Foundations of Logic Programming in Hybridised Logics

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Abstract. The present paper sets the foundation of logic programming in hybridised logics. The basic logic programming semantic concepts such as query and solutions, and the fundamental results such as the existence of initial models and Herbrand's theorem, are developed over a very general hybrid logical system. We employ the hybridisation process proposed by Diaconescu over an arbitrary logical system captured as an institution to define the logic programming framework.

1 Introduction

Hybrid logics [1] are a brand of modal logics that allows direct reference to the possible worlds/states in a simple and very natural way through the so-called nominals. This feature has several advantages from the point of view of logic and formal specification. For example, it becomes considerably simpler to define proof systems in hybrid logics [2], and one can prove results of a generality that is not available in non-hybrid modal logic. In specifications of dynamic systems the possibility of explicit reference to specific states of the model is an essential feature.

The hybridisation of a logic is the process of developing the features of hybrid logic on top of the base logic both at the syntactic level (i.e. modalities, nominals, etc.) and semantics (i.e. possible worlds). By a hybridised institution (or hybrid institution) we mean the result of this process when logics are treated abstractly as institutions [7]. The hybridisation development in [13, 6] abstracts away the details, both at the syntactic and semantic levels, that are independent of the very essence of the hybrid logic idea. One great advantage of this approach is the clarity of the theoretical developments that are not hindered by the irrelevant details of the concrete logics. Another practical benefit is the applicability of the results to a wide variety of concrete instances.

In this paper we investigate a series of model-theoretic properties of hybrid logics in an institution-independent setting such as basic set of sentences [3], substitution [4] and reachable model [11, 10]. While the definition of basic set of sentences is a straightforward extension from a base institution to its hybrid counterpart, the notion of substitution needs much consideration. Establishing an appropriate concept of substitution is the most difficult part of the whole enterprise of constructing an initial model of a given hybrid theory and proving a variant of Herbrand's theorem. The notion of substitution is closely related to quantification. Our abstract results are applicable to hybrid logical systems where the variables may be interpreted differently across distinct worlds, which amounts to the world-line semantics of [14]. Our paper does not cover the rigid quantification of [2] when the possible worlds share the same domain and the variables are interpreted the same in all worlds.

Initial semantics [8] is closely related to good computational properties of logical systems and it plays a crucial role for the semantics of abstract data types. For example, initiality supports the execution of specification languages through rewriting, thus integrating efficiently formal verification of software systems into modelling. The initial semantics methodology has spread much beyond its original context, that of traditional equational specification, to a variety of modern and more sophisticated logical contexts. Moreover, initial semantics plays a foundational role in logic programming. For example, in [12], initial models are known as "least Herbrand models". Our approach to initiality is layered and is intimately linked to the structure of sentences, in the style of [9]. The existence of initial models of sets of atomic sentences is assumed in abstract setting but is developed in concrete examples; then the initiality property is shown to be closed under certain sentence building operators.

The second main contribution of the paper is a variant of Herbrand's theorem for hybrid institutions, which reduces the satisfiability of a query with respect to a hybrid theory to the search of a suitable substitution. The logic programming paradigm [12], in its classical form, can be described as follows: Given a program (Σ, Γ) (that consists of a signature Σ and a set of Horn clauses Γ) and a query $(\exists Y)\rho$ (that consists of an existentially quantified conjunction of atoms) find a solution θ, i.e. values for the variables Y such that the corresponding instance $\theta(\rho)$ of ρ is satisfied by (Σ, Γ). The essence of this paradigm is however independent of any logical system of choice. The basic logic programming concepts, query, solutions, and the the fundamental results, such as Herbrand's theorem, are developed over an arbitrary institution (satisfying certain hypotheses) in [4] by employing institution-independent concepts of variables, substitution, quantifiers and atomic formulas. Our work sets foundation for a uniform development of logic programming over a large variety of hybrid logics as we employ the hybridisation process over an arbitrary institution [13, 6] to prove the desired results.

The institution-independent status of the present study makes the results applicable to a multitude of concrete hybrid logics including those obtained from hybridisation of non-conventional logics used in computer science.

The paper is organised as follows: in Section 2 we recall the definition of institution and the related notions such as substitution, reachable model and basic set of sentences. In Section 3 we recall the institution-independent process of hybridisation of a logical system and we lift the notions discussed in the previous section to the hybrid setting. Section 4 is dedicated to the development of the layered initiality result. In Section 5 we present an institution-independent version of Herbrand's theorem and its applications to concrete hybrid logics. Section 6 concludes the paper and discusses the future work.

2 Institutions

The concept of institution formalises the intuitive notion of logical system, and has been defined by Goguen and Burstall in the seminal paper [7].

Definition 1. *An* institution $\mathtt{I} = (\mathbb{S}ig^{\mathtt{I}}, \mathbb{S}en^{\mathtt{I}}, \mathbb{M}od^{\mathtt{I}}, \models^{\mathtt{I}})$ *consists of*

(1) a category $\mathbb{S}ig^{\mathtt{I}}$*, whose objects are called* signatures*,*
(2) a functor $\mathbb{S}en^{\mathtt{I}} : \mathbb{S}ig^{\mathtt{I}} \to \mathbb{S}et$*, providing for each signature* Σ *a set whose elements are called* (Σ-)sentences*,*
(3) a functor $\mathbb{M}od^{\mathtt{I}} : (\mathbb{S}ig^{\mathtt{I}})^{op} \to \mathbb{CAT}$*, providing for each signature* Σ *a category whose objects are called* (Σ-)models *and whose arrows are called* (Σ-)morphisms*,*
(4) a relation $\models^{\mathtt{I}}_{\Sigma} \subseteq |\mathbb{M}od^{\mathtt{I}}(\Sigma)| \times \mathbb{S}en^{\mathtt{I}}(\Sigma)$ *for each signature* $\Sigma \in |\mathbb{S}ig^{\mathtt{I}}|$*, called* ($\Sigma$-)satisfaction*, such that for each morphism* $\varphi : \Sigma \to \Sigma'$ *in* $\mathbb{S}ig^{\mathtt{I}}$*, the following* satisfaction condition *holds:*

$$M' \models^{\mathtt{I}}_{\Sigma'} \mathbb{S}en^{\mathtt{I}}(\varphi)(e) \text{ iff } \mathbb{M}od^{\mathtt{I}}(\varphi)(M') \models^{\mathtt{I}}_{\Sigma} e$$

for all $M' \in |\mathbb{M}od^{\mathtt{I}}(\Sigma')|$ *and* $e \in \mathbb{S}en^{\mathtt{I}}(\Sigma)$*.*

When there is no danger of confusion, we omit the superscript from the notations of the institution components; for example $\mathbb{S}ig^{\mathtt{I}}$ may be simply denoted by $\mathbb{S}ig$. We denote the *reduct* functor $\mathbb{M}od(\varphi)$ by $\_ \upharpoonright_{\varphi}$ and the sentence translation $\mathbb{S}en(\varphi)$ by $\varphi(\_)$. When $M = M' \upharpoonright_{\varphi}$ we say that M is the φ-*reduct* of M' and M' is a φ-*expansion* of M. We say that φ is *conservative* if each Σ-model has a φ-expansion. Given a signature Σ and two sets of Σ-sentences E_1 and E_2, we write $E_1 \models\!\mid E_2$ whenever $E_1 \models E_2$ and $E_2 \models E_1$.

The literature shows myriads of logical systems from computing or mathematical logic captured as institutions (see, for example, [5]).

*Example 1 (First-Order Logic (***FOL***) [7]).* The signatures are triplets (S, F, P), where S is the set of sorts, $F = \{F_{\underline{ar}\to s}\}_{(\underline{ar},s)\in S^*\times S}$ is the ($S^*\times S$ -indexed) set of operation symbols, and $P = \{P_{\underline{ar}}\}_{\underline{ar}\in S^*}$ is the (S^*-indexed) set of relation symbols. If $\underline{ar} = \epsilon$, where ϵ denotes the empty arity, an element of $F_{\underline{ar}\to s}$ is called a *constant symbol*, or a *constant*. By a slight notational abuse, we let F and P also denote $\bigcup_{(\underline{ar},s)\in S^*\times S} F_{\underline{ar}\to s}$ and $\bigcup_{\underline{ar}\in S^*} P_{\underline{ar}}$, respectively. A signature morphism between (S, F, P) and (S', F', P') is a triplet $\varphi = (\varphi^{st}, \varphi^{op}, \varphi^{rl})$, where $\varphi^{st} : S \to S'$, $\varphi^{op} : F \to F'$, $\varphi^{rl} : P \to P'$ such that for all $(\underline{ar}, s) \in S^* \times S$ we have $\varphi^{op}(F_{\underline{ar}\to s}) \subseteq F'_{\varphi^{st}(\underline{ar})\to\varphi^{st}(s)}$, and for all $\underline{ar} \in S^*$ we have $\varphi^{rl}(P_{\underline{ar}}) \subseteq P'_{\varphi^{st}(\underline{ar})}$. When there is no danger of confusion, we may let φ denote each of φ^{st}, φ^{op}, φ^{rl}. Given a signature $\Sigma = (S, F, P)$, a Σ-model is a triplet $M = (\{s_M\}_{s\in S}, \{\sigma_M\}_{(\underline{ar},s)\in S^*\times S, \sigma\in F_{\underline{ar}\to s}}, \{\pi_M\}_{\underline{ar}\in S^*, \pi\in P_{\underline{ar}}})$ interpreting each sort s as a set s_M, each operation symbol $\sigma \in F_{\underline{ar}\to s}$ as a function $\sigma_M : \underline{ar}_M \to$

s_M (where $\underline{ar}_M$ stands for $(s_1)_M \times \ldots \times (s_n)_M$ if $\underline{ar} = s_1 \ldots s_n$), and each relation symbol $\pi \in P_{\underline{ar}}$ as a relation $\pi_M \subseteq \underline{ar}_M$. Morphisms between models are the usual Σ-morphisms, i.e., S-sorted functions that preserve the structure. The Σ-algebra of terms is denoted by T_Σ. The Σ-sentences are obtained from (a) equality atoms (e.g. $t_1 = t_2$, where $t_1, t_2 \in s_{T_\Sigma}$, $s \in S$) or (b) relational atoms (e.g. $\pi(t_1, \ldots, t_n)$, where $\pi \in P_{s_1 \ldots s_n}$, $t_i \in (s_i)_{T_\Sigma}$, $s_i \in S$ and $i \in \{1, \ldots, n\}$) by applying for a finite number of times Boolean connectives and quantification over finite sets of variables. Satisfaction is the usual first-order satisfaction and is defined using the natural interpretations of ground terms t as elements t_M in models M. The definitions of functors $\mathbb{S}en$ and $\mathbb{M}od$ on morphisms are the natural ones: for any signature morphism $\varphi : \Sigma \to \Sigma'$, $\mathbb{S}en(\varphi) : \mathbb{S}en(\Sigma) \to \mathbb{S}en(\Sigma')$ translates sentences symbol-wise, and $\mathbb{M}od(\varphi) : \mathbb{M}od(\Sigma') \to \mathbb{M}od(\Sigma)$ is the forgetful functor.

*Example 2 (***REL***).* The institution **REL** is the sub-institution of single-sorted first-order logic with signatures having only constants and relational symbols.

*Example 3 (Propositional Logic (***PL***)).* The institution **PL** is the fragment of **FOL** determined by signatures with empty sets of sort symbols.

