal{D}$ is the class of signature extensions with a finite number of constants. A model M is $\mathcal{D}$-reachable iff its elements are exactly the interpretations of terms.*

Proof. In **FOL**, let $\Sigma = (S, F, P)$ be a signature, X and Y two finite sets of constants that are different from the constants in F, and (M, h) a $\Sigma(Y)$-model with elements that are interpretation of terms, i.e. the unique extension $h^\# : T_{(S,F)}(Y) \to M$ of h to a Σ-morphism is surjective. Then for every $\Sigma(X)$-model (M, g) there exists a function $\theta : X \to T_{(S,F)}(Y)$ such that $\theta; h^\# = g$. Note that for any $x \in X$ we have $((M, h)\upharpoonright_\theta)_x = h^\#(\theta(x)) = g(x) = (M, g)_x$. Hence, $(M, h)\upharpoonright_\theta = (M, g)$.

For the converse implication, let $\Sigma = (S, F, P)$ be a signature and assume a Σ-model M that is $\mathcal{D}$-reachable. We prove that $T_{(S,F)} \to M$ is surjective, i.e. for every $m \in M$ there exists $t \in T_{(S,F)}$ such that $M_t = m$. Let $m \in M_s$ be an arbitrary element of M. Consider a new constant x of sort s, and let N be an expansion of M along $\Sigma \hookrightarrow \Sigma(x)$ (where $\Sigma(x) = (S, F \cup \{x\}, P)$) that interprets x as m. Since M is $\mathcal{D}$-reachable, there exists a substitution $\theta : \{x\} \to T_{(S,F)}$ such that $M\upharpoonright_\theta = N$. Take $t = \theta(x)$, and we have $M_t = M_{\theta(x)} = (M\upharpoonright_\theta)_x = N_x = m$.

One can replicate the above arguments for **POA** too.

□

In concrete institutions, $\mathcal{D}$ consists of signature extensions with a finite number of constants. Since $\mathcal{D}$-reachable models have elements that consist of interpretations of ground terms, we may refer to $\mathcal{D}$-reachable models as

[8] $\mathcal{C}'$ is a broad subcategory of $\mathcal{C}$ if $\mathcal{C}'$ is a subcategory of $\mathcal{C}$ and $|\mathcal{C}'| = |\mathcal{C}|$.

ground reachable models. The following remark is a direct consequence of Lemma 2.4 and Proposition 2.5.

Remark 2.1. In **FOL**, the basic models of the sets of atoms are ground reachable.

3. Sufficient Completeness

In this section we provide sufficient conditions for the existence of pushouts of *sufficient complete* presentation morphisms. Let $(S, F, F^c, P) \in |\mathbb{S}ig^{\mathbf{CFOL}}|$ and $M \in |\mathbb{M}od^{\mathbf{FOL}}(S, F, P)|$. We denote by

(1) F^{S^c} the family of sets of operations of constrained sorts, i.e. for all $s \in S$ and $w \in S^*$ we have $F^{S^c}_{w\to s} = \begin{cases} F_{w\to s} & \text{if} \quad s \in S^c \\ \emptyset & \text{if} \quad s \in S^l \end{cases}$,

(2) $Loose(M)$ the elements of loose sorts of M, i.e. for all $s \in S$ we have $Loose(M)_s = \begin{cases} \emptyset & \text{if} \quad s \in S^c \\ M_s & \text{if} \quad s \in S^l \end{cases}$,

(3) $reach_M : T_{(S,F,P)}(Loose(M)) \to M$ the unique extension of the inclusion $(Loose(M)_s \hookrightarrow M_s)_{s\in S}$ to a (S, F, P)-morphism,

(4) $Reach(M)$ the sub-model of M which consists of elements that are reachable by the operations in F^{S^c}, i.e. the image of $T_{(S,F,P)}(Loose(M))$ through $reach_M$,

(5) $con_M : T_{(S,F^c)}(Loose(M)) \to M\upharpoonright_{(S,F^c)}$ the unique extension of the inclusion $(Loose(M)_s \hookrightarrow M_s)_{s\in S}$ to a (S, F^c)-morphism,

(6) $Con(M)$ the sub-algebra of $M\upharpoonright_{(S,F^c)}$ which consists of elements that are reachable by the constructors in F^c, i.e. the image of $T_{(S,F^c)}(Loose(M))$ through con_M.

We prove some useful properties of **CFOL** models and signature morphisms.

Lemma 3.1. *Let* $(S, F, F^c, P) \in |\mathbb{S}ig^{\mathbf{CFOL}}|$ *and* $M \in |\mathbb{M}od^{\mathbf{FOL}}(S, F, P)|$.

(1) For all $E \subseteq \mathbb{S}en^{\mathbf{FOL}}(S, F, P)$, $M \models E$ *implies* $Reach(M) \models E$.

(2) If $M \in |\mathbb{M}od^{\mathbf{CFOL}}(S, F, F^{S^c}, P)|$ and $Con(M)$ is a (S, F)-algebra then $M \in |\mathbb{M}od^{\mathbf{CFOL}}(S, F, F^c, P)|$.

Let $(S_0, F_0, F_0^c, P_0) \xrightarrow{\varphi} (S, F, F^c, P) \in \mathbb{S}ig^{\mathbf{CFOL}}$ and $M_0 = M{\restriction}_{(S_0,F_0,P_0)}$.

(3) If φ is a $(**e*)$-morphism then $Con(M_0) = Con(M){\restriction}_{(S_0,F_0^c)}$.

(4) If φ is a $(****)$-morphism then $Con(M){\restriction}_{(S_0,F_0^c)} \subseteq Con(M_0)$.

Proof. Let $(S, F, F^c, P) \in |\mathbb{S}ig^{\mathbf{CFOL}}|$ and $M \in |\mathbb{M}od^{\mathbf{FOL}}(S, F, P)|$.

(1) The proof of the first statement is straightforward by the definition of the satisfaction relation.

(2) We show that for all terms $t^r \in T_{(S,F,P)}(Loose(M))$ there exists a term $t^c \in T_{(S,F^c)}(Loose(M))$ such that $reach_M(t^r) = con_M(t^c)$. We proceed by induction on the structure of the terms in $T_{(S,F,P)}(Loose(M))$.

   (a) The base case is trivial.

   (b) For the induction step, let $\sigma \in F_{w\to s}$, $t^r \in T_{(S,F,P)}(Loose(M))_w$, and assume there exists $t^c \in T_{(S,F^c)}(Loose(M))_w$ such that $con_M(t^c) = reach_M(t^r)$. We have that $reach_M(\sigma(t^r)) = M_\sigma(reach_M(t^r)) = M_\sigma(con_M(t^c))$, and since $Con(M)$ is a (S, F)-algebra, we obtain $M_\sigma(con_M(t^c)) \in Con(M)$. It follows that there exists a term $t \in T_{(S,F^c)}(Loose(M))$ such that $con_M(t) = M_\sigma(con_M(t^c))$. We get that $con_M(t) = reach_M(\sigma(t^r))$.

   Since $reach_M$ is surjective, con_M is surjective, which implies that M is a (S, F, F^c, P)-model.

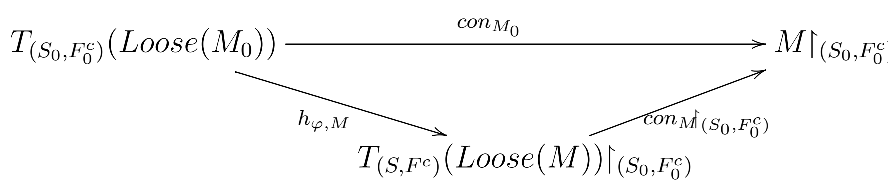
(3) Let $\varphi : (S_0, F_0, F_0^c, P_0) \to (S, F, F^c, P)$ be a $(**e*)$-morphism. Since any $(**e*)$-morphism preserves the loose sorts, $Loose(M){\restriction}_{S_0} = Loose(M_0)$, where $Loose(M){\restriction}_{S_0}$ is the S_0-sorted set such that for all sorts $s_0 \in S_0$ we have $(Loose(M){\restriction}_{S_0})_{s_0} = Loose(M)_{\varphi(s_0)}$. It follows that $Loose(M_0) \subseteq T_{(S,F^c)}(Loose(M)){\restriction}_{(S_0,F_0^c)}$.