Example 4 (Constrained Institutions). Let $\mathtt{I} = (\mathbb{S}ig^{\mathtt{I}}, \mathbb{S}en^{\mathtt{I}}, \mathbb{M}od^{\mathtt{I}}, \models^{\mathtt{I}})$ be an institution. A *constrained model functor* $\mathbb{M}od^{\mathtt{CI}} : (\mathbb{S}ig^{\mathtt{CI}})^{op} \to \mathbb{CAT}$ is a subfunctor of $\mathbb{M}od^{\mathtt{I}} : (\mathbb{S}ig^{\mathtt{I}})^{op} \to \mathbb{CAT}$, i.e. $\mathbb{S}ig^{\mathtt{CI}} \subseteq \mathbb{S}ig^{\mathtt{I}}$, for each $\Sigma \in |\mathbb{S}ig^{\mathtt{CI}}|$ we have $\mathbb{M}od^{\mathtt{CI}}(\Sigma) \subseteq \mathbb{M}od^{\mathtt{I}}(\Sigma)$, and for each $\Sigma \overset{\varphi}{\to} \Sigma \in \mathbb{S}ig^{\mathtt{CI}}$ the functor $\mathbb{M}od^{\mathtt{CI}}(\varphi) : \mathbb{M}od^{\mathtt{CI}}(\Sigma') \to \mathbb{M}od^{\mathtt{CI}}(\Sigma)$ is defined by $\mathbb{M}od^{\mathtt{CI}}(\varphi)(h) = \mathbb{M}od^{\mathtt{I}}(\varphi)(h)$ for all $h \in \mathbb{M}od^{\mathtt{CI}}(\Sigma')$. We say that $\mathtt{CI} = (\mathbb{S}ig^{\mathtt{CI}}, \mathbb{S}en^{\mathtt{CI}}, \mathbb{M}od^{\mathtt{CI}}, \models^{\mathtt{CI}})$ is *a constrained institution*, where (a) $\mathbb{S}en^{\mathtt{CI}} : \mathbb{S}ig^{\mathtt{CI}} \to \mathbb{S}et$ is the restriction of $\mathbb{S}en^{\mathtt{I}} : \mathbb{S}ig^{\mathtt{I}} \to \mathbb{S}et$ to $\mathbb{S}ig^{\mathtt{CI}}$, and (b) $\models^{\mathtt{CI}}_\Sigma \subseteq |\mathbb{M}od^{\mathtt{CI}}(\Sigma)| \times \mathbb{S}en^{\mathtt{I}}(\Sigma)$ is the restriction of $\models^{\mathtt{I}}_\Sigma \subseteq |\mathbb{M}od^{\mathtt{I}}(\Sigma)| \times \mathbb{S}en^{\mathtt{I}}(\Sigma)$ to $|\mathbb{M}od^{\mathtt{CI}}(\Sigma)|$ for all $\Sigma \in |\mathbb{S}ig^{\mathtt{CI}}|$.

2.1 Quantification Subcategory

Let $\mathtt{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution. A *broad subcategory* [1] $\mathcal{Q} \subseteq \mathbb{S}ig$ is called *quantification subcategory* [6] when for each $\Sigma \overset{\chi}{\to} \Sigma' \in \mathcal{Q}$ and $\Sigma \overset{\varphi}{\to} \Sigma_1 \in \mathbb{S}ig$ there is a designated pushout

$$\begin{array}{ccc} \Sigma' & \xrightarrow{\varphi[\chi]} & \Sigma'_1 \\ {\scriptstyle\chi}\uparrow & & \uparrow{\scriptstyle\chi(\varphi)} \\ \Sigma & \xrightarrow[\varphi]{} & \Sigma_1 \end{array}$$

with $\chi(\varphi) \in \mathcal{Q}$ which is a *weak amalgamation square* [2] and such that the horizontal composition of such designated pushouts is again a designated pushout, i.e. $\chi(1_\Sigma) = \chi$,

[1] A category $\mathcal{C}$ is a broad subcategory of $\mathcal{C}'$ if $\mathcal{C}$ is a subcategory of $\mathcal{C}'$ and $\mathcal{C}$ contains all objects of $\mathcal{C}'$, i.e. $|\mathcal{C}| = |\mathcal{C}'|$.

[2] For all $M' \in |\mathbb{M}od(\Sigma')|$ and $M_1 \in |Mod(\Sigma_1)|$ such that $M'\!\upharpoonright_\chi = M_1\!\upharpoonright_\varphi$ there exists $M'_1 \in |\mathbb{M}od(\Sigma'_1)|$ such that $M'_1\!\upharpoonright_{\varphi[\chi]} = M'$ and $M'_1\!\upharpoonright_{\chi(\varphi)} = M_1$.

$1_\Sigma[\chi] = 1_{\Sigma'}$, and for the following pushouts

$$\begin{array}{ccccc} \Sigma' & \xrightarrow{\varphi[\chi]} & \Sigma'_1 & \xrightarrow{\theta[\chi(\varphi)]} & \Sigma'_2 \\ {\scriptstyle\chi}\uparrow & & \uparrow{\scriptstyle\chi(\varphi)} & & \uparrow{\scriptstyle\chi(\varphi)(\theta)} \\ \Sigma & \xrightarrow[\varphi]{} & \Sigma_1 & \xrightarrow[\theta]{} & \Sigma_2 \end{array}$$

we have $\varphi[\chi];\theta[\chi(\varphi)] = (\varphi;\theta)[\chi]$ and $\chi(\varphi)(\theta) = \chi(\varphi;\theta)$.

A variable for a **FOL** signature $\Sigma = (S, F, P)$ is a triple (x, s, Σ), where x is the name of the variable and $s \in S$ is the sort of the variable. Let $\chi : \Sigma \hookrightarrow \Sigma[X]$ be a signature extension with variables from X, where $X = \{X_s\}_{s\in S}$ is a S-sorted set of variables, $\Sigma[X] = (S, F \cup X, P)$ and for all $(\underline{ar}, s) \in S^* \times S$ we have $(F \cup X)_{\underline{ar}\to s} = \begin{cases} F_{\underline{ar}\to s} & \text{if } \underline{ar} \in S^+, \\ F_{\underline{ar}\to s} \cup X_s & \text{if } \underline{ar} = \epsilon. \end{cases}$ The quantification subcategory $\mathcal{Q}^{\mathbf{FOL}}$ for **FOL** consists of signature extensions with a finite set of variables. Given a signature morphism $\varphi : \Sigma \to \Sigma_1$, where $\Sigma_1 = (S_1, F_1, P_1)$, then

- $\chi(\varphi) : \Sigma_1 \hookrightarrow \Sigma_1[X^\varphi]$, where $X^\varphi = \{(x, \varphi(s), \Sigma_1) \mid (x, s, \Sigma) \in X\}$,
- $\varphi[\chi]$ is the canonical extension of φ that maps each (x, s, Σ) to $(x, \varphi(s), \Sigma_1)$.

It is straightforward to check that $\mathcal{Q}^{\mathbf{FOL}}$ defined above is a quantification subcategory.

2.2 Substitutions

We recall the notion of substitution in institutions.

Definition 2. *[4] Let* $\mathtt{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ *be an institution and* $\Sigma \in |\mathbb{S}ig|$. *For any signature morphisms* $\chi_1 : \Sigma \to \Sigma_1$ *and* $\chi_2 : \Sigma \to \Sigma_2$, *a* Σ-substitution $\theta : \chi_1 \to \chi_2$ *consists of a pair* $(\mathbb{S}en(\theta), \mathbb{M}od(\theta))$ *, where*

- $\mathbb{S}en(\theta) : \mathbb{S}en(\Sigma_1) \to \mathbb{S}en(\Sigma_2)$ *is a function and*
- $\mathbb{M}od(\theta) : \mathbb{M}od(\Sigma_2) \to \mathbb{M}od(\Sigma_1)$ *is a functor.*

such that both of them preserve Σ, *i.e. the following diagrams commute:*

$$\begin{array}{ccc} \mathbb{S}en(\Sigma_1) & \xrightarrow{\mathbb{S}en(\theta)} & \mathbb{S}en(\Sigma_2) \\ {\scriptstyle\mathbb{S}en(\chi_1)}\uparrow & \nearrow{\scriptstyle\mathbb{S}en(\chi_2)} & \\ \mathbb{S}en(\Sigma) & & \end{array} \qquad \begin{array}{ccc} \mathbb{M}od(\Sigma_1) & \xleftarrow{\mathbb{M}od(\theta)} & \mathbb{M}od(\Sigma_2) \\ & {\scriptstyle\mathbb{M}od(\chi_1)}\searrow & \downarrow{\scriptstyle\mathbb{M}od(\chi_2)} \\ & & \mathbb{M}od(\Sigma) \end{array}$$

and such that the following satisfaction condition *holds:*

$$\mathbb{M}od(\theta)(M_2) \models \rho_1 \textit{ iff } M_2 \models \mathbb{S}en(\theta)(\rho_1)$$

for each Σ_2*-model* M_2 *and each* Σ_1*-sentence* ρ_1.

Note that a substitution $\theta : \chi_1 \to \chi_2$ is uniquely identified by its domain χ_1, codomain χ_2 and the pair $(\mathbb{S}en(\theta), \mathbb{M}od(\theta))$. We sometimes let $\_\!\upharpoonright_\theta$ denote the functor $\mathbb{M}od(\theta)$, and let θ denote the sentence translation $\mathbb{S}en(\theta)$.

*Example 5 (***FOL** *substitutions [4]).* Consider two signature extensions with constants $\chi_1 : \Sigma \hookrightarrow \Sigma[C_1]$ and $\chi_2 : \Sigma \hookrightarrow \Sigma[C_2]$, where $\Sigma = (S, F, P) \in |\mathbb{S}ig^{\mathbf{FOL}}|$, C_i is a set of constant symbols different from the the symbols in Σ. A function $\theta : C_1 \to T_\Sigma(C_2)$ represents a substitution between χ_1 and χ_2. On the syntactic side, θ can be canonically extended to a function $\mathbb{S}en(\theta) : \mathbb{S}en(\Sigma[C_1]) \to \mathbb{S}en(\Sigma[C_2])$ as follows:

– $\mathbb{S}en(\theta)(t_1 = t_2)$ is defined as $\theta^{term}(t) = \theta^{term}(t')$ for each $\Sigma[C_1]$-equation $t_1 = t_2$, where $\theta^{term} : T_\Sigma(C_1) \to T_\Sigma(C_2)$ is the unique extension of θ to a Σ-morphism.
– $\mathbb{S}en(\theta)(\pi(t_1, \ldots, t_n))$ is defined as $\pi(\theta^{term}(t_1), \ldots, \theta^{term}(t_n))$ for each $\Sigma[C_1]$-relational atom $\pi(t_1, \ldots, t_n)$.
– $\mathbb{S}en(\theta)(\bigwedge E)$ is defined as $\bigwedge \mathbb{S}en(\theta)(E)$ for each conjuction $\bigwedge E$ of $\Sigma[C_1]$-sentences, and similarly for the case of any other Boolean connectives.
– $\mathbb{S}en(\theta)((\forall X)\rho)$ is defined as $(\forall X^\theta)\mathbb{S}en(\theta')(\rho)$ for each $\Sigma[C_1]$-sentence $(\forall X)\rho$, where $X^\theta = \{(x, s, \Sigma[C_2]) \mid (x, s, \Sigma[C_1]) \in X\}$ and the substitution $\theta' : C_1 \cup X \to T_\Sigma(C_2 \cup X^\theta)$ extends θ by mapping each variable $(x, s, \Sigma[C_1]) \in X$ to $(x, s, \Sigma[C_2]) \in X^\theta$.

On the semantics side, θ determines a functor $\mathbb{M}od(\theta) : \mathbb{M}od(\Sigma[C_2]) \to \mathbb{M}od(\Sigma[C_1])$ such that for all $\Sigma[C_2]$-models M we have

– $\mathbb{M}od(\theta)(M)_x = M_x$, for each sort $x \in S$, or operation symbol $x \in F$, or relation symbol $x \in P$, and
– $\mathbb{M}od(\theta)(M)_x = M_{\theta(x)}$ for each $x \in C_1$.

Category of substitutions. Let $\mathtt{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution and $\Sigma \in |\mathbb{S}ig|$ a signature. Σ-substitutions form a category $\mathbb{S}ubst^{\mathtt{I}}(\Sigma)$, where the objects are signature morphisms $\Sigma \xrightarrow{\chi} \Sigma' \in |\Sigma/\mathbb{S}ig|$, and the arrows are substitutions $\theta : \chi_1 \to \chi_2$ as described in Definition 2. For any substitutions $\theta : \chi_1 \to \chi_2$ and $\theta' : \chi_2 \to \chi_3$ the composition $\theta; \theta'$ consists of the pair $(\mathbb{S}en(\theta; \theta'), \mathbb{M}od(\theta; \theta'))$, where $\mathbb{S}en(\theta; \theta') = \mathbb{S}en(\theta); \mathbb{S}en(\theta')$ and $\mathbb{M}od(\theta; \theta') = \mathbb{M}od(\theta'); \mathbb{M}od(\theta)$.

Given a signature morphism $\varphi : \Sigma_0 \to \Sigma$ there exists a reduct functor $\mathbb{S}ubst^{\mathtt{I}}(\varphi) : \mathbb{S}ubst^{\mathtt{I}}(\Sigma) \to \mathbb{S}ubst^{\mathtt{I}}(\Sigma_0)$ that maps any Σ-substitution $\theta : \chi_1 \to \chi_2$ to the Σ_0-substitution $\mathbb{S}ubst(\varphi)(\theta) : \varphi; \chi_1 \to \varphi; \chi_2$ such that $\mathbb{S}en(\mathbb{S}ubst^{\mathtt{I}}(\varphi)(\theta)) = \mathbb{S}en(\theta)$ and $\mathbb{M}od(\mathbb{S}ubst^{\mathtt{I}}(\varphi)(\theta)) = \mathbb{M}od(\theta)$. It follows that $\mathbb{S}ubst^{\mathtt{I}} : \mathbb{S}ig^{op} \to \mathbb{CAT}$ is a functor. In applications not all substitutions are of interest, and it is often assumed a substitution sub-functor $\mathbb{S}ub^{\mathtt{I}} : \mathcal{D}^{op} \to \mathbb{CAT}$ of $\mathbb{S}ubst^{\mathtt{I}} : \mathbb{S}ig^{op} \to \mathbb{CAT}$ to work with, where $\mathcal{D} \subseteq \mathbb{S}ig$ is a subcategory of signature morphisms. When there is no danger of confusion we may drop the superscript $\mathtt{I}$ from the notations; for example $\mathbb{S}ub^{\mathtt{I}}$ may be simply denoted by $\mathbb{S}ub$.

*Example 6 (***FOL** *substitution functor).* Given a signature $\Sigma \in |\mathbb{S}ig^{\mathbf{FOL}}|$, only Σ-substitutions represented by functions $\theta : C_1 \to T_\Sigma(C_2)$ are relevant for the present study, where C_1 and C_2 are finite sets of new constants for Σ. Let $\mathbb{S}ub^{\mathbf{FOL}} : (\mathcal{D}^{\mathbf{FOL}})^{op} \to \mathbb{CAT}$ denote the substitution functor which maps each signature Σ to the subcategory of Σ-substitutions represented by functions of the form $\theta : C_1 \to T_\Sigma(C_2)$ as above.

Example 7. (**PL** substitution functor) Let $\mathcal{D}^{\mathbf{PL}}$ be the subcategory of **PL** signature morphisms consisting of identities, and $\mathbb{S}ub^{\mathbf{PL}} : (\mathcal{D}^{\mathbf{PL}})^{op} \to \mathbb{CAT}$ the trivial substitution functor consisting also of identities.

2.3 Reachable Models

This subsection is devoted to the institution-independent characterisation of the models that consist of interpretations of terms.

Definition 3. *Let $\mathtt{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution, $\mathcal{D} \subseteq \mathbb{S}ig$ a broad subcategory of signature morphisms, and $\mathbb{S}ub : \mathcal{D}^{op} \to \mathbb{CAT}$ a substitution functor. A model $M \in |\mathbb{M}od(\Sigma)|$, where $\Sigma \in |\mathbb{S}ig|$, is* $\mathbb{S}ub$-reachable *if for every signature morphism $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$ and each χ-expansion M' of M there exists a substitution $\theta : \chi \to 1_\Sigma \in \mathbb{S}ub(\Sigma)$ such that $M\upharpoonright_\theta = M'$.*

This notion of reachable model is the parametrisation of the one in [10] with substitutions.

Proposition 1. *In* **FOL***, a model is $\mathbb{S}ub^{\mathbf{FOL}}$-reachable iff its elements consist of interpretations of terms.*

The proof of Proposition 1 is a slight generalisation of the one in [10]. Note that in **PL**, all models are $\mathbb{S}ub^{\mathbf{PL}}$-reachable.

2.4 Basic Sentences

A set of sentences $B \subseteq Sen(\Sigma)$ is *basic* [3] if there exists a Σ-model M^B such that, for all Σ-models M, $M \models B$ iff there exists a morphism $M^B \to M$. We say that M^B is a *basic model* of B. If in addition the morphism $M^B \to M$ is unique then the set B is called *epi basic*; in this case, M^B is the initial model of B.

Lemma 1. *Any set of atoms in* **FOL** *is epi basic and the corresponding basic models consist of interpretations of terms, i.e. are $\mathbb{S}ub^{\mathbf{FOL}}$-reachable.*

Proof. Let B be a set of atomic (S, F, P)-sentences in **FOL**. The basic model M^B is the initial model of B and it is constructed as follows: on the quotient $T_{(S,F)}/_{\equiv_B}$ of the term model $T_{(S,F)}$ by the congruence generated by the equational atoms of B, we interpret each relation symbol $\pi \in P$ by $\pi_{M^B} = \{(\widehat{t_1}, \ldots, \widehat{t_n}) \mid \pi(t_1, \ldots, t_n) \in B\}$, where $\widehat{t}$ is the congruence class of t for all terms $t \in T_{(S,F)}$. $\square$

The proof of Lemma 1 is well known, and it can be found, for example, in [3] or [5], but since it constitutes the foundation of the initiality property, we include it for the convenience of the reader. Since **PL** is obtained from **FOL** by restricting the category of signatures, every set of **PL** atoms is epi basic.

3 Hybrid Institutions

We recall the institution-independent process of hybridisation that has been introduced in [13, 6]. Consider an institution $\mathtt{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ with a quantification subcategory $\mathcal{Q} \subseteq \mathbb{S}ig$.

The category of HI signatures. The category of hybrid signatures of $\mathbb{S}ig$ is defined as the following cartesian product of categories: $\mathbb{S}ig^{\mathtt{HI}} = \mathbb{S}ig^{\mathtt{I}} \times \mathbb{S}ig^{\mathbf{REL}}$. The **REL** signatures are denoted by (Nom, Λ), where Nom is a set of constants called nominals and Λ is a set of relational symbols called modalities; Λ_n stands for the set of modalities of arity n. Hybrid signatures morphisms $\varphi = (\varphi_{\mathbb{S}ig}, \varphi_{\mathbb{N}om}, \varphi_{\mathbb{R}el}) : (\Sigma, Nom, \Lambda) \to (\Sigma', Nom, \Lambda')$ are triples such that $\varphi_{\mathbb{S}ig} : \Sigma \to \Sigma' \in \mathbb{S}ig^{\mathtt{I}}$ and $(\varphi_{\mathbb{N}om}, \varphi_{\mathbb{R}el}) : (Nom, \Lambda) \to (Nom', \Lambda') \in \mathbb{S}ig^{\mathbf{REL}}$. When there is no danger of confusion we may drop the subscripts from notations and denote $\varphi_{\mathbb{S}ig}$, $\varphi_{\mathbb{N}om}$ and $\varphi_{\mathbb{R}el}$ simply by φ.

HI sentences. Let us denote by $\mathcal{Q}^{\mathtt{HI}}$ the subcategory $\mathcal{Q}^{\mathtt{HI}} \subseteq \mathbb{S}ig^{\mathtt{HI}}$ which consists of signature morphisms of the form $\chi : (\Sigma, Nom, \Lambda) \to (\Sigma', Nom, \Lambda)$ such that $\chi_{\mathbb{S}ig} \in \mathcal{Q}$, $\chi_{\mathbb{N}om} = 1_{Nom}$ and $\chi_{\mathbb{R}el} = 1_{\Lambda}$.

Theorem 1. *[13, 6] If $\mathcal{Q}$ is a quantification subcategory for* I *then $\mathcal{Q}^{\mathtt{HI}}$ is a quantification subcategory for* HI.

The satisfaction condition for hybridised institutions relies upon Theorem 1. A nominal variable for a hybrid signature $\Delta = (\Sigma, Nom, \Lambda)$ is a pair of the form (x, Δ), where x is the name of the variable and Δ is the qualification of the variable. Given a hybrid signature $\Delta = (\Sigma, Nom, \Lambda)$, the set of sentences $\mathbb{S}en^{\mathtt{HI}}(\Delta)$ is the least set such that

- $Nom \subseteq \mathbb{S}en^{\mathtt{HI}}(\Delta)$,
- $\lambda(k_1, \dots, k_n) \in \mathbb{S}en^{\mathtt{HI}}(\Delta)$ for any $\lambda \in \Lambda_{n+1}$, $k_i \in Nom$, $i \in \{1, \dots, n\}$;
- $\mathbb{S}en^{\mathtt{I}} \subseteq \mathbb{S}en(\Delta)$;
- $\rho_1 \star \rho_2 \in \mathbb{S}en^{\mathtt{HI}}(\Delta)$ for any $\rho_1, \rho_2 \in \mathbb{S}en^{\mathtt{HI}}(\Delta)$ and $\star \in \{\wedge, \Rightarrow\}$;
- $\neg\rho \in \mathbb{S}en^{\mathtt{HI}}(\Delta)$ for any $\rho \in \mathbb{S}en^{\mathtt{HI}}(\Delta)$;
- $@_k\rho \in \mathbb{S}en^{\mathtt{HI}}(\Delta)$ for any $\rho \in \mathbb{S}en^{\mathtt{HI}}(\Delta)$ and $k \in Nom$;
- $[\lambda](\rho_1, \dots, \rho_n)$ for any $\lambda \in \Lambda_{n+1}$, $\rho_i \in \mathbb{S}en^{\mathtt{HI}}(\Delta)$ and $i \in \{1, \dots, n\}$;
- $(\forall\chi)\rho' \in \mathbb{S}en^{\mathtt{HI}}(\Delta)$ for any $\chi : (\Sigma, Nom, \Lambda) \to (\Sigma', Nom, \Lambda) \in \mathcal{Q}^{\mathtt{HI}}$ and $\rho' \in \mathbb{S}en^{\mathtt{HI}}(\Sigma', Nom, \Lambda)$;
- $(\forall J)\rho$ for any set J of nominal variables for Δ and $\rho \in \mathbb{S}en^{\mathtt{HI}}(\Sigma, Nom \cup J, \Lambda)$;
- $(\downarrow j)\rho$ for any nominal variable j for Δ and $\rho \in \mathbb{S}en^{\mathtt{HI}}(\Sigma, Nom \cup \{j\}, \Lambda)$.