   Let $h_{\varphi,M} : T_{(S_0,F_0^c)}(Loose(M_0)) \to T_{(S,F^c)}(Loose(M)){\restriction}_{(S_0,F_0^c)}$ be the unique extension of the inclusion $Loose(M_0) \hookrightarrow T_{(S,F^c)}(Loose(M)){\restriction}_{(S_0,F_0^c)}$ to a (S_0, F_0^c)-morphism. We show that for all sorts $s \in S$ and terms $t \in T_{(S,F^c)}(Loose(M))_{\varphi(s)}$ there exists a term $t_0 \in T_{(S_0,F_0^c)}(Loose(M_0))_s$ such that $h_{\varphi,M}(t_0) = t$. We proceed by induction on the structure of the terms in $T_{(S,F^c)}(Loose(M))$.

(a) Since $Loose(M)\restriction_{S_0} = Loose(M_0)$, the base case is trivial.

(b) For the induction step, assume $s_0 \in S_0$, $\sigma \in F^c_{w \to \varphi(s_0)}$ and $t \in T_{(S,F^c)}(Loose(M))_w$. There exists a constructor $\sigma_0 \in (F^c_0)_{w_0 \to s_0}$ such that $\varphi(\sigma_0) = \sigma$. By the induction hypothesis, there exists $t_0 \in T_{(S_0,F^c_0)}(Loose(M_0))_{w_0}$ such that $h_{\varphi,M}(t_0) = t$. We have that $h_{\varphi,M}(\sigma_0(t_0)) = \sigma(h_{\varphi,M}(t_0)) = \sigma(t)$.

It follows that $h_{\varphi,M}$ is surjective. Note that both model morphisms con_{M_0} and $h_{\varphi,M}; con_M\restriction_{(S_0,F^c_0)}$ are identities on $Loose(M_0)$, which implies $con_{M_0} = h_{\varphi,M}; con_M\restriction_{(S_0,F^c_0)}$.



Since $h_{\varphi,M}$ is surjective, the image of $T_{(S_0,F^c_0)}(Loose(M_0))$ through $h_{\varphi,M}$ is $T_{(S,F^c)}(Loose(M))\restriction_{(S_0,F^c_0)}$. It follows that $con_{M_0}(T_{(S_0,F^c_0)}(Loose(M_0))) = con_M(T_{(S,F^c)}(Loose(M)))\restriction_{(S_0,F^c_0)}$. Hence, $Con(M_0) = Con(M)\restriction_{(S_0,F^c_0)}$.

(4) Let $\varphi : (S_0, F_0, F^c_0, P_0) \to (S, F, F^c, P)$ be a $(****)$-morphism. We prove that for all $s_0 \in S_0$ and $t \in T_{(S,F^c)}(Loose(M))_{\varphi(s_0)}$ there exists $t_0 \in T_{(S_0,F^c_0)}(Loose(M_0))_{s_0}$ such that $con_{M_0}(t_0) = con_M(t)$. We proceed by induction on the structure of the terms in $T_{(S,F^c)}(Loose(M))$.

(a) For the base case, since $Loose(M)\restriction_{S_0} \subseteq Loose(M_0)$, for every $s_0 \in S_0$ and $t \in Loose(M)_{\varphi(s_0)}$ we have $con_{M_0}(t) = con_M(t) = t$.

(b) Let $s_0 \in S_0$, $\sigma \in F^c_{w \to \varphi(s_0)}$ and $t \in T_{(S,F^c)}(Loose(M))_w$. There are two subcases:

(i) $s_0 \in S^c_0$. There exists $\sigma_0 \in (F^c_0)_{w_0 \to s_0}$ such that $\varphi(\sigma_0) = \sigma$. By induction hypothesis, there exists $t_0 \in T_{(S_0,F^c_0)}(Loose(M_0))_{w_0}$ such that $con_M(t) = con_{M_0}(t_0)$. It follows that $con_M(\sigma(t)) = M_\sigma(con_M(t)) = M_{\varphi(\sigma_0)}(con_{M_0}(t_0)) = (con_{M_0}(\sigma_0(t_0)))$.

(ii) $s_0 \in S^l_0$. Take $t_0 = con_M(\sigma(t))$, and we have $con_{M_0}(t_0) = t_0 = con_M(\sigma(t))$.

□

Definition 3.1. A **CFOL** presentation $((S,F,F^c,P),E)$ is *sufficient complete* if for all models $M \in |\mathbb{M}od^{\mathbf{CFOL}}(S,F,F^{S^c},P)|$ that satisfy E, we have $M \in |\mathbb{M}od^{\mathbf{CFOL}}(S,F,F^c,P)|$.

In other words, a presentation $((S,F,F^c,P),E)$ is sufficient complete when for all (S,F,P)-models M that are reachable by the operations in F^{S^c}, M satisfies E implies that M is reachable by the constructors in F^c.

Let $\mathbb{S}ig^{\mathbf{CFOL}^{sc}} \subseteq \mathbb{S}ig^{\mathbf{CFOL}^{pres}}$ be the full subcategory of sufficient complete presentations: $|\mathbb{S}ig^{\mathbf{CFOL}^{sc}}|$ consists of presentations $((S,F,F^c,P),E)$ that are sufficient complete. We define the *institution of sufficient complete presentations* $\mathbf{CFOL}^{sc}$ as the restriction of $\mathbf{CFOL}^{pres}$ to the sufficient complete presentations.

Proposition 3.2. *The inclusion functor* $\mathbb{S}ig^{\mathbf{CFOL}^{sc}} \hookrightarrow \mathbb{S}ig^{\mathbf{CFOL}^{pres}}$ *lifts the* $((iee*)^{pres}, (****)^{pres})$*-pushouts.*

Proof. Consider the following span of sufficient complete presentation morphisms

$$((S_2,F_2,F_2^c,P_2),E_2) \xleftarrow{\chi} ((S_0,F_0,F_0^c,P_0),E_0) \xrightarrow{\varphi} ((S_1,F_1,F_1^c,P_1),E_1)$$

such that φ is a $(iee*)^{pres}$-morphism and χ is a $(****)^{pres}$-morphism. By Proposition 2.1, there exists a pushout of **CFOL** signature morphisms

$$\begin{array}{ccc}
(S_2,F_2,F_2^c,P_2) & \overset{\varphi_2}{\dashrightarrow} & (S,F,F^c,P) \\
\chi \uparrow & & \uparrow \chi_1 \\
(S_0,F_0,F_0^c,P_0) & \xrightarrow[\varphi]{} & (S_1,F_1,F_1^c,P_1)
\end{array}$$

such that φ_2 is a $(iee*)$-morphism and χ_1 is a $(****)$-morphism. By Proposition 2.3, the following square of presentation morphisms is a pushout.

$$\begin{array}{ccc}
((S_2,F_2,F_2^c,P_2),E_2) & \overset{\varphi_2}{\dashrightarrow} & ((S,F,F^c,P),\chi_1(E_1)\cup\varphi_2(E_2)) \\
\chi \uparrow & & \uparrow \chi_1 \\
((S_0,F_0,F_0^c,P_0),E_0) & \xrightarrow[\varphi]{} & ((S_1,F_1,F_1^c,P_1),E_1)
\end{array}$$

If we prove that $((S,F,F^c,P),\chi_1(E_1)\cup\varphi_2(E_2))$ is sufficient complete then the above square of presentation morphisms is a pushout in $\mathbb{S}ig^{\mathbf{CFOL}^{sc}}$.

Assume a model $M \in |\mathbb{M}od((S,F,F^{S^c},P),\chi_1(E_1)\cup\varphi_2(E_2))|$. Let $M_1 = M\restriction_{(S_1,F_1,P_1)}$ and $M_2 = M\restriction_{(S_2,F_2,P_2)}$.