Translation of HI sentences. Let $\varphi : (\Sigma, Nom, \Lambda) \to (\Sigma', Nom', \Lambda')$ be a morphism of HI signatures. The translation $\mathbb{S}en^{\mathtt{HI}}(\varphi)$ is defined as follows:

- $\mathbb{S}en^{\mathtt{HI}}(\varphi)(k) = \varphi_{\mathbb{N}om}(k)$;

- $\mathbb{S}en^{\mathtt{HI}}(\varphi)(\lambda(k_1,\ldots,k_n)) = \varphi_{\mathbb{R}el}(\lambda)(\varphi_{\mathbb{N}om}(k_1),\ldots,\varphi_{\mathbb{N}om}(k_n))$ for any $\lambda \in \Lambda_{n+1}$, $k_i \in Nom$, $i \in \{1,\ldots,n\}$;
- $\mathbb{S}en^{\mathtt{HI}}(\varphi)(\rho) = \mathbb{S}en^{\mathtt{I}}(\varphi_{\mathbb{S}ig})(\rho)$ for any $\rho \in \mathbb{S}en^{\mathtt{I}}(\Sigma)$;
- $\mathbb{S}en^{\mathtt{HI}}(\rho_1 \star \rho_2) = \mathbb{S}en^{\mathtt{HI}}(\rho_1) \star \mathbb{S}en^{\mathtt{HI}}(\rho_2)$, where $\star \in \{\wedge, \Rightarrow\}$;
- $\mathbb{S}en^{\mathtt{HI}}(\neg\rho) = \neg\mathbb{S}en^{\mathtt{HI}}(\rho)$;
- $\mathbb{S}en^{\mathtt{HI}}(@_k\rho) = @_{\varphi_{\mathbb{N}om}(k)}\mathbb{S}en^{\mathtt{HI}}(\rho)$;
- $\mathbb{S}en^{\mathtt{HI}}([\lambda](\rho_1,\ldots,\rho_n)) = [\varphi_{\mathbb{R}el}(\lambda)](\mathbb{S}en^{\mathtt{HI}}(\rho_1),\ldots,\mathbb{S}en^{\mathtt{HI}}(\rho_n))$;
- $\mathbb{S}en^{\mathtt{HI}}((\forall\chi)\rho') = (\forall\chi(\varphi))\mathbb{S}en^{\mathtt{HI}}(\varphi[\chi])(\rho')$, where the signature morphisms $\chi : (\Sigma, Nom, \Lambda) \to (\Sigma', Nom, \Lambda)$ is in $\mathcal{Q}^{\mathtt{HI}}$, $\chi(\varphi) = (\chi_{\mathbb{S}ig}(\varphi_{\mathbb{S}ig}), 1_{Nom'}, 1_{\Lambda'})$ and $\varphi[\chi] = (\varphi_{\mathbb{S}ig}[\chi_{\mathbb{S}ig}], \varphi_{\mathbb{N}om}, \varphi_{\mathbb{R}el})$;
- $\mathbb{S}en^{\mathtt{HI}}((\forall J)\rho) = (\forall J^{\varphi})\mathbb{S}en^{\mathtt{HI}}(\varphi[J])(\rho)$, where $J^{\varphi} = \{(x,(\Sigma', Nom', \Lambda')) \mid (x,(\Sigma, Nom, \Lambda)) \in J\}$ and $\varphi[J] : (\Sigma, Nom \cup J, \Lambda) \to (\Sigma, Nom' \cup J^{\varphi}, \Lambda')$ is canonical extension of φ that maps each variable $(x,(\Sigma, Nom, \Lambda)) \in J$ to $(x,(\Sigma', Nom', \Lambda'))$;
- $\mathbb{S}en^{\mathtt{HI}}((\downarrow j)\rho) = (\downarrow j^{\varphi})\mathbb{S}en^{\mathtt{HI}}(\varphi[j])(\rho)$, where $j^{\varphi} = (x,(\Sigma', Nom', \Lambda'))$ and $\varphi^j : (\Sigma, Nom \cup \{j\}, \Lambda) \to (\Sigma', Nom' \cup \{j^{\varphi}\}, \Lambda')$ is the canonical extension of φ mapping j to j^{φ}.

`HI` **models.** The (Σ, Nom, Λ)-models are paris $(\mathcal{M}, R)$ where

- R is a (Nom, Λ)-model in **REL**. The carrier set $|R|$ forms the set of states of the model $(\mathcal{M}, R)$. The relations $\{\lambda_R \mid \lambda \in \Lambda_n, n \in \mathbb{N}\}$ represent the interpretation of the modalities Λ.
- $\mathcal{M}$ is a function $|R| \to \mathbb{M}od^{\mathtt{I}}(\Sigma)$. For each $s \in |R|$, we denote $\mathcal{M}(s)$ simply by $\mathcal{M}_s$.

A (Σ, Nom, Λ)-homomorphism $h : (\mathcal{M}, R) \to (\mathcal{M}', R')$ consists of

- a (Nom, Λ)-homomorphism in **REL**, $h^{st} : R \to R'$, and
- a natural transformation $h^{mod} : \mathcal{M} \Rightarrow \mathcal{M}' \circ h^{st}$. [3]

When there is no danger of confusion we may drop the superscripts *st* and *mod* from the notations h^{st} and h^{mod}, respectively. The composition of `HI` homomorphisms is defined canonically as $h_1; h_2 = ((h_1^{st}; h_2^{st}), h_1^{mod}; (h_2^{mod} \circ h_1^{st}))$.

Reducts of `HI` models. Let $\Delta = (\Sigma, Nom, \Lambda)$ and $\Delta' = (\Sigma', Nom', \Lambda')$ be two `HI` signatures, $\Delta \xrightarrow{\varphi} \Delta'$ a `HI` signature morphism, and $(\mathcal{M}', R')$ a Δ'-model. The reduct $(\mathcal{M}, R) = \mathbb{M}od^{\mathtt{HI}}(\varphi)(\mathcal{M}', R')$ of $(\mathcal{M}', R')$ along φ denoted by $(\mathcal{M}', R')\restriction_{\varphi}$, is the Δ-model such that $|R| = |R'|$, $k_R = \varphi_{\mathbb{N}om}(k)_{R'}$ for all $k \in Nom$, $\lambda_R = \varphi_{\mathbb{R}el}(\lambda)_{R'}$ for all $\lambda \in \Lambda$, and $\mathcal{M}_s = \mathbb{M}od^{\mathtt{I}}(\varphi_{\mathbb{S}ig})(\mathcal{M}'_s)$ for all $s \in |R|$.

[3] h^{mod} is a $|R|$-indexed family of Σ-homomorphisms $h^{mod} = \{h_s^{mod} : \mathcal{M}_s \to \mathcal{M}'_{h^{st}(s)}\}_{s \in |R|}$.

Satisfaction Relation. For any signature $\Delta = (\Sigma, Nom, \Lambda)$, model $(\mathcal{M}, R) \in |\mathbb{M}od^{\mathtt{HI}}(\Delta)|$ and state $s \in |R|$ we define:

- $(\mathcal{M}, R) \models^s k$ iff $k_R = s$, for any $k \in Nom$;
- $(\mathcal{M}, R) \models^s \lambda(k_1, \ldots, k_n)$ iff $(s, (k_1)_R, \ldots, (k_n)_R) \in \lambda_R$, for any $\lambda \in \Lambda_{n+1}$, $k_i \in Nom$, $i \in \{1, \ldots, n\}$;
- $(\mathcal{M}, R) \models^s \rho$ iff $\mathcal{M}_s \models^{\mathtt{I}} \rho$ for any $\rho \in \mathbb{S}en^{\mathtt{I}}(\Sigma)$;
- $(\mathcal{M}, R) \models^s \rho_1 \wedge \rho_2$ iff $(\mathcal{M}, R) \models^s \rho_1$ and $(\mathcal{M}, R) \models^s \rho_2$;
- $(\mathcal{M}, R) \models^s \rho_1 \Rightarrow \rho_2$ iff $(\mathcal{M}, R) \models^s \rho_1$ implies $(\mathcal{M}, R) \models^s \rho_2$;
- $(\mathcal{M}, R) \models^s \neg\rho$ iff $(\mathcal{M}, R) \not\models^s \rho$;
- $(\mathcal{M}, R) \models^s @_k\rho$ iff $(\mathcal{M}, R) \models^{k_R} \rho$;
- $(\mathcal{M}, R) \models^s [\lambda](\rho_1, \ldots, \rho_n)$ iff for every $(s, s_1, \ldots, s_n) \in \lambda_R$, $(\mathcal{M}, R) \models^{s_i} \rho_i$ for some $i \in \{1, \ldots, n\}$;
- $(\mathcal{M}, R) \models^s (\forall\chi)\rho$ iff for every expansion $(\mathcal{M}', R)$ along $\chi : (\Sigma, Nom, \Lambda) \to (\Sigma', Nom, \Lambda)$ we have $(\mathcal{M}', R) \models^s \rho$;
- $(\mathcal{M}, R) \models^s (\forall J)\rho$ iff for every expansion $(\mathcal{M}, R')$ along $\iota_J : (\Sigma, Nom, \Lambda) \hookrightarrow (\Sigma, Nom \cup J, \Lambda)$ we have $(\mathcal{M}, R') \models^s \rho$;
- $(\mathcal{M}, R) \models^s (\downarrow j)\rho$ iff $(\mathcal{M}, R') \models^s \rho$, where $(M\, R')$ is the expansion of $(\mathcal{M}, R)$ along $\iota_j : (\Sigma, Nom, \Lambda) \to (\Sigma, Nom \cup \{j\}, \Lambda)$ such that $j_R = s$.

$\lambda(k_1, \ldots, k_n)$ is introduced in this paper but a semantically equivalent sentence can be obtained by combining the remaining sentence operators. However, in certain fragments of hybrid logics the sentence operators are restricted making the present approach more useful. The sentence building operator @ is called *retrieve* since it changes the point of evaluation in the model. The sentence building operator $\downarrow$ is called *store* since it gives a name to the current state and it allows a reference to it. The global satisfaction holds when the satisfaction holds locally in all states, i.e. $(\mathcal{M}, R) \models^{\mathtt{HI}} \rho$ iff $(\mathcal{M}, R) \models^s \rho$ for all $s \in |R|$. Given a signature $\Delta \in |\mathbb{S}ig^{\mathtt{HI}}|$ and two sets of sentences $\Gamma, E \in \mathbb{S}en^{\mathtt{HI}}(\Delta)$, we write $\Gamma \models^{\mathtt{HI}} E$ iff for all models $(\mathcal{M}, R) \in |\mathbb{M}od^{\mathtt{HI}}(\Delta)|$ such that $(\mathcal{M}, R) \models^{\mathtt{HI}} \Gamma$ we have $(\mathcal{M}, R) \models^{\mathtt{HI}} E$. Note that variables may be interpreted differently across distinct worlds, which amounts to the world-line semantics of [14].

Satisfaction Condition. The satisfaction condition for hybrid institutions is a direct consequence of the following local satisfaction condition.

Theorem 2. *[6] Let $\Delta = (\Sigma, Nom, \Lambda)$ and $\Delta' = (\mathbb{S}ig', Nom', \Lambda')$ be two* HI *signatures and $\varphi : \Delta \to \Delta'$ a signature morphism. For any $\rho \in \mathbb{S}en^{\mathtt{HI}}(\Delta)$, $(\mathcal{M}', R') \in |\mathbb{M}od^{\mathtt{HI}}(\Delta')|$ and $s \in |R'|$ we have*

$$\mathbb{M}od^{\mathtt{HI}}(\varphi)(\mathcal{M}', R') \models^s \rho \text{ iff } (\mathcal{M}', R') \models^s \mathbb{S}en^{\mathtt{HI}}(\varphi)(\rho)$$

The result of the hybridisation process is an institution.

Corollary 1. *[6]* $\mathtt{HI} = (\mathbb{S}ig^{\mathtt{HI}}, \mathbb{S}en^{\mathtt{HI}}, \mathbb{M}od^{\mathtt{HI}}, \models^{\mathtt{HI}})$ *is an institution.*

A myriad of examples of hybrid institutions may be generated by applying the construction described above to various parameters: (1) the base institution I together with the quantification category $\mathcal{Q}$, and (2) by considering different constrained model functors $(\mathbb{M}od^{\mathtt{CHI}} : \mathbb{S}ig^{\mathtt{CHI}} \to \mathbb{CAT})$ for HI.

*Example 8 (Hybrid first-order logic (**HFOL**)).* This institution is obtained by applying the hybridisation process to **FOL** with the quantification subcategory consisting of signature extensions with a finite number of variables.

*Example 9 (Hybrid Propositional Logic (**HPL**)).* This institution is obtained by applying the hybridisation process to **PL** with the quantification category consisting only of identity signature morphisms. In applications, the category $\mathbb{S}ig^{\mathbf{HPL}}$ is restricted to the *full subcategory* [4] $\mathbb{S}ig^{\mathbf{HPL}'}$ which consists of signatures (P, Nom, Λ), where P is a set of propositional variables, Nom is a set of nominals and Λ is the family of modalities such that $\Lambda_2 = \{\lambda\}$ and $\Lambda_n = \emptyset$ for all $n \neq 2$. In this case we denote $[\Lambda]$ simply by $\Box$. Let $\mathbf{HPL}' = (\mathbb{S}ig^{\mathbf{HPL}'}, \mathbb{S}en^{\mathbf{HPL}}, \mathbb{M}od^{\mathbf{HPL}}, \models^{\mathbf{HPL}})$.

Example 10 (Constrained Hybridisation). Let $\mathtt{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be a base institution, and $\mathbb{M}od^{\mathtt{CHI}} : \mathbb{S}ig^{\mathtt{CHI}} \to \mathbb{CAT}$ a constrained model functor for $\mathtt{HI}$. The *constrained hybridised institution* $\mathtt{CHI} = (\mathbb{S}ig^{\mathtt{CHI}}, \mathbb{S}en^{\mathtt{CHI}}, \mathbb{M}od^{\mathtt{CHI}}, \models^{\mathtt{CHI}})$ is obtained similarly to the case of base institutions:

(a) $\mathbb{S}en^{\mathtt{CHI}} : \mathbb{S}ig^{\mathtt{CHI}} \to \mathbb{S}et$ is the restriction of $\mathbb{S}en^{\mathtt{HI}} : \mathbb{S}ig^{\mathtt{HI}} \to \mathbb{S}et$ to $\mathbb{S}ig^{\mathtt{CHI}}$,
(b) for each signature $\Delta \in |\mathbb{S}ig^{\mathtt{CHI}}|$ and model $(\mathcal{M}, R) \in |\mathbb{M}od^{\mathtt{CHI}}(\Delta)|$,

$$(\mathcal{M}, R) \models^{\mathtt{CHI}} \rho \text{ iff } (\mathcal{M}, R) \models^{s} \rho \text{ for all } s \in |R|.$$

Note that $(\mathcal{M}, R) \models^{\mathtt{HI}} \rho$ iff $(\mathcal{M}, R) \models^{\mathtt{CHI}} \rho$. Given a signature $\Delta \in |\mathbb{S}ig^{\mathtt{CHI}}|$ and two sets of sentences $\Gamma, E \in \mathbb{S}en^{\mathtt{CHI}}(\Delta)$, we write $\Gamma \models^{\mathtt{CHI}} E$ iff for each model $(\mathcal{M}, R) \in |\mathbb{M}od^{\mathtt{CHI}}(\Delta)|$ such that $(\mathcal{M}, R) \models^{\mathtt{CHI}} \Gamma$ we have $(\mathcal{M}, R) \models^{\mathtt{CHI}} E$.

Remark 1. $\Gamma \models^{\mathtt{HI}} E$ implies $\Gamma \models^{\mathtt{CHI}} E$ but the converse implication may not hold.

Example 11 (Injective Hybridisation). Let $\mathtt{I} = (\mathbb{S}ig^{\mathtt{I}}, \mathbb{S}en^{\mathtt{I}}, \mathbb{M}od^{\mathtt{I}}, \models^{\mathtt{I}})$ be a base institution. The *injective hybridisation* $\mathtt{IHI} = (\mathbb{S}ig^{\mathtt{IHI}}, \mathbb{S}en^{\mathtt{IHI}}, \mathbb{M}od^{\mathtt{IHI}}, \models^{\mathtt{IHI}})$ of the base institution $\mathtt{I}$ is a constrained hybridised institution obtained from $\mathtt{HI} = (\mathbb{S}ig^{\mathtt{HI}}, \mathbb{S}en^{\mathtt{HI}}, \mathbb{M}od^{\mathtt{HI}}, \models^{\mathtt{HI}})$ and its constrained model functor $\mathbb{M}od^{\mathtt{IHI}} : \mathbb{S}ig^{\mathtt{IHI}} \to \mathbb{CAT}$ that do not allow confusion among nominals: (a) $\mathbb{S}ig^{\mathtt{IHI}}$ is the broad subcategory of $\mathbb{S}ig^{\mathtt{HI}}$ consisting of signature morphisms injective on nominals, i.e. $\varphi_{\mathbb{N}om}$ is injective for all $\varphi \in \mathbb{S}ig^{\mathtt{IHI}}$, and (b) $\mathbb{M}od^{\mathtt{IHI}}(\Sigma, Nom, \Lambda)$ is the full subcategory of $\mathbb{M}od^{\mathtt{HI}}(\Sigma, Nom, \Lambda)$ consisting of models that do not allow confusion among nominals, i.e. $j_R = k_R$ implies $j = k$ for all $j, k \in Nom$.

Our results are not applicable directly to hybrid institutions but rather to their restriction to models that do not allow confusion among nominals. The following results can be instantiated, for example, to the injective hybridisation of **FOL**. However, when the quantification subcategory $\mathcal{Q}$ consists of identities (take for example **PL**) then the semantic restriction of the hybridised logic is no longer required. This means that the following results are applicable to **HPL**.

[4] A category $\mathcal{C}$ is a full subcategory of $\mathcal{C}'$ if $\mathcal{C}$ is a subcategory of $\mathcal{C}'$ and for all objects $A, B \in |C|$ we have $\mathcal{C}(A, B) = \mathcal{C}'(A, B)$.

Example 12 (Quantifier-free Injective Hybridisation). The quantifier-free injective hybridisation $\mathtt{QIHI} = (\mathbb{S}ig^{\mathtt{IHI}}, \mathbb{S}en^{\mathtt{QIHI}}, \mathbb{M}od^{\mathtt{IHI}}, \models^{\mathtt{IHI}})$ of a base institution $\mathtt{I} = (\mathbb{S}ig^{\mathtt{I}}, \mathbb{S}en^{\mathtt{I}}, \mathbb{M}od^{\mathtt{I}}, \models^{\mathtt{I}})$ is obtained from the injective hybridisation $\mathtt{IHI} = (\mathbb{S}ig^{\mathtt{IHI}}, \mathbb{S}en^{\mathtt{IHI}}, \mathbb{M}od^{\mathtt{IHI}}, \models^{\mathtt{IHI}})$ by restricting the syntax to quantifier-free sentences, i.e. for each $(\Sigma, Nom, \Lambda) \in |\mathbb{S}ig^{\mathtt{IHI}}|$ the set $\mathbb{S}en^{\mathtt{QIHI}}(\Sigma, Nom, \Lambda)$ consists of sentences obtained from nominal sentences (e.g. $k \in Nom$), hybrid relational atoms (e.g. $\lambda(k_1, \ldots, k_n) \in \mathbb{S}en^{\mathtt{IHI}}(\Sigma, Nom, \Lambda)$) and the sentences in $\mathbb{S}en(\Sigma)$ by applying Boolean connectives and the operator @. This institution is useful for defining hybrid substitutions that do not involve any form of quantification (see Section 3.1).

3.1 Hybrid Substitutions

We extend the notion of substitution from a base institution to its hybridisation. In this subsection we assume a base institution $\mathtt{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$, a broad subcategory of signature morphisms $\mathcal{D} \subseteq \mathbb{S}ig$, and a substitution functor $\mathbb{S}ub : \mathcal{D}^{op} \to \mathbb{CAT}$ for the base institution $\mathtt{I}$. Let $\mathcal{D}^{\mathtt{HI}} \subseteq \mathbb{S}ig^{\mathtt{HI}}$ be the broad subcategory of hybrid signature morphisms of the form $\varphi : (\Sigma, Nom, \Lambda) \to (\Sigma_1, Nom, \Lambda)$ such that $\Sigma \overset{\varphi_{\mathbb{S}ig}}{\to} \Sigma_1 \in \mathcal{D}$, $\varphi_{\mathbb{N}om} = 1_{Nom}$ and $\varphi_{\mathbb{R}el} = 1_{\Lambda}$.

Inherited Substitutions. Hybrid substitutions can be obtained from combinations of substitutions in the base institution. Let $(\Sigma, Nom, \Lambda) \in |\mathbb{S}ig^{\mathtt{IHI}}|$ be a signature and $\Theta = \{\theta_k : (\Sigma \overset{\varphi_1}{\to} \Sigma_1) \to (\Sigma \overset{\varphi_2}{\to} \Sigma_2)\}_{k \in Nom}$ a family of substitutions in $\mathbb{S}ub$. On the the syntactic side, Θ determines a function

$$\Theta^k : \mathbb{S}en^{\mathtt{QIHI}}(\Sigma_1, Nom, \Lambda) \to \mathbb{S}en^{\mathtt{QIHI}}(\Sigma_2, Nom, \Lambda)$$

for each nominal $k \in Nom$:

- $\Theta^k(j) = j$, for all $j \in Nom$;
- $\Theta^k(\lambda(k_1, \ldots, k_n)) = \lambda(k_1, \ldots, k_n)$ for all $\lambda \in \Lambda_{n+1}$ and $k_i \in Nom$;
- $\Theta^k(\rho) = \theta_k(\rho)$ for any $\rho \in \mathbb{S}en^{\mathtt{I}}(\Sigma)$;
- $\Theta^k(\rho \star \rho') = \Theta^k(\rho) \star \Theta^k(\rho')$, $\star \in \{\wedge, \Rightarrow\}$;
- $\Theta^k(\neg\rho) = \neg\Theta^k(\rho)$;
- $\Theta^k(@_j \rho) = \Theta^j(\rho)$;

Since $\mathbb{S}en^{\mathtt{I}}(\varphi_1); \mathbb{S}en(\theta_k) = \mathbb{S}en^{\mathtt{I}}(\varphi_2)$ for all nominals $k \in Nom$, the following result holds.

Lemma 2. *The diagram below is commutative*

$$\begin{array}{ccc} \mathbb{S}en^{\mathtt{QIHI}}(\Sigma_1, Nom, \Lambda) & \xrightarrow{\quad\Theta^k\quad} & \mathbb{S}en^{\mathtt{QIHI}}(\Sigma_2, Nom, \Lambda) \\ {\scriptstyle \mathbb{S}en^{\mathtt{QIHI}}(\varphi_1)} \nwarrow & & \nearrow {\scriptstyle \mathbb{S}en^{\mathtt{QIHI}}(\varphi_2)} \\ & \mathbb{S}en^{\mathtt{QIHI}}(\Sigma, Nom, \Lambda) & \end{array}$$

for all nominals $k \in Nom$.

On the semantic side, Θ determines a functor

$$\mathbb{M}od^{\mathtt{IHI}}(\Theta^k) : \mathbb{M}od^{\mathtt{IHI}}(\Sigma_2, Nom, \Lambda) \to \mathbb{M}od^{\mathtt{IHI}}(\Sigma_1, Nom, \Lambda)$$

often denoted by $\_\!\upharpoonright_{\Theta^k}$ for all nominals $k \in Nom$:

- for every $(\mathcal{M}^2, R) \in |\mathbb{M}od^{\mathtt{IHI}}(\Sigma_2, Nom, \Lambda)|$, $(\mathcal{M}^2, R) \upharpoonright_{\Theta^k} = (\mathcal{M}^2 \upharpoonright_{\Theta^k}, R)$, where $\mathcal{M}^2 \upharpoonright_{\Theta^k}$ is defined by
  - $(\mathcal{M}^2 \upharpoonright_{\Theta^k})_{j_R} = \mathcal{M}^2_{j_R} \upharpoonright_{\theta_j}$ for all nominals $j \in Nom$, and
  - $(\mathcal{M}^2 \upharpoonright_{\Theta^k})_s = \mathcal{M}^2_s \upharpoonright_{\theta_k}$ for all $s \in (|R| - Nom_R)$.
- for every $h^2 : (\mathcal{M}^2, R) \to (\mathcal{N}^2, P) \in \mathbb{M}od^{\mathtt{IHI}}(\Sigma_2, Nom, \Lambda)$ we have
  - $(h^2 \upharpoonright_{\Theta^k})_{j_R} = h^2_{j_R} \upharpoonright_{\theta_j}$ for all nominals $j \in Nom$, and
  - $(h^2 \upharpoonright_{\Theta^k})_s = h^2_s \upharpoonright_{\theta_k}$ for all $s \in (|R| - Nom_R)$.