Firstly, we show that con_M is surjective on $\varphi_2(S_2)$. Since φ_2 encapsulates all operations, $\varphi_r : (S_2,F_2,F_2^{S_2^c},P_2) \to (S,F,F^{S^c},P)$ is a **CFOL** signature morphism, where $\varphi_r^{st} = \varphi_2^{st}$, $\varphi_r^{op} = \varphi_2^{op}$ and $\varphi_r^{rl} = \varphi_2^{rl}$. It follows that $M_2 \in |\mathbb{M}od(S_2,F_2,F_2^{S_2^c},P_2)|$. By the satisfaction condition, $M_2 \models E_2$, and since $((S_2,F_2,F_2^c,P_2),E_2)$ is sufficient complete, we obtain $M_2 \in |\mathbb{M}od(S_2,F_2,F_2^c,P_2)|$. It follows that $M_2\restriction_{(S_2,F_2^c)} = Con(M_2)$. Since φ_2 is a $(**e*)$-morphism, by Lemma 3.1 (3), $Con(M_2) = Con(M)\restriction_{(S_2,F_2^c)}$. We have $M\restriction_{(S_2,F_2^c)} = Con(M)\restriction_{(S_2,F_2^c)}$. Hence, con_M is surjective on $\varphi_2(S_2)$.

Secondly, we show that for all $s_1 \in S_1$ and $t_1 \in T_{(S_1,F_1^c)}(Loose(M_1))_{s_1}$ there exists $t \in T_{(S,F^c)}(Loose(M))_{\chi_1(s_1)}$ such that $con_{M_1}(t_1) = con_M(t)$. We proceed by induction on the structure of the terms in $T_{(S_1,F_1^c)}(Loose(M_1))$.

1. Let $t_1 \in Loose(M_1)_{s_1}$. There are two sub-cases:

   (a) $\chi_1(s_1) \in \varphi_2(S_2)$. Since con_M is surjective on $\varphi_2(S_2)$, there exists $t \in T_{(S,F^c)}(Loose(M))$ such that $con_M(t) = t_1$.

   (b) $\chi_1(s_1) \notin \varphi_2(S_2)$. Since $s_1 \in S_1^l$ and $\chi_1(s_1) \notin \varphi_2(S_2)$, we have $\chi_1(s_1) \in S^l$ (indeed, if $\chi_1(s_1) \in S^c$ then since $\chi_1(s_1) \notin \varphi_2(S_2)$ and $F^c = \chi_1(F_1^c) \cup \varphi_2(F_2^c)$, there exists $\sigma_1 \in (F_1^c)_{w_1\to s_1'}$ such that $\chi_1(s_1) = \chi_1(s_1')$; by the pushout construction, $\chi_1(s_1) = \chi_1(s_1') \notin \varphi_2(S_2)$ implies $s_1 = s_1'$; we obtain $s_1 \in S_1^c$, which is a contradiction). Take $t = t_1$, and we have $con_{M_1}(t_1) = t_1 = t = con_M(t)$.

2. Let $\sigma_1 \in (F_1^c)_{w_1\to s_1}$, $t_1 \in T_{(S_1,F_1^c)}(Loose(M_1))_{w_1}$, and assume that there exists $t \in T_{(S,F^c)}(Loose(M'))_{\chi_1(w_1)}$ such that $con_{M_1}(t_1) = con_M(t)$. We have $con_{M_1}(\sigma_1(t_1)) = (M_1)_{\sigma_1}(con_{M_1}(t_1)) = M_{\chi_1(\sigma_1)}(con_M(t)) = con_M(\chi_1(\sigma_1)(t))$.

Thirdly, we prove that $Con(M)$ is a (S,F)-algebra. Let $\sigma \in F_{w\to s}$ and $m \in Con(M)_w$. There are two cases:

1. There exists $\sigma_1 \in (F_1)_{w_1\to s_1}$ such that $\chi_1(\sigma_1) = \sigma$. By Lemma 3.1 (4), $Con(M)\restriction_{(S_1,F_1^c)} \subseteq Con(M_1)$, which implies $Con(M)_{\chi_1(w_1)} \subseteq Con(M_1)_{w_1}$, and we obtain $m \in Con(M_1)_{w_1}$. Since $((S_1,F_1,F_1^c,P_1),E_1)$ is sufficient complete, $M_\sigma(m) = (M_1)_{\sigma_1}(m) \in Con(M_1)_{s_1}$. By the second part of the proof, there exists $t \in T_{(S,F^c)}(Loose(M))_{\chi_1(s_1)}$ such that $con_M(t) = M_\sigma(m)$. Hence, $M_\sigma(m) \in Con(M)_{\chi_1(s_1)}$.

2. There exists $\sigma_2 \in (F_2)_{w_2 \to s_2}$ such that $\varphi_2(\sigma_2) = \sigma$. By Lemma 3.1 (3), $Con(M)\upharpoonright_{(S_2,F_2^c)} = Con(M_2)$, which implies $Con(M)_{\varphi_2(w_2)} = Con(M_2)_{w_2}$, and we obtain $m \in Con(M_2)_{w_2}$. Since $((S_2, F_2, F_2^c, P_2), E_2)$ is sufficient complete, $M_\sigma(m) = (M_2)_{\sigma_2}(m) \in Con(M_2)_{s_2}$. Since $Con(M_2)_{s_2} = Con(M)_{\varphi_2(s_2)}$, we get $M_\sigma(m) \in Con(M)_{\varphi_2(s_2)}$.

□

Proposition 3.3. *The inclusion functor $\mathbb{S}ig^{\mathbf{CFOL}^{sc}} \hookrightarrow \mathbb{S}ig^{\mathbf{CFOL}^{pres}}$ lifts the $((**e*)^{pres}, (**e*)^{pres})$-pushouts.*

Proof. Consider the following span of $(**e*)^{pres}$-morphisms in $\mathbf{CFOL}^{sc}$:

$$((S_2, F_2, F_2^c, P_2), E_2) \stackrel{\chi}{\leftarrow} ((S_0, F_0, F_0^c, P_0), E_0) \stackrel{\varphi}{\rightarrow} ((S_1, F_1, F_1^c, P_1), E_1)$$

By Proposition 2.2, there exists a pushout of **CFOL** signature morphisms

$$\{(S_2, F_2, F_2^c, P_2) \stackrel{\chi}{\leftarrow} (S_0, F_0, F_0^c, P_0) \stackrel{\varphi}{\rightarrow} (S_1, F_1, F_1^c, P_1),$$
$$(S_2, F_2, F_2^c, P_2) \stackrel{\varphi_2}{\rightarrow} (S, F, F^c, P) \stackrel{\chi_1}{\leftarrow} (S_1, F_1, F_1^c, P_1)\}$$

such that both χ_1 and φ_2 are $(**e*)$-morphisms. By Proposition 2.3, the following square of presentation morphisms is a pushout.

$$\begin{array}{ccc} ((S_2, F_2, F_2^c, P_2), E_2) & \stackrel{\varphi_2}{\dashrightarrow} & ((S, F, F^c, P), \chi_1(E_1) \cup \varphi_2(E_2)) \\ \chi \uparrow & & \uparrow \chi_1 \\ ((S_0, F_0, F_0^c, P_0), E_0) & \xrightarrow[\varphi]{} & ((S_1, F_1, F_1^c, P_1), E_1) \end{array}$$

If we prove that $((S, F, F^c, P), \chi_1(E_1) \cup \varphi_2(E_2))$ is sufficient complete then the above square of presentation morphisms is a pushout in $\mathbb{S}ig^{\mathbf{CFOL}^{sc}}$. Let $M \in |\mathbb{M}od((S, F, F^{S^c}, P), \chi_1(E_1) \cup \varphi_2(E_2))|$. It follows that $reach_M : T_{(S,F,P)}(Loose(M)) \to M$ is a surjection.