The definition of $\_\!\upharpoonright_{\Theta^k}$ is consistent because no confusion of nominals is allowed inside of the models $(\mathcal{M}^2, R)$. Since $\mathbb{M}od(\theta_k); \mathbb{M}od^{\mathtt{I}}(\varphi_1) = \mathbb{M}od^{\mathtt{I}}(\varphi_2)$ for all nominals $k \in Nom$, the following result holds.

Lemma 3. *The diagram below is commutative*

$$\begin{array}{ccc} \mathbb{M}od^{\mathtt{IHI}}(\Sigma_1, Nom, \Lambda) & \xleftarrow{\ \upharpoonright_{\Theta^k}\ } & \mathbb{M}od^{\mathtt{IHI}}(\Sigma_2, Nom, \Lambda) \\ & {\scriptstyle \mathbb{M}od^{\mathtt{IHI}}(\varphi_1)} \searrow \quad \swarrow {\scriptstyle \mathbb{M}od^{\mathtt{IHI}}(\varphi_2)} & \\ & \mathbb{M}od^{\mathtt{IHI}}(\Sigma, Nom, \Lambda) & \end{array}$$

for all nominals $k \in Nom$.

Next result can be regarded as the satisfaction condition for the substitutions inherited from the base institution.

Proposition 2 (Satisfaction Condition). *Given a signature $(\Sigma, Nom, \Lambda) \in |\mathbb{S}ig^{\mathtt{HI}}|$, for every model $(\mathcal{M}^2, R) \in \mathbb{M}od^{\mathtt{IHI}}(\Sigma_2, Nom, \Lambda)$ and each sentence $\rho \in \mathbb{S}en^{\mathtt{QIHI}}(\Sigma, Nom, \Lambda)$*

$$(\mathcal{M}^2, R) \models^{k_R} \Theta^k(\rho) \text{ iff } (\mathcal{M}^2, R) \upharpoonright_{\Theta^j} \models^{k_R} \rho$$

for all nominals $j, k \in Nom$.

Proposition 2 stands at the basis of proving initiality and Herbrand's theorem in hybrid logics where the variables may be interpreted differently across distinct worlds. The following is a corollary of Proposition 2 which allows one to infer new sentences from initial axioms by applying substitutions inherited from the base institution.

Corollary 2. *Assume a signature $\Delta = (\Sigma, Nom, \Lambda) \in |\mathbb{S}ig^{\mathtt{IHI}}|$ and a hybrid substitution $\Theta = \{\theta_j : (\Sigma \overset{\varphi_1}{\to} \Sigma_1) \to (\Sigma \overset{\varphi_2}{\to} \Sigma_2)\}_{j \in Nom}$. For all sentences $\rho \in \mathbb{S}en^{\mathtt{QIHI}}(\Delta)$ ρ we have*

$$(\forall \varphi_1)\rho \models^{\mathtt{IHI}} @_k (\forall \varphi_2) \Theta^k(\rho)$$

for all nominals $k \in Nom$.

Nominal substitutions. Nominal substitutions are captured by the notion of signature morphisms in the hybridised institution. Let $\iota_j : (\Sigma, Nom, \Lambda) \hookrightarrow (\Sigma, Nom \cup \{j\}, \Lambda)$ be a signature extension with the nominal variable j. A nominal substitution is represented by a function $\varphi_{\mathbb{N}om} : \{j\} \to Nom$ which can be canonically extended to a signature morphism $\varphi : (\Sigma, Nom \cup \{j\}, \Lambda) \to (\Sigma, Nom, \Lambda)$. The following result is a consequence of the satisfaction condition for the hybridised institution.

Lemma 4. *Let $(\forall j)\rho \in \mathbb{S}en^{\mathtt{HI}}(\Sigma, Nom, \Lambda)$ and $k \in Nom$.*

(1) $(\forall j)\rho \models^{\mathtt{HI}} \rho[j \leftarrow k]$; moreover, $(\mathcal{M}, R) \models^s (\forall j)\rho$ implies $(M, R) \models^s \rho[j \leftarrow k]$ for all models $(\mathcal{M}, R) \in |\mathbb{M}od^{\mathtt{HI}}(\Sigma, Nom, \Lambda)|$ and states $s \in |R|$;
(2) $(\downarrow j)\rho \models^{\mathtt{HI}} @_k\rho[j \leftarrow k]$. [5]

3.2 Reachable Hybrid Models

In this subsection we extend the notion of reachability to hybrid institutions. Let $\mathtt{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be a base institution, $\mathcal{D} \subseteq \mathbb{S}ig$ a broad subcategory of signature morphisms, and $\mathbb{S}ub : \mathcal{D}^{op} \to \mathbb{CAT}$ a substitution functor for $\mathtt{I}$.

Definition 4. *A model $(\mathcal{M}, R) \in |\mathbb{M}od^{\mathtt{HI}}(\Sigma, Nom, \Lambda)|$, where $(\Sigma, Nom, \Lambda) \in |\mathbb{S}ig^{\mathtt{HI}}|$, is $\mathbb{S}ub$-reachable if (a) $|R| = Nom_R$, where $Nom_R = \{k_R \mid k \in Nom\}$, and (b) $\mathcal{M}_{k_R}$ is $\mathbb{S}ub$-reachable in $\mathtt{I}$ for all nominals $k \in Nom$.*

In the injective hybridisation, the expansions of reachable models along signature morphisms in $\mathcal{D}^{\mathtt{HI}}$ generate hybrid substitutions.

Proposition 3. *Given a signature $(\Sigma, Nom, \Lambda) \in |\mathbb{S}ig^{\mathtt{HI}}|$ and a $\mathbb{S}ub$-reachable model $(\mathcal{M}, R) \in |\mathbb{M}od^{\mathtt{IHI}}(\Sigma, Nom, \Lambda)|$ then for every signature morphism $\chi : (\Sigma, Nom, \Lambda) \to (\Sigma', Nom, \Lambda)$ with $\Sigma \overset{\chi}{\to} \Sigma' \in \mathcal{D}$ and each χ-expansion $(\mathcal{M}', R)$ of $(\mathcal{M}, R)$ there exists a hybrid substitution $\Theta = \{\chi \overset{\theta_k}{\to} 1_\Sigma\}_{k \in Nom}$ such that $(\mathcal{M}, R)\restriction_{\Theta^j} = (\mathcal{M}', R)$ for all nominals $j \in Nom$.*

This definition of reachability is used in the context of injective hybridisations and their constrained sub-institutions.

4 Initiality

The following results on the existence of initial models depend on multiple parameters that can be instantiated in the same context in many ways producing different results. We will focus largely on parameter instantiation of the abstract theorems to concrete hybrid logical systems to obtain the desired applications. However, the interested reader may find other useful applications as well. In this section we assume a base institution $\mathtt{I} = (\mathbb{S}ig^{\mathtt{I}}, \mathbb{S}en^{\mathtt{I}}, \mathbb{M}od^{\mathtt{I}}, \models^{\mathtt{I}})$, a broad subcategory $\mathcal{D} \subseteq \mathbb{S}ig$ of signature morphisms and a substitution functor $\mathbb{S}ub : \mathcal{D}^{op} \to \mathbb{CAT}$ for the base institution $\mathtt{I}$.

[5] We denote by $\rho[j \leftarrow k]$ the sentence $\varphi(\rho)$, where the signature morphism $\varphi : (\Sigma, Nom \cup \{j\}, \Lambda) \to (\Sigma, Nom, \Lambda)$ is the canonical extension of the function $\varphi_{\mathbb{N}om} : \{j\} \to Nom$ defined by $\varphi_{\mathbb{N}om}(j) = k$.

4.1 Basic Hybrid Sentences

In addition to the assumptions made at the beginning of this section, let us consider a sub-functor $(\mathbb{S}en_0^{\mathtt{I}} : \mathbb{S}ig \to \mathbb{S}et)$ of $\mathbb{S}en^{\mathtt{I}}$. We define the sentence functor $(\mathbb{S}en_0^{\mathtt{HI}} : \mathbb{S}ig^{\mathtt{HI}} \to \mathbb{S}et)$ of $\mathbb{S}en^{\mathtt{HI}}$ for each signature $(\Sigma, Nom, \Lambda) \in |\mathbb{S}ig^{\mathtt{HI}}|$,

(1) $@_j k \in \mathbb{S}en_0^{\mathtt{HI}}(\Sigma, Nom, \Lambda)$ for all $j, k \in Nom$,
(2) $@_j \lambda(k_1, \ldots, k_n) \in \mathbb{S}en_0^{\mathtt{HI}}(\Sigma, Nom, \Lambda)$ for all $\lambda \in \Lambda_{n+1}$, $j \in Nom$ and $k_i \in Nom$,
(3) $@_j \rho \in \mathbb{S}en_0^{\mathtt{HI}}(\Sigma, Nom, \Lambda)$ for all $j \in Nom$ and $\rho \in \mathbb{S}en_0^{\mathtt{I}}(\Sigma)$.

In concrete examples of institutions, $\mathtt{I}_0 = (\mathbb{S}ig^{\mathtt{I}}, \mathbb{S}en_0^{\mathtt{I}}, \mathbb{M}od^{\mathtt{I}}, \models^{\mathtt{I}})$ is the restriction of the base institution to atomic sentences, and the institution $\mathtt{HI}_0 = (\mathbb{S}ig^{\mathtt{HI}}, \mathbb{S}en_0^{\mathtt{HI}}, \mathbb{M}od^{\mathtt{HI}}, \models^{\mathtt{HI}})$ gives the building bricks for constructing theories that have initial models in the hybridised institution.

Theorem 3. *If every set of sentences of* $\mathtt{I}_0$ *is epi basic then every set of sentences of* $\mathtt{HI}_0$ *is epi basic. Moreover, if each set of sentences of* $\mathtt{I}_0$ *has a basic model that is* $\mathbb{S}ub$*-reachable then each set of sentences of* $\mathtt{HI}_0$ *has a basic model that is* $\mathbb{S}ub$*-reachable.*

We apply Theorem 3 to **HFOL**. Let $\mathbb{S}en_0^{\mathbf{FOL}} : \mathbb{S}ig^{\mathbf{FOL}} \to \mathbb{S}et$ be the sub-functor of $\mathbb{S}en^{\mathbf{FOL}}$ such that for any signature $\Sigma \in |\mathbb{S}ig^{\mathbf{FOL}}|$ the set $\mathbb{S}en_0^{\mathbf{FOL}}(\Sigma)$ consists of atoms. We define $\mathbf{FOL}_0 = (\mathbb{S}ig^{\mathbf{FOL}}, \mathbb{S}en_0^{\mathbf{FOL}}, \mathbb{M}od^{\mathbf{FOL}}, \models^{\mathbf{FOL}})$ and $\mathbf{HFOL}_0 = (\mathbb{S}ig^{\mathbf{HFOL}}, \mathbb{S}en_0^{\mathbf{HFOL}}, \mathbb{M}od^{\mathbf{HFOL}}, \models^{\mathbf{HFOL}})$ using the general pattern described above.

Corollary 3. *All sets of* $\mathbf{HFOL}_0$ *sentences are epi basic and the corresponding basic models are* $\mathbb{S}ub^{\mathbf{FOL}}$*-reachable.*

Proof. By Lemma 1, any set of **FOL** atoms is epi basic. By Proposition 1, the corresponding basic models are $\mathbb{S}ub^{\mathbf{FOL}}$-reachable. By Theorem 3, any set of $\mathbf{HFOL}_0$ sentences is epi basic and the corresponding basic models are $\mathbb{S}ub^{\mathbf{FOL}}$-reachable. □

We return to the general setting and we define the sub-functor $(\mathbb{S}en_0^{\mathtt{IHI}} : \mathbb{S}ig^{\mathtt{IHI}} \to \mathbb{S}et)$ of $\mathbb{S}en^{\mathtt{IHI}}$ for each signature $(\Sigma, Nom, \Lambda) \in |\mathbb{S}ig^{\mathtt{HI}}|$,

(1) $@_j \lambda(k_1, \ldots, k_n) \in \mathbb{S}en_0^{\mathtt{IHI}}(\Sigma, Nom, \Lambda)$ for any $\lambda \in \Lambda_{n+1}$, $j \in Nom$ and $k_i \in Nom$,
(2) $@_j \rho \in \mathbb{S}en_0^{\mathtt{IHI}}(\Sigma, Nom, \Lambda)$ for any $j \in Nom$ and $\rho \in \mathbb{S}en_0^{\mathtt{I}}(\Sigma)$.

The institution $\mathtt{IHI}_0 = (\mathbb{S}ig^{\mathtt{IHI}}, \mathbb{S}en_0^{\mathtt{IHI}}, \mathbb{M}od^{\mathtt{IHI}}, \models^{\mathtt{IHI}})$ gives the building bricks for constructing theories that have initial models in the injective hybridisation.

Theorem 4. *If every set of sentences of* $\mathtt{I}_0$ *is epi basic then every set of sentences of* $\mathtt{IHI}_0$ *is epi basic. Moreover, if each set of sentences of* $\mathtt{I}_0$ *has a basic model that is* $\mathbb{S}ub$*-reachable then each set of sentences of* $\mathtt{IHI}_0$ *has a basic model that is* $\mathbb{S}ub$*-reachable.*

We apply Theorem 4 to **IHFOL**. Using the general pattern described above, let us define $\mathbf{IHFOL}_0 = (\mathbb{S}ig^{\mathbf{IHFOL}}, \mathbb{S}en_0^{\mathbf{IHFOL}}, \mathbb{M}od^{\mathbf{IHFOL}}, \models^{\mathbf{IHFOL}})$ as the injective hybridisation of **FOL**.

Corollary 4. *All sets of* $\mathbf{IHFOL}_0$ *sentences are epi basic and the corresponding basic models are* $\mathbb{S}ub^{\mathbf{FOL}}$*-reachable.*

Proof. By Lemma 1, any set of atoms in **FOL** is epi basic. By Proposition 1, the corresponding basic models are $\mathbb{S}ub^{\mathbf{FOL}}$-reachable. By Theorem 4, any set of sentences in $\mathbf{HFOL}_0$ is epi basic and the corresponding basic models are $\mathbb{S}ub^{\mathbf{FOL}}$-reachable. $\square$

In the next subsections we prove that the initiality property is closed under the following sentence building operators: logical implication $\Rightarrow$, universal quantification $\forall$, store $\downarrow$, and box $\Box$.

4.2 Implication

In addition to the assumptions made at the beginning of this section, let us consider a constrained model functor $\mathbb{M}od^{\mathtt{CHI}} : \mathbb{S}ig^{\mathtt{CHI}} \to \mathbb{CAT}$ for HI, and three sub-functors $(\mathbb{S}en_*^{\mathtt{CHI}} : \mathbb{S}ig^{\mathtt{CHI}} \to \mathbb{S}et)$, $(\mathbb{S}en_\bullet^{\mathtt{CHI}} : \mathbb{S}ig^{\mathtt{CHI}} \to \mathbb{S}et)$ and $(\mathbb{S}en_1^{\mathtt{CHI}} : \mathbb{S}ig^{\mathtt{CHI}} \to \mathbb{S}et)$ of $\mathbb{S}en^{\mathtt{CHI}}$ such that any sentence of $\mathbb{S}en_1^{\mathtt{CHI}}(\Delta)$, where $\Delta \in |\mathbb{S}ig^{\mathtt{CHI}}|$, is semantically equivalent in CHI to a sentence of the form $\bigwedge H \Rightarrow C$, where $H \subseteq \mathbb{S}en_*^{\mathtt{CHI}}(\Delta)$ and $C \in \mathbb{S}en_\bullet^{\mathtt{CHI}}(\Delta)$.

Theorem 5. *If for each signature* $\Delta \in |\mathbb{S}ig^{\mathtt{CHI}}|$,

(1) any set $B \subseteq \mathbb{S}en_*^{\mathtt{CHI}}(\Delta)$ *is basic in* HI, [6] *and*
(2) any set $\Gamma \subseteq \mathbb{S}en_\bullet^{\mathtt{CHI}}(\Delta)$ *has an initial* $\mathbb{S}ub$*-reachable model* $(\mathcal{M}^\Gamma, R^\Gamma) \in |\mathbb{M}od^{\mathtt{CHI}}(\Delta)|$,

then any set of sentences of the institution $\mathtt{CHI}_1 = (\mathbb{S}ig^{\mathtt{CHI}}, \mathbb{S}en_1^{\mathtt{CHI}}, \mathbb{M}od^{\mathtt{CHI}}, \models^{\mathtt{CHI}})$ *has an initial* $\mathbb{S}ub$*-reachable model.*

We apply Theorem 5 to **HFOL**. The constrained model functor $\mathbb{M}od^{\mathtt{CHI}}$ is $\mathbb{M}od^{\mathbf{HFOL}} : \mathbb{S}ig^{\mathbf{HFOL}} \to \mathbb{CAT}$. The functors $\mathbb{S}en_*^{\mathtt{CHI}}$ and $\mathbb{S}en_\bullet^{\mathtt{CHI}}$ are both instantiated to $\mathbb{S}en_0^{\mathbf{HFOL}} : \mathbb{S}ig^{\mathbf{HFOL}} \to \mathbb{S}et$. The institution $\mathtt{CHI}_1$ is $\mathbf{HFOL}_1$, the restriction of **HFOL** to sentences of the form $\bigwedge H \Rightarrow C$, where $H \cup \{C\}$ is a set of $\mathbf{HFOL}_0$ sentences.

Corollary 5. *Any set of* $\mathbf{HFOL}_1$ *sentences has an initial* $\mathbb{S}ub^{\mathbf{FOL}}$*-reachable model.*

We apply Theorem 5 to **IHFOL**. The institution CHI is **IHFOL**. The functor $\mathbb{S}en_*^{\mathtt{CHI}}$ is the restriction of $(\mathbb{S}en_0^{\mathbf{HFOL}} : \mathbb{S}ig^{\mathbf{HFOL}} \to \mathbb{S}et)$ to $\mathbb{S}ig^{\mathbf{IHFOL}}$. The functor $\mathbb{S}en_\bullet^{\mathtt{CHI}}$ is $(\mathbb{S}en_0^{\mathbf{IHFOL}} : \mathbb{S}ig^{\mathbf{IHFOL}} \to \mathbb{S}et)$. The sentence functor $\mathbb{S}en_1^{\mathtt{CHI}}$ is $(\mathbb{S}en_1^{\mathbf{IHFOL}} : \mathbb{S}ig^{\mathbf{IHFOL}} \to \mathbb{S}et)$ such that for all $\Delta \in |\mathbb{S}ig^{\mathbf{IHFOL}}|$ the set $\mathbb{S}en_1^{\mathbf{IHFOL}}(\Delta)$ consists of sentences of the form $\bigwedge H \Rightarrow C$, where $H \subseteq \mathbb{S}en_0^{\mathbf{HFOL}}(\Delta)$ and $C \in \mathbb{S}en_0^{\mathbf{IHFOL}}(\Delta)$.

[6] This condition implies that there exists a basic model $(\mathcal{M}^B, R^B) \in |\mathbb{M}od^{\mathtt{HI}}(\Delta)|$, but it is also possible that $(\mathcal{M}^B, R^B) \notin |\mathbb{M}od^{\mathtt{CHI}}(\Delta)|$.

Corollary 6. *Any set of* $\mathbf{IHFOL}_1$ *sentences has an initial* $\mathbb{S}ub^{\mathbf{FOL}}$*-reachable model, where* $\mathbf{IHFOL}_1 = (\mathbb{S}ig^{\mathbf{IHFOL}}, \mathbb{S}en_1^{\mathbf{IHFOL}}, \mathbb{M}od^{\mathbf{IHFOL}}, \models^{\mathbf{IHFOL}})$.

4.3 Nominal Quantification

In addition to the assumptions made at the beginning of this section, let us consider a constrained model functor $\mathbb{M}od^{\mathtt{CHI}} : \mathbb{S}ig^{\mathtt{CHI}} \to \mathbb{CAT}$ for $\mathtt{HI}$, and two sub-functors $(\mathbb{S}en_1^{\mathtt{CHI}} : \mathbb{S}ig^{\mathtt{CHI}} \to \mathbb{S}et)$ and $(\mathbb{S}en_2^{\mathtt{CHI}} : \mathbb{S}ig^{\mathtt{CHI}} \to \mathbb{S}et)$ of $\mathbb{S}en^{\mathtt{CHI}}$ such that all sentences of $\mathtt{CHI}_2 = (\mathbb{S}ig^{\mathtt{CHI}}, \mathbb{S}en_2^{\mathtt{CHI}}, \mathbb{M}od^{\mathtt{CHI}}, \models^{\mathtt{CHI}})$ are semantically equivalent in $\mathtt{CHI}$ to a sentence of the form $(\forall j)\rho$, where j is a nominal variable and ρ is a sentence of $\mathtt{CHI}_1 = (\mathbb{S}ig^{\mathtt{CHI}}, \mathbb{S}en_1^{\mathtt{CHI}}, \mathbb{M}od^{\mathtt{CHI}}, \models^{\mathtt{CHI}})$.

Theorem 6. *If every set of sentences of* $\mathtt{CHI}_1$ *has an initial* $\mathbb{S}ub$*-reachable model then each set of sentences of* $\mathtt{CHI}_2$ *has an initial* $\mathbb{S}ub$*-reachable model.*

The following result is essential for applying Theorem 6 to concrete examples of institutions.

Lemma 5. *In the institution* $\mathtt{CHI}$*, any sentence* $\bigwedge H \Rightarrow C$ *is semantically equivalent to* $(\forall j) \bigwedge\{@_j h \mid h \in H\} \Rightarrow @_j C$*, and any sentence* $@_j @_k \rho$ *is semantically equivalent to* $@_k \rho$.

We apply Theorem 6 on top of $\mathbf{HFOL}_1$ defined in Subsection 4.2. The institution $\mathtt{CHI}$ is $\mathbf{HFOL}$, and the institution $\mathtt{CHI}_1$ is $\mathbf{HFOL}_1$. The sentence functor $\mathbb{S}en_2^{\mathtt{CHI}}$ is $(\mathbb{S}en_2^{\mathbf{HFOL}} : \mathbb{S}ig^{\mathbf{HFOL}} \to \mathbb{S}et)$ which associates to each signature $\Delta = (\Sigma, Nom, \Lambda) \in |\mathbb{S}ig^{\mathbf{HFOL}}|$ the set of sentences of the form $\bigwedge H \Rightarrow C$, where $H \cup \{C\}$ consists of sentences obtained from nominal sentences (e.g. $k \in Nom$), hybrid relational atoms (e.g. $\lambda(k_1, \ldots, k_n) \in \mathbb{S}en^{\mathbf{IHFOL}}(\Delta)$) and **FOL** atoms (e.g. $t_1 = t_2 \in \mathbb{S}en_0^{\mathbf{FOL}}(\Sigma)$ and $\pi(t_1, \ldots, t_n) \in \mathbb{S}en_0^{\mathbf{FOL}}(\Sigma)$) by applying the sentence building operator @. [7]

Corollary 7. *Any set of sentences in* $\mathbf{HFOL}_2$ *has an initial* $\mathbb{S}ub^{\mathbf{FOL}}$*-reachable model.*

Proof. By Lemma 5, any sentence in $\mathbf{HFOL}_2$ is semantically equivalent to a sentence of the form $(\forall j)\rho$, where j is a nominal variable and ρ is a sentence of $\mathbf{HFOL}_1$. By Corollary 6, any set of sentences in $\mathbf{HFOL}_1$ has an initial $\mathbb{S}ub^{\mathbf{FOL}}$-reachable model. By Theorem 6, any set of sentences in $\mathbf{HFOL}_2$ has an initial $\mathbb{S}ub^{\mathbf{FOL}}$-reachable model. □

We apply Theorem 6 on top of $\mathbf{HFOL}_2$. Let $\mathbf{HFOL}_3$ be the institution obtained from **HFOL** by restricting the syntax to sentences of the form $(\forall J)\rho$, where J is a finite set of nominal variables and ρ is a quantifier-free sentence of $\mathbf{HFOL}_2$.