By Lemma 3.1 (2), it suffices to prove that $Con(M)$ is a (S, F)-algebra. Let $\sigma \in F_{w \to s}$ and $m \in Con(M)_w$. Since $F = \chi_1(F_1) \cup \varphi_2(F_2)$, without loss of generality we assume that there exists $\sigma_1 \in (F_1)_{w_1 \to s_1}$ such that $\chi_1(\sigma_1) = \sigma$. The case when there exists $\sigma_2 \in (F_2)_{w_2 \to s_2}$ such that $\varphi_2(\sigma_2) = \sigma$ is similar. Let $M_1 = M\upharpoonright_{(S_1,F_1,P_1)}$. By Lemma 3.1 (3), we have $Con(M)\upharpoonright_{(S_1,F_1^c)} = Con(M_1)$, which implies $m \in Con(M_1)_{w_1}$. By the satisfaction condition, $M_1 \models E_1$. By Lemma 3.1 (1), $Reach(M_1) \models E_1$. Since $((S_1, F_1, F_1^c, P_1), E_1)$ is sufficient complete, $Reach(M_1) \in \mathbb{M}od(S_1, F_1, F_1^c, P_1)$, which implies that $Reach(M_1)\upharpoonright_{(S_1,F_1^c)} = Con(M_1)$. Since $(M_1)_{\sigma_1}(m) \in Reach(M_1)_{s_1}$, $(M_1)_{\sigma_1}(m) \in Con(M_1)_{s_1}$. Hence, $M_\sigma(m) \in Con(M)_{\chi(s_1)}$. □

The results of this section are obtained in **CFOL**, but the method of constructing the pushouts can be used to other constructor-based institutions such as **CPOA**.

4. Borrowing Interpolation

In this section we prove a general result concerning Horn institutions that provides sufficient conditions to carry out the interpolation from a base institution to its constructor-based counterpart across the "forgetful" institution morphisms. Below we recall the concept of Craig interpolation at the abstract level of institutions.

Definition 4.1 (Craig Interpolation)**.** In any institution a commuting square of signature morphisms

$$\begin{array}{ccc} \Sigma_2 & \xrightarrow{\varphi_2} & \Sigma \\ \chi \uparrow & & \uparrow \chi_1 \\ \Sigma_0 & \xrightarrow[\varphi]{} & \Sigma_1 \end{array}$$

is a *Craig Interpolation square (CI square)* iff for each set E_1 of Σ_1-sentences and any set E_2 of Σ_2-sentences such that $\chi_1(E_1) \models \varphi_2(E_2)$ there exists $E_0 \subseteq \mathbb{S}en(\Sigma_0)$ such that $E_1 \models \varphi(E_0)$ and $\chi(E_0) \models E_2$.

The Craig interpolation property can be strengthened by adding to the initial premises E_1 a set Γ_2 of Σ_2-sentences as secondary premises.

Definition 4.2 (Craig-Robinson Interpolation)**.** In any institution we say that a commuting square of signature morphisms

$$\begin{array}{ccc} \Sigma_2 & \xrightarrow{\varphi_2} & \Sigma \\ \chi \uparrow & & \uparrow \chi_1 \\ \Sigma_0 & \xrightarrow[\varphi]{} & \Sigma_1 \end{array}$$

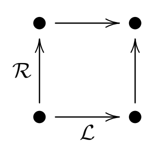
is a *Craig-Robinson Interpolation square (CRI square)* iff for each set E_1 of Σ_1-sentences and any sets E_2 and Γ_2 of Σ_2-sentences such that $\chi_1(E_1) \cup \varphi_2(\Gamma_2) \models \varphi_2(E_2)$ there exists a set E_0 of Σ_0-sentences such that $E_1 \models \varphi(E_0)$ and $\chi(E_0) \cup \Gamma_2 \models E_2$.

The name "Craig-Robinson" interpolation has been used for instances of this property in [40, 19] and "strong Craig interpolation" has been used in [18]. Note that a CRI square is also a CI square, and under certain conditions, such as compactness and the presence of implications, CI is equivalent to CRI. For a proof of the following lemma see for example [14].

Lemma 4.1. *In any compact institution* $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ *with implications a pushout square of signature morphisms is a CI square iff it is a CRI square.*

The above lemma does not hold in Horn institutions such as **HCL** because they do not have implications.

Definition 4.3. An institution has *Craig* $(\mathcal{L}, \mathcal{R})$*-interpolation (*$(\mathcal{L}, \mathcal{R})$*-CI)*, respectively, *Craig-Robinson* $(\mathcal{L}, \mathcal{R})$*-interpolation (*$(\mathcal{L}, \mathcal{R})$*-CRI)*, for two subcategories of signature morphisms $\mathcal{L}$ and $\mathcal{R}$ if each pushout square of signature morphisms of the form



is a CI square, respectively, CRI square.

Remark 4.1. Below they are some interpolation results from the literature.

- according to [14], **HCL** has $((ie*), (***))$-CRI,
- according to [12, 35], **HCL** has $((***), (iii))$-CI.
- according to [14], **HPOA** has $((ie), (**))$-CRI, and
- according to [12, 35], **HPOA** has $((**), (ii))$-CI.

4.1. Horn Institutions

We show that in Horn institutions one may not consider all models but a restricted class of models to establish that a sentence is a semantic consequence of a set of sentences. More concrete, for any signature Σ we identify a proper class $\mathcal{M}_\Sigma$ of Σ-models such that

$$\Gamma \models_\Sigma e \text{ iff for all } M \in \mathcal{M}_\Sigma,\ M \models_\Sigma \Gamma \text{ implies } M \models_\Sigma e$$

for any set of Σ-sentences Γ and each Σ-sentence e.

Below we formalize the concept of Horn institution [9].

Definition 4.4. An institution $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ is a $\mathcal{D}$-Horn institution over $\mathcal{I}_0 = (\mathbb{S}ig, \mathbb{S}en_0, \mathbb{M}od, \models)$, where $\mathbb{S}en_0 \subseteq \mathbb{S}en$ and $\mathcal{D} \subseteq \mathbb{S}ig$, when

- for all $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$, $H \subseteq \mathbb{S}en_0(\Sigma')$ finite, and $C \in \mathbb{S}en_0(\Sigma')$ we have $(\forall\chi) \bigwedge H \Rightarrow C \in \mathbb{S}en(\Sigma)$, and
- any sentence of $\mathcal{I}$ is of the form $(\forall\chi) \bigwedge H \Rightarrow C$ as above.

For example **HCL** is a $\mathcal{D}$-Horn institution over its restriction $\mathbf{HCL}_0$ to atomic sentences, where $\mathcal{D}$ consists of signature extensions with a finite number of constants.

We consider the following result fundamental for the class of Horn institutions, and as we will see later, it is playing a key role in carrying out the interpolation properties from the class of Horn institutions to their constructor-based variants.

Theorem 4.2. *Let $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be a $\mathcal{D}$-Horn institution over $\mathcal{I}_0 = (\mathbb{S}ig, \mathbb{S}en_0, \mathbb{M}od, \models)$ such that*

(1) all sets of sentences in $\mathcal{I}_0$ are basic, and

(2) for each signature $\Sigma \in |\mathbb{S}ig|$ and set of sentences $B \subseteq \mathbb{S}en_0(\Sigma)$ there exists a basic model M_B that is $\mathcal{D}$-reachable.

For all sets of sentences $\Gamma \subseteq \mathbb{S}en(\Sigma)$ and any sentence $(\forall\chi) \bigwedge H \Rightarrow C \in \mathbb{S}en(\Sigma)$, where $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$, we have

(i) $M_{\Gamma_0}\restriction_{\chi} \models_{\Sigma} \Gamma$, and

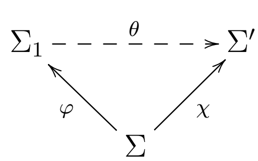
(ii) $M_{\Gamma_0} \models_{\Sigma'} C$ implies $\Gamma \models_{\Sigma} (\forall\chi) \bigwedge H \Rightarrow C$.

where

(3) $\Gamma_0 = \{e' \in \mathbb{S}en_0(\Sigma') \mid \chi(\Gamma) \cup H \models_{\Sigma'} e'\}$, and

(4) M_{Γ_0} is the basic model of Γ_0.

Proof. Firstly, we show that $M_{\Gamma_0}\restriction_{\chi} \models_{\Sigma} \Gamma$. Let $(\forall\varphi) \bigwedge E \Rightarrow e \in \Gamma$, where $\Sigma \xrightarrow{\varphi} \Sigma_1 \in \mathcal{D}$, and N a φ-expansion of $M_{\Gamma_0}\restriction_{\chi}$ such that $N \models_{\Sigma_1} E$. Since M_{Γ_0} is $\mathcal{D}$-reachable, there exists $\theta : \varphi \to \chi$ such that $M_{\Gamma_0}\restriction_{\theta} = N$.



By the satisfaction condition, $M_{\Gamma_0} \models_{\Sigma'} \theta(E)$ which implies $\Gamma_0 \models_{\Sigma'} \theta(E)$. Since $\chi(\Gamma) \cup H \models_{\Sigma'} \Gamma_0$, we have $\chi(\Gamma) \cup H \models_{\Sigma'} \theta(E)$, which implies $\chi(\Gamma) \cup H \models_{\Sigma'} \chi(\Gamma) \cup \theta(E)$. On the other hand $(\forall\varphi) \bigwedge E \Rightarrow e \models_{\Sigma} (\forall\chi) \bigwedge \theta(E) \Rightarrow \theta(e)$, and we obtain $\Gamma \models (\forall\chi) \bigwedge \theta(E) \Rightarrow \theta(e)$ which implies $\chi(\Gamma) \cup \theta(E) \models_{\Sigma'} \theta(e)$. Since $\chi(\Gamma) \cup H \models_{\Sigma'} \chi(\Gamma) \cup \theta(E)$, we have $\chi(\Gamma) \cup H \models_{\Sigma'} \theta(e)$ meaning that $\theta(e) \in \Gamma_0$. Hence, $M_{\Gamma_0} \models \theta(e)$ and by satisfaction condition $N \models_{\Sigma_1} e$.

Secondly, assuming $\Gamma \not\models_{\Sigma} (\forall\chi) \bigwedge H \Rightarrow C$, we show $M_{\Gamma_0} \not\models_{\Sigma'} C$. This is a direct consequence of the fact that $\Gamma_0 \not\models_{\Sigma'} C$. Indeed if $\Gamma_0 \models_{\Sigma'} C$ then $\chi(\Gamma) \cup H \models_{\Sigma'} C$, and we obtain $\Gamma \models_{\Sigma} (\forall\chi) \bigwedge H \Rightarrow C$, which is a contradiction. □

Proposition 4.3. *Note that under the conditions of Theorem 4.2 we have: $\Gamma \models_{\Sigma} e$ iff for all $\Sigma \xrightarrow{\varphi} \Sigma' \in \mathcal{D}$ and $\mathcal{D}$-reachable $M' \in \mathbb{M}od(\Sigma')$, we have $M'{\upharpoonright_{\varphi}} \models_{\Sigma} \Gamma$ implies $M'{\upharpoonright_{\varphi}} \models_{\Sigma} e$.*

Proof. The implication from left to right is obvious. For the converse implication assume that $\Gamma \not\models_{\Sigma} (\forall\chi) \bigwedge H \Rightarrow C$, where $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$. By Theorem 4.2, $M_{\Gamma_0}{\upharpoonright_{\chi}} \models_{\Sigma} \Gamma$ and $M_{\Gamma_0} \not\models_{\Sigma'} C$, where $\Gamma_0 = \{e' \in \mathbb{S}en_0(\Sigma') \mid \chi(\Gamma) \cup H \models_{\Sigma'} e'\}$. We obtain $M_{\Gamma_0}{\upharpoonright_{\chi}} \models_{\Sigma} \Gamma$ and $M_{\Gamma_0}{\upharpoonright_{\chi}} \not\models_{\Sigma} (\forall\chi) \bigwedge H \Rightarrow C$, for a $\mathcal{D}$-reachable Σ'-model M_{Γ_0}. □

The result below is a corollary of Theorem 4.2.

Corollary 4.4. *For all sufficient complete presentations $((S, F, F^c, P), E)$ and any (S, F, F^c, P)-sentence $(\forall Y) \bigwedge H \Rightarrow C$ such that the sorts of the variables in Y are loose, we have:*

$$E \models^{\mathbf{HCL}}_{(S,F,P)} (\forall Y) \bigwedge H \Rightarrow C \textit{ iff } E \models^{\mathbf{CHCL}}_{(S,F,F^c,P)} (\forall Y) \bigwedge H \Rightarrow C$$

Proof. It is obvious that $E \models^{\mathbf{HCL}}_{(S,F,P)} (\forall Y) \bigwedge H \Rightarrow C$ implies $E \models^{\mathbf{CHCL}}_{(S,F,F^c,P)} (\forall Y) \bigwedge H \Rightarrow C$. For the converse implication, assume that $E \not\models^{\mathbf{HCL}}_{(S,F,P)} (\forall Y) \bigwedge H \Rightarrow C$. We set the parameters of Theorem 4.2: $\mathcal{I}$ is **HCL**, $\mathcal{I}_0$ is $\mathbf{HCL}_0$, the restriction of **HCL** to atomic sentences, and $\mathcal{D}$ consists of signature extensions with a finite number of constants.

Let $\iota_Y : \Sigma \hookrightarrow \Sigma(Y)$ be the inclusion such that $\Sigma = (S, F, P)$ and $\Sigma(Y) = (S, F \cup Y, P)$. We define $E_0 = \{e \text{ is an atom} \mid E \cup H \models_{\Sigma(Y)} e\}$. There exists a basic model $M_{E_0} \in |\mathbb{M}od^{\mathbf{HCL}}(\Sigma(Y))|$ that is ground reachable (see Remark 2.1). By Theorem 4.2, $M_{E_0}{\upharpoonright_{\iota_Y}} \models^{\mathbf{HCL}}_{\Sigma} E$ and $M_{E_0} \not\models^{\mathbf{HCL}}_{\Sigma(Y)} C$. Note that $M_{E_0} \not\models^{\mathbf{HCL}}_{\Sigma(Y)} C$ implies $M_{E_0}{\upharpoonright_{\iota_Y}} \not\models^{\mathbf{HCL}}_{\Sigma} (\forall Y) \bigwedge H \Rightarrow C$.

Since Y consists of variables of loose sorts, $M_{E_0}\upharpoonright_{\iota_Y} \in |\mathbb{M}od(S, F, F^{S^c}, P)|$. Because $((S, F, F^c, P), E)$ is sufficient complete, $M_{E_0}\upharpoonright_{\iota_Y} \in |\mathbb{M}od(S, F, F^c, P)|$. We have that $M_{E_0}\upharpoonright_{\iota_Y} \models^{\mathbf{CHCL}}_{(S,F,F^c,P)} E$ and $M_{E_0}\upharpoonright_{\iota_Y} \not\models^{\mathbf{CHCL}}_{(S,F,F^c,P)} (\forall Y) \bigwedge H \Rightarrow C$, which implies $E \not\models^{\mathbf{CHCL}}_{(S,F,F^c,P)} (\forall Y) \bigwedge H \Rightarrow C$. □

4.2. Borrowing Result

Institution morphisms were introduced in [20] and are suitable to formalise "forgetful" mappings between more complex institutions to simpler ones.

Definition 4.5. Let $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ and $\mathcal{I}' = (\mathbb{S}ig', \mathbb{S}en', \mathbb{M}od', \models')$ be two institutions. An *institution morphism* $(\phi, \alpha, \beta) : \mathcal{I} \to \mathcal{I}'$ consists of

- a functor $\phi : \mathbb{S}ig \to \mathbb{S}ig'$, and
- two natural transformations $\alpha : \phi; \mathbb{S}en' \Rightarrow \mathbb{S}en$ and $\beta : \mathbb{M}od \Rightarrow \phi; \mathbb{M}od'$ such that the following satisfaction condition for institution morphisms holds:

  $$M \models_\Sigma \alpha_\Sigma(e') \text{ iff } \beta_\Sigma(M) \models'_{\phi(\Sigma)} e'$$

  for all signatures $\Sigma \in |\mathbb{S}ig|$, Σ-models M, and $\phi(\Sigma)$-sentences e'.

We define the institution morphism $\Delta_{\mathbf{CHCL}} = (\phi, \alpha, \beta) : \mathbf{CHCL} \to \mathbf{HCL}$ as follows:

(1) the functor ϕ maps every **CHCL** signature morphism $(S, F, F^c, P) \xrightarrow{\varphi} (S', F', F'^c, P')$ to the **HCL** signature morphism $(S, F, P) \xrightarrow{\varphi} (S', F', P')$;

(2) α is the identity natural transformation, i.e. for every **CHCL** signature (S, F, F^c, P) we have $\alpha_{(S,F,F^c,P)} = 1_{\mathbb{S}en(S,F,P)}$;

(3) β is the inclusion natural transformation, i.e. for every **CHCL** signature (S, F, F^c, P), $\beta_{(S,F,F^c,P)} : \mathbb{M}od(S, F, F^c, P) \hookrightarrow \mathbb{M}od(S, F, P)$ is the inclusion functor.

We use the previous results to borrow interpolation across the "forgetful" institution morphisms.

Theorem 4.5 (Borrowing Interpolation)**.** *Let $\mathcal{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution, $\mathbb{S}ig_n \subseteq \mathbb{S}ig$ a broad subcategory of signature morphisms, and $\mathbb{S}en_n : \mathbb{S}ig_n \to \mathbb{S}et$ a sub-functor of $\mathbb{S}en : \mathbb{S}ig_n \to \mathbb{S}et$ such that*

(1) for all $\rho \in \mathbb{S}en(\Sigma)$ we have $\rho \models\!\mid \Gamma_\rho$ for some $\Gamma_\rho \subseteq \mathbb{S}en_n(\Sigma)$.

Let $\mathbb{S}ig^{sc} \subseteq \mathbb{S}ig^{pres}$ be a subcategory of presentation morphisms such that

(2) $(\Sigma, \Gamma) \in |\mathbb{S}ig^{sc}|$ whenever $(\Sigma, E) \in |\mathbb{S}ig^{sc}|$ and $E \subseteq \Gamma \subseteq \mathbb{S}en(\Sigma)$.

Let $(\phi, \alpha, \beta) : \mathcal{I} \to \mathcal{I}'$ be an institution morphism as in Definition 4.5 such that

(3) for all $\Sigma \in |\mathbb{S}ig|$, α_Σ is surjective, and

(4) for all $\Sigma \in |\mathbb{S}ig|$, $\Gamma' \subseteq \mathbb{S}en'(\phi(\Sigma))$ and $\rho' \in \mathbb{S}en'(\phi(\Sigma))$ such that $(\Sigma, \alpha_\Sigma(\Gamma')) \in |\mathbb{S}ig^{sc}|$ and $\alpha_\Sigma(\rho') \in \mathbb{S}en_n(\Sigma)$ we have $\alpha_\Sigma(\Gamma') \models_\Sigma \alpha_\Sigma(\rho')$ iff $\Gamma' \models'_{\phi(\Sigma)} \rho'$.

Let $\mathcal{L} \subseteq \mathbb{S}ig_n$ and $\mathcal{R} \subseteq \mathbb{S}ig$ be two broad subcategories of signature morphisms such that

(5) $\mathbb{S}ig$ has $(\mathcal{L}, \mathcal{R})$-pushouts that are preserved by ϕ,

(6) $\mathbb{S}ig_n$ is closed to $(\mathcal{L}, \mathcal{R})$-pushouts, i.e. for all pushouts of signature morphisms $\{\Sigma_2 \stackrel{\chi}{\leftarrow} \Sigma_0 \stackrel{\varphi}{\rightarrow} \Sigma_1, \Sigma_1 \stackrel{\chi_1}{\rightarrow} \Sigma \stackrel{\varphi_2}{\leftarrow} \Sigma_2\}$ such that $\varphi \in \mathcal{L} \subseteq \mathbb{S}ig_n$ and $\chi \in \mathcal{R}$ we have $\varphi_2 \in \mathbb{S}ig_n$,

(7) the inclusion functor $\mathbb{S}ig^{sc} \hookrightarrow \mathbb{S}ig^{pres}$ lifts $(\mathcal{L}^{pres}, \mathcal{R}^{pres})$-pushouts.

The institution $\mathcal{I}^{sc}$ has

(i) $(\mathcal{L}^{pres}, \mathcal{R}^{pres})$-CRI whenever $\mathcal{I}'$ has $(\phi(\mathcal{L}), \phi(\mathcal{R}))$-CRI, and

(ii) $(\mathcal{L}^{pres}, \mathcal{R})$-CI whenever $\mathcal{I}'$ has $(\phi(\mathcal{L}), \phi(\mathcal{R}))$-CI.

Proof. Let $\{(\Sigma_2, E_2) \stackrel{\chi}{\leftarrow} (\Sigma_0, E_0) \stackrel{\varphi}{\rightarrow} (\Sigma_1, E_1)\}$ be a span of presentation morphisms in $\mathbb{S}ig^{sc}$ such that $\chi \in \mathcal{L}$ and $\varphi \in \mathcal{R}$. By condition (5), there exists a $(\mathcal{L}, \mathcal{R})$-pushout $\{\Sigma_2 \stackrel{\chi}{\leftarrow} \Sigma_0 \stackrel{\varphi}{\rightarrow} \Sigma_1, \Sigma_1 \stackrel{\chi_1}{\rightarrow} \Sigma \stackrel{\varphi_2}{\leftarrow} \Sigma_2\}$ in $\mathbb{S}ig$. By Proposition 2.3, the following square of presentation morphisms is a pushout in $\mathbb{S}ig^{pres}$.

$$\begin{array}{ccc}
(\Sigma_2, E_2) & \xrightarrow{\varphi_2} & (\Sigma, \chi_1(E_1) \cup \varphi_2(E_2)) \\
\chi \uparrow & & \uparrow \chi_1 \\
(\Sigma_0, E_0) & \xrightarrow[\varphi]{} & (\Sigma_1, E_1)
\end{array}$$

By (7), the above pushout of presentation morphisms is also a pushout in $\mathbb{S}ig^{sc}$. Without loss of generality we assume that $(\mathcal{L}^{pres}, \mathcal{R}^{pres})$-pushouts in $\mathbb{S}ig^{sc}$ are of the form $\{(\Sigma_2, E_2) \xleftarrow{\chi} (\Sigma_0, E_0) \xrightarrow{\varphi} (\Sigma_1, E_1), (\Sigma_1, E_1) \xrightarrow{\chi_1} (\Sigma, \chi_1(E_1) \cup \varphi_2(E_2)) \xleftarrow{\varphi_2} (\Sigma_2, E_2)\}$, where $\{\Sigma_2 \xleftarrow{\chi} \Sigma_0 \xrightarrow{\varphi} \Sigma_1, \Sigma_1 \xrightarrow{\chi_1} \Sigma \xleftarrow{\varphi_2} \Sigma_2\}$ is a $(\mathcal{L}, \mathcal{R})$-pushout in $\mathbb{S}ig$.

We prove that $\mathcal{I}^{sc}$ has $(\mathcal{L}^{pres}, \mathcal{R}^{pres})$-CRI whenever $\mathcal{I}'$ has $(\phi(\mathcal{L}), \phi(\mathcal{R}))$-CRI. Consider the following $(\mathcal{L}^{pres}, \mathcal{R}^{pres})$-pushout in $\mathbb{S}ig^{sc}$

$$\{(\Sigma_2, E_2) \xleftarrow{\chi} (\Sigma_0, E_0) \xrightarrow{\varphi} (\Sigma_1, E_1), (\Sigma_1, E_1) \xrightarrow{\chi_1} (\Sigma, E) \xleftarrow{\varphi_2} (\Sigma_2, E_2)\}$$

such that $E = \chi_1(E_1) \cup \varphi_2(E_2)$. Let $\Gamma_1 \subseteq \mathbb{S}en(\Sigma_1)$ and $\Gamma_2, \Delta_2 \subseteq \mathbb{S}en(\Sigma_2)$ such that

$$\chi_1(\Gamma_1) \cup \varphi_2(\Delta_2) \models_{(\Sigma,E)} \varphi_2(\Gamma_2)$$

It follows that $\chi_1(\Gamma_1 \cup E_1) \cup \varphi_2(\Delta_2 \cup E_2) \models_\Sigma \varphi_2(\Gamma_2)$. By (1), there exists $\Gamma_n \subseteq \mathbb{S}en_n(\Sigma_2)$ such that $\Gamma_2 \models\!\mid \Gamma_n$, which implies $\chi_1(\Gamma_1 \cup E_1) \cup \varphi_2(\Delta_2 \cup E_2) \models_\Sigma \varphi_2(\Gamma_n)$. We define $\Gamma'_1 = \alpha^{-1}_{\Sigma_1}(\Gamma_1 \cup E_1)$, $\Delta'_2 = \alpha^{-1}_{\Sigma_2}(\Delta_2 \cup E_2)$ and $\Gamma'_n = \alpha^{-1}_{\Sigma_2}(\Gamma_n)$. By (3), $\alpha_{\Sigma_1}(\Gamma'_1) = \Gamma_1 \cup E_1$, $\alpha_{\Sigma_2}(\Delta'_2) = \Delta_2 \cup E_2$ and $\alpha_{\Sigma_2}(\Gamma'_n) = \Gamma_n$, which implies $\alpha_\Sigma(\phi(\chi_1)(\Gamma'_1) \cup \phi(\varphi_2)(\Delta'_2)) = \chi_1(\Gamma_1 \cup E_1) \cup \varphi_2(\Delta_2 \cup E_2)$ and $\alpha_\Sigma(\phi(\varphi_2)(\Gamma'_n)) = \varphi_2(\Gamma_n)$. It follows that

$$\alpha_\Sigma(\phi(\chi_1)(\Gamma'_1) \cup \phi(\varphi_2)(\Delta'_2)) \models_\Sigma \alpha_\Sigma(\phi(\varphi_2)(\Gamma'_n)) \tag{4a}$$

We have $(\Sigma, \chi_1(E_1) \cup \varphi_2(E_2)) \in |\mathbb{S}ig^{sc}|$ and by (2), $(\Sigma, \chi_1(\Gamma_1 \cup E_1) \cup \varphi_2(\Delta_2 \cup E_2)) \in |\mathbb{S}ig^{sc}|$. It follows that

$$(\Sigma, \alpha_\Sigma(\phi(\chi_1)(\Gamma'_1) \cup \phi(\varphi_2)(\Delta'_2))) \in |\mathbb{S}ig^{sc}| \tag{4b}$$

We have $\varphi \in \mathcal{L}$, and in particular, $\varphi \in \mathbb{S}ig_n$. By (6), $\varphi_2 \in \mathbb{S}ig_n$, which implies $\varphi_2(\Gamma_n) \in \mathbb{S}en_n(\Sigma)$. It follows that

$$\alpha_\Sigma(\phi(\varphi_2)(\Gamma'_n)) \in \mathbb{S}en_n(\Sigma) \tag{4c}$$

By (4) applied to (4a), (4b) and (4c) we obtain

$$\phi(\chi_1)(\Gamma'_1) \cup \phi(\varphi_2)(\Delta'_2) \models'_{\phi(\Sigma)} \phi(\varphi_2)(\Gamma'_n)$$

Since $\mathcal{I}'$ has $(\phi(L), \phi(\mathcal{R}))$-CRI, there exists $\Gamma'_0 \subseteq \mathbb{S}en'(\phi(\Sigma_0))$ such that

$$\Gamma'_1 \models'_{\phi(\Sigma_1)} \phi(\varphi)(\Gamma'_0) \text{ and } \Delta'_2 \cup \phi(\chi)(\Gamma'_0) \models'_{\phi(\Sigma_2)} \Gamma'_n$$

We define $\Gamma_0 = \alpha_{\Sigma_0}(\Gamma'_0)$, and by the satisfaction condition for institution morphisms, $\Gamma_1 \cup E_1 \models_{\Sigma_1} \varphi(\Gamma_0)$ and $\Delta_2 \cup E_2 \cup \chi(\Gamma_0) \models_{\Sigma_2} \Gamma_n$. It follows that $\Gamma_1 \models_{(\Sigma_1,E_1)} \varphi(\Gamma_0)$ and $\Delta_2 \cup \chi(\Gamma_0) \models_{(\Sigma_2,E_2)} \Gamma_n$. Since $\Gamma_n \mathrel{\models\!\!\!\mid} \Gamma_2$, we get

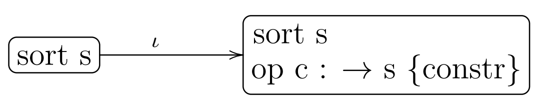
$$\Gamma_1 \models_{(\Sigma_1,E_1)} \varphi(\Gamma_0) \text{ and } \Delta_2 \cup \chi(\Gamma_0) \models_{(\Sigma_2,E_2)} \Gamma_2$$

The case when $\mathcal{I}^{sc}$ has $(\mathcal{L}^{pres}, \mathcal{R})$-CI under the assumption that $\mathcal{I}'$ has $(\phi(\mathcal{L}), \phi(\mathcal{R}))$-CI is similar. The only significant difference is that we consider $(\mathcal{L}^{pres}, \mathcal{R})$-pushouts of the form $\{(\Sigma_2, \chi(E_0)) \xleftarrow{\chi} (\Sigma_0, E_0) \xrightarrow{\varphi} (\Sigma_1, E_1), (\Sigma_1, E_1) \xrightarrow{\chi_1} (\Sigma, \chi_1(E_1)) \xleftarrow{\varphi_2} (\Sigma_2, \chi(E_0))\}$, where $\{\Sigma_2 \xleftarrow{\chi} \Sigma_0 \xrightarrow{\varphi} \Sigma_1, \Sigma_1 \xrightarrow{\chi_1} \Sigma \xleftarrow{\varphi_2} \Sigma_2\}$ is a $(\mathcal{L}, \mathcal{R})$-pushout in $\mathbb{S}ig$, for proving interpolation. □

In concrete examples, $\mathcal{I}$ is a constructor-based institution, such as **CHCL**. (ϕ, α, β) is the "forgetful" institution morphism defined from the constructor-based institution to its base institution, such as $\Delta_{\mathbf{CHCL}}$. $\mathbb{S}ig_n$ is the broad subcategory of signature morphisms that encapsulate the constructors. $\mathbb{S}en_n$ is the sentence sub-functor that maps each signature to the set of sentences free of quantification over variables of constrained sorts. $\mathbb{S}ig^{sc}$ is the full subcategory of sufficient complete presentations.

The following example shows that if we do not restrict $\mathbb{S}ig_n$ such that its signature morphisms encapsulate constructors then $\mathbb{S}en_n : \mathbb{S}ig_n \to \mathbb{S}et$ is not a functor, and our results may not hold.

Example 4.1. Consider the following example of signature extension with the constructor $c :\to s$



If $\Sigma = dom(\iota)$, and $\Sigma' = codom(\iota)$, and $\mathbb{S}en_n$ is the sub-functor that maps each signature to the set of sentences free of quantification over variables of constrained sorts then $(\forall x, y : s)x = y \in \mathbb{S}en_n(\Sigma)$ but $\iota((\forall x, y : s)x = y) \notin \mathbb{S}en_n(\Sigma')$.

In the following we justify the applicability of Theorem 4.5 by instantiating it to **CHCL** and **CHPOA**.

Corollary 4.6. *We have the following interpolation results:*

- $\mathbf{CHCL}^{sc}$ *has* $((iee*)^{pres}, (* * **)^{pres})$*-CRI,*

- $\mathbf{CHCL}^{sc}$ *has* $((**e*)^{pres}, (iiei))$*-CI,*
- $\mathbf{CHPOA}^{sc}$ *has* $((iee)^{pres}, (***))^{pres}$*-CRI, and*
- $\mathbf{CHPOA}^{sc}$ *has* $((**e)^{pres}, (iie))$*-CI.*

Proof. We set the parameters of Theorem 4.5. $\mathcal{I} = \mathbf{CHCL}$, $\mathcal{I}' = \mathbf{HCL}$, and $(\phi, \alpha, \beta) = \Delta_{\mathbf{CHCL}}$. $\mathbb{S}ig_n$ is the broad subcategory of signature morphisms consisting of $(**e*)$-morphisms. $\mathbb{S}en_n$ is the sub-functor that maps each signature to the set of sentences free of quantification over variables of constrained sorts.

We prove that for all $(\forall X)\rho \in \mathbb{S}en^{\mathbf{CHCL}}(\Sigma)$, where $\Sigma = (S, F, F^c, P)$ and X is a finite set of variables of constrained sorts, we have $(\forall X)\rho \models\!\mid \Gamma_{(\forall X)\rho}$, where $\Gamma_{(\forall X)\rho}$ is defined as follows:

$$\Gamma_{(\forall X)\rho} = \{(\forall Y)\theta(\rho) \mid \theta : X \to T_{(S,F^c)}(Y),\ Y \text{ is a finite set of loose variables}\}$$

Since $\Gamma_{(\forall X)\rho}$ consists of sentences obtained from $(\forall X)\rho$ by substituting terms for variables, we have $(\forall X)\rho \models \Gamma_{(\forall X)\rho}$. For the converse implication, we assume that $M \models_\Sigma \Gamma_{(\forall X)\rho}$. Let $f : X \to M$ be a valuation. There exists a set Y of constants of loose sorts and a valuation $g : Y \to M$ such that $g^\# : T_{(S,F^c)}(Y) \to M\!\upharpoonright_{(S,F^c)}$ is surjective, where $g^\#$ is the unique extension of g to a (S, F^c)-morphism. Since $g^\#$ is surjective, there exists $\theta : X \to T_{(S,F^c)}(Y)$ such that $\theta; g^\# = f$. Because X is finite, the set of variables $Y_f \subseteq Y$ occurring in $\theta(X)$ is finite too. We define $g_f : Y_f \to M$ as the restriction of g to Y_f, and $\theta_f : X \to T_{(S,F^c)}(Y_f)$ as the co-restriction of θ to $T_{(S,F^c)}(Y_f)$. We have $(\forall Y_f)\theta_f(\rho) \in \Gamma_{(\forall X)\rho}$ and $M \models_\Sigma \Gamma_{(\forall X)\rho}$, which implies $(M, g_f) \models_{\Sigma(Y_f)} \theta_f(\rho)$. By the satisfaction condition we get $(M, (\theta_f; g_f^\#)) \models_{\Sigma(X)} \rho$, where $g_f^\# : T_{(S,F^c)}(Y_f) \to M\!\upharpoonright_{(S,F^c)}$ is the unique extension of g_f to a (S, F^c)-morphism. Since $\theta_f; g_f^\# = f$ we get that $(M, f) \models_{\Sigma(X)} \rho$.

$\mathbb{S}ig^{sc}$ is the full subcategory of sufficient complete presentations. The condition (2) of Theorem 4.5 holds because the sufficient complete property of a presentation is not be changed by adding new sentences. For all **CHCL** signatures (S, F, F^c, P), $\alpha_{(S,F,F^c,P)}$ is the identity, and in particular, a surjection. By Corollary 4.4, the condition (4) of Theorem 4.5 is satisfied.

By Proposition 2.1, **CHCL** has $((iee*), (****))$-pushouts which are mapped to $((ie*), (***))$-pushouts by ϕ. By Proposition 2.1 again, $(**e*)$-morphisms are closed to $((iee*), (****))$-pushouts. By Proposition 3.2, the inclusion functor $\mathbb{S}ig^{sc} \hookrightarrow \mathbb{S}ig^{pres}$ lifts $((iee*)^{pres}, (****)^{pres})$-pushouts. By

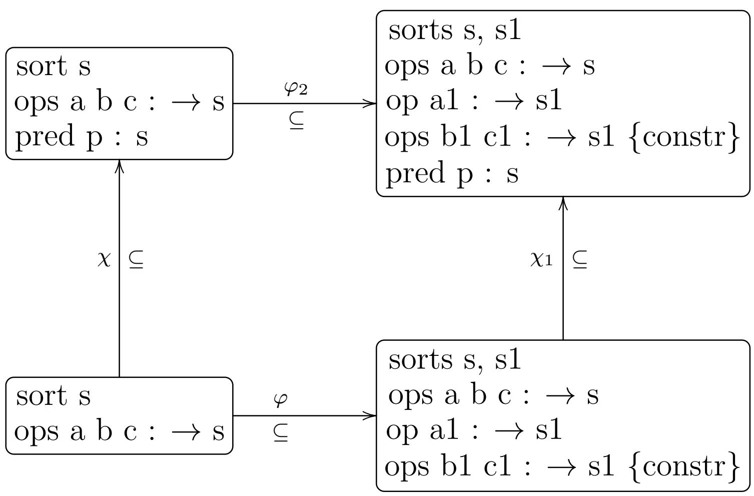
Remark 4.1, **HCL** has $((ie*), (***))$-CRI, and by Theorem 4.5, $\mathbf{CHCL}^{sc}$ has $((iee*)^{pres}, (****)^{pres})$-CRI.

By Proposition 2.2, **CHCL** has $((**e*), (iiei))$-pushouts which are mapped to $((***), (iii))$-pushouts by ϕ. By Proposition 2.2 again, $(**e*)$-morphisms are closed to $((**e*), (iiei))$-pushouts. By Proposition 3.3, the inclusion functor $\mathbb{S}ig^{sc} \hookrightarrow \mathbb{S}ig^{pres}$ lifts $((**e*)^{pres}, iiei)$-pushouts. By Remark 4.1, **HCL** has $((***), (iii))$-CI. By Theorem 4.5, $\mathbf{CHCL}^{sc}$ has $((**e*)^{pres}, (iiei))$-CI.

By defining a "forgetful" institution morphism $\Delta_{\mathbf{CHPOA}} : \mathbf{CHPOA} \to \mathbf{HPOA}$, one can replicate the above arguments for **CHPOA** too. □

The following example shows that without sufficient completeness assumption, an interpolant may not be found.

Example 4.2. Consider the following pushout of **CHCL** signature morphisms:



Let $\Sigma_0 = dom(\varphi) = dom(\chi)$, $\Sigma_1 = codom(\varphi)$, $\Sigma_2 = codom(\chi)$ and $\Sigma = codom(\chi_1) = codom(\varphi_2)$. Note that $\varphi : \Sigma_0 \to \Sigma_1$ is a $(iee*)$-morphism as no "new" (ordinary) operation and constructor symbols are introduced for "old" sorts.

Take $\Gamma_1 = \{(a_1 = b_1 \Rightarrow a = b), (a_1 = c_1 \Rightarrow a = c)\}$, $\Delta_2 = \{p(b), p(c)\}$ and $\Gamma_2 = \{p(a)\}$. The presentation (Σ_1, Γ_1) is not sufficient complete because there are no equations to define the value of a_1. The presentation (Σ_2, Δ_2) is sufficient complete because the signature Σ_2 has no constructors. Since all Σ-models M have the carrier sets for the sort s_1 consisting of interpretations of b_1 and c_1, we have $M \models_\Sigma a_1 = b_1$ or $M \models_\Sigma a_1 = c_1$. If $M \models_\Sigma \Gamma_1$ then

$M \models_\Sigma a = b$ or $M \models_\Sigma a = c$. If $M \models_\Sigma \Gamma_1 \cup \Delta_2$ we get $M \models_\Sigma p(a)$. Since M was arbitrarily chosen, $\Gamma_1 \cup \Delta_2 \models_\Sigma \Gamma_2$. Because $\Gamma_1 \not\models_{\Sigma_1} a = b$ and $\Gamma_1 \not\models_{\Sigma_1} a = c$, there is no $\Gamma_0 \subseteq \mathbb{S}en^{\mathbf{CHCL}}(\Sigma_0)$ such that $\Gamma_1 \models_{\Sigma_1} \Gamma_0$ and $\Gamma_0 \cup \Delta_2 \models_{\Sigma_2} \Gamma_2$. Note that in **CFOL** the interpolant is $a = b \vee a = c$.

5. Conclusions

We have conducted an institution-independent study of the interpolation properties in logics with Horn sentences and constructors in the signatures. We have showed that the interpolation property holds without the need of infinitary sentences. According to Example 4.2, sufficient completeness assumption is needed for CRI. One important application of CRI in algebraic specifications is lifting the completeness result in [23, 22] (which also depends on the sufficient completeness) at the level of specifications in the Borzyszkowski's style [6]. We have illustrated the applicability power of our method by deriving interpolation results for **CHCL** and **CHPOA**, but instances of our work are expected also for the constructor-based Horn variants of membership algebra [30], higher-order logic [8, 26] with intensional Henkin semantics, and partial algebra [36, 7]. We believe that these results can be naturally extended to institutions with sort generation constraints, such as the CASL institution.

In the future we are planning to investigate interpolation for the constructor-based variants of first-order institutions, such as the institution of first-order logic.

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