Corollary 8. *Any set of* $\mathbf{HFOL}_3$ *sentences has an initial* $\mathbb{S}ub^{\mathbf{FOL}}$*-reachable model.*

[7] The institution $\mathbf{HFOL}_2$ contains also sentences that are free of @. It follows that $\mathbb{S}en_1^{\mathbf{HFOL}}(\Delta) \subsetneq \mathbb{S}en_2^{\mathbf{HFOL}}(\Delta)$ for all $\Delta \in |\mathbb{S}ig^{\mathbf{HFOL}}|$.

We call the $\mathbf{HFOL}_3$ sentences *hybrid Horn clauses of the institution* **HFOL**. Since **PL** is obtained from **FOL** by restricting the category of signatures, Corollary 8 holds also for **HPL**. We apply Theorem 6 on top of $\mathbf{IHFOL}_1$ defined in Subsection 4.2. The sentence functor $\mathbb{S}en_1^{\mathtt{CHI}}$ is $(\mathbb{S}en_1^{\mathbf{IHFOL}} : \mathbb{S}ig^{\mathbf{IHFOL}} \to \mathbb{S}et)$. The functor $\mathbb{S}en_2^{\mathtt{CHI}}$ is $(\mathbb{S}en_2^{\mathbf{IHFOL}} : \mathbb{S}ig^{\mathbf{IHFOL}} \to \mathbb{S}et)$ which associates to each signature the set of sentences of the form $\bigwedge H \Rightarrow C$, where

(a) H consists of sentences obtained from nominal sentences, hybrid relational atoms, and **FOL** atoms by applying the sentence building operator @, and
(b) C is a sentence obtained from hybrid relational atoms and **FOL** atoms by applying @.

Corollary 9. *Any set of sentences in* $\mathbf{IHFOL}_2$ *has an initial* $\mathbb{S}ub^{\mathbf{FOL}}$*-reachable model.*

Another application of Theorem 6 can be found in Subsection 4.4.

4.4 Inherited Quantification

In addition to the assumptions made at the beginning of this section, let us consider a constrained model functor $\mathbb{M}od^{\mathtt{CIHI}} : \mathbb{S}ig^{\mathtt{CIHI}} \to \mathbb{CAT}$ for the injective hybridisation IHI, a quantification subcategory $\mathcal{Q} \subseteq \mathcal{D}$, and two sub-functors $(\mathbb{S}en_2^{\mathtt{CIHI}} : \mathbb{S}ig^{\mathtt{CIHI}} \to \mathbb{S}et)$ and $(\mathbb{S}en_3^{\mathtt{CIHI}} : \mathbb{S}ig^{\mathtt{CIHI}} \to \mathbb{S}et)$ of $\mathbb{S}en^{\mathtt{CIHI}}$ such that

(1) the sentences of $\mathtt{CIHI}_2 = (\mathbb{S}ig^{\mathtt{CIHI}}, \mathbb{S}en_2^{\mathtt{CIHI}}, \mathbb{M}od^{\mathtt{CIHI}}, \models^{\mathtt{CIHI}})$ are semantically closed to @, i.e. for all $\Delta \in |\mathbb{S}ig^{\mathtt{CIHI}}|$, $k \in Nom$ and $\rho \in \mathbb{S}en_2^{\mathtt{CIHI}}(\Delta)$ there exists $\varepsilon \in \mathbb{S}en_2^{\mathtt{CIHI}}(\Delta)$ such that $@_k\rho \models\!\mid^{\mathtt{CIHI}} \varepsilon$,
(2) In CIHI, any sentence of $\mathbb{S}en_3(\Sigma, Nom, \Lambda)$, where $(\Sigma, Nom, \Lambda) \in |\mathbb{S}ig^{\mathtt{CIHI}}|$, is semantically equivalent to a sentence of the form $(\forall\chi)\rho$, where $(\Sigma, Nom, \Lambda) \overset{\chi}{\to} (\Sigma', Nom, \Lambda) \in \mathcal{Q}^{\mathtt{HI}}$ and $\rho \in \mathbb{S}en_2^{\mathtt{CIHI}}(\Sigma', Nom, \Lambda)$,
(3) for any $(\Sigma, Nom, \Lambda) \in |\mathbb{S}ig^{\mathtt{CIHI}}|$ and $\Sigma \overset{\chi}{\to} \Sigma' \in \mathcal{D}$ we have $(\Sigma', Nom, \Lambda) \in |\mathbb{S}ig^{\mathtt{CIHI}}|$, and
(4) for any hybrid substitution $\Theta = \{(\Sigma \overset{\chi_1}{\to} \Sigma_1) \overset{\theta_k}{\to} (\Sigma \overset{\chi_2}{\to} \Sigma_2)\}_{k \in Nom}$ and sentence $\rho \in \mathbb{S}en_2^{\mathtt{CIHI}}(\Sigma_1, Nom, \Lambda)$ we have $\Theta^k(\rho) \in \mathbb{S}en_2^{\mathtt{CIHI}}(\Sigma_2, Nom, \Lambda)$.

Theorem 7. *If every set of sentences of* $\mathtt{CIHI}_2$ *has an initial* $\mathbb{S}ub$*-reachable model then each set of sentences of* $\mathtt{CIHI}_3$ *has an initial* $\mathbb{S}ub$*-reachable model.*

We apply Theorem 7 on top of $\mathbf{IHFOL}_2$ defined in Subsection 4.3. The institution CIHI is **IHFOL**, and the institution $\mathtt{CIHI}_2$ is $\mathbf{IHFOL}_2$. Note that $\mathbf{IHFOL}_2$ is closed to @, which means that assumption (1) of this subsection holds. The institution $\mathtt{CIHI}_3$ is $\mathbf{IHFOL}_3$, the restriction of **IHFOL** to sentences of the form $(\forall X)\rho$, where X is a finite set of first-order variables and ρ is a sentence in $\mathbf{IHFOL}_2$. [8] This implies that assumption (2) of this subsection holds.

[8] Note that $(\forall X)\rho$ is an abbreviation for $(\forall\chi)\rho$, where $\chi : (\Sigma, Nom, \Lambda) \hookrightarrow (\Sigma[X], Nom, \Lambda) \in \mathcal{Q}^{\mathbf{HFOL}}$ is a signature extension with the finite set of first-order variables X and $\rho \in \mathbb{S}en_2^{\mathbf{IHFOL}}(\Sigma[X], Nom, \Lambda)$.

Since $\mathcal{D}^{\mathtt{HI}} \subseteq \mathbb{S}ig^{\mathbf{IHFOL}}$, assumption (3) of this subsection holds. All sentences of $\mathbf{IHFOL}_2$ are quantifier-free and modal-free, and by applying a hybrid substitution to a $\mathbf{IHFOL}_2$ sentence, the result is also a $\mathbf{IHFOL}_2$ sentence. It follows that assumption (4) of this subsection holds.

Corollary 10. *Any set of* $\mathbf{IHFOL}_3$ *sentences has an initial* $\mathbb{S}ub^{\mathbf{FOL}}$*-reachable model.*

We apply Theorem 6 on top of the institution $\mathbf{IHFOL}_3$ defined above. Let $\mathbf{IHFOL}_4$ be the institution obtained from **IHFOL** by restricting the syntax to sentences of the form $(\forall J)\rho$, where J is a finite set of nominal variables and ρ is a sentence of $\mathbf{IHFOL}_3$.

Corollary 11. *Every set of* $\mathbf{IHFOL}_4$ *sentences has an initial* $\mathbb{S}ub^{\mathbf{FOL}}$*-reachable model.*

We call the $\mathbf{IHFOL}_4$ sentences *hybrid Horn clauses of the institution* **IHFOL**. Defining a paramodulation procedure for $\mathbf{IHFOL}_4$ is future research. However, the results obtained in this paper set the foundation for this direction of research.

Any sentence of the form $(\downarrow j)\rho$ is semantically equivalent to $(\forall j) j \Rightarrow \rho$. It follows that initiality is closed under store $\downarrow$. If $\lambda \in \Lambda_2$ then $[\lambda](\rho)$ is semantically equivalent to $(\forall k)\lambda(k) \Rightarrow @_k\rho$. It follows that initiality is closed under box $\Box$ when $\Lambda_n = \emptyset$ for all $n \neq 2$.

5 Herbrand's Theorem

We prove a version of Herbrand's theorem in the framework of hybrid institutions.

Theorem 8. *Let* $\mathtt{I} = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ *be an institution,* $\mathcal{D} \subseteq \mathbb{S}ig$ *a broad subcategory of signature morphisms,* $\mathbb{S}ub : \mathcal{D}^{op} \to \mathbb{CAT}$ *a substitution functor for* $\mathtt{I}$ *and* $\mathcal{Q} \subseteq \mathcal{D}$ *a quantification subcategory. Consider a constrained model functor* $\mathbb{M}od^{\mathtt{CIHI}} : \mathbb{S}ig^{\mathtt{CIHI}} \to \mathbb{CAT}$ *for the injective hybridisation* $\mathtt{IHI}$ *such that*

(1) for any $(\Sigma, Nom, \Lambda) \in |\mathbb{S}ig^{\mathtt{CIHI}}|$ *and* $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$ *we have* $(\Sigma', Nom, \Lambda) \in |\mathbb{S}ig^{\mathtt{CIHI}}|$.

Assume a sub-functor $(\mathbb{S}en_b^{\mathtt{CIHI}} : \mathbb{S}ig^{\mathtt{CIHI}} \to \mathbb{S}et)$ *of* $\mathbb{S}en^{\mathtt{CIHI}}$ *such that*

(2) any $B \subseteq \mathbb{S}en_b^{\mathtt{CIHI}}(\Sigma, Nom, \Lambda)$ *is basic in* $\mathtt{HI}$*, where* $(\Sigma, Nom, \Lambda) \in |\mathbb{S}ig^{\mathtt{CIHI}}|$.

Let $\Delta = (\Sigma, Nom, \Lambda) \in |\mathbb{S}ig^{\mathtt{CIHI}}|$ *be a signature,* $k \in Nom$ *a nominal,* $\Gamma \subseteq \mathbb{S}en^{\mathtt{CIHI}}(\Delta)$ *a set of sentences that has an initial* $\mathbb{S}ub$*-reachable model* $(\mathcal{M}^{\Gamma}, R^{\Gamma}) \in |\mathbb{M}od^{\mathtt{CIHI}}(\Delta)|$*, and* $(\exists J)(\exists \chi)\rho \in \mathbb{S}en^{\mathtt{CIHI}}(\Delta)$ *a sentence such that (a)* J *is a set of nominal variables, (b)* $\Delta \xrightarrow{\chi} \Delta' \in \mathcal{Q}^{\mathtt{HI}}$ *with* $\Delta' = (\Sigma', Nom, \Lambda)$*, and (c)* $\rho \in \mathbb{S}en_b^{\mathtt{CIHI}}(\Delta'[J])$ *with* $\Delta'[J] = (\Sigma', Nom \cup J, \Lambda)$*. Then the following statements are equivalent:*

(i) $\Gamma \models^{\texttt{CIHI}} @_k(\exists J)(\exists \chi)\rho$,
(ii) $(\mathcal{M}^\Gamma, R^\Gamma) \models^{k_{(R^\Gamma)}} (\exists J)(\exists \chi)\rho$,
(iii) *there exists a hybrid substitution* $\Theta = \{\theta_j : (\Sigma \xrightarrow{\chi} \Sigma') \to (\Sigma \xrightarrow{\varphi} \Sigma'')\}_{j \in Nom}$ *and a nominal substitution* $\psi : J \to Nom$ *such that* $\Gamma \models^{\texttt{CIHI}} @_k(\forall \varphi)\Theta^k(\psi(\rho))$ *and* $\varphi : (\Sigma, Nom, \Lambda) \to (\Sigma'', Nom, \Lambda)$ *is conservative in* CIHI.

The pair of substitutions $\langle \psi, \Theta \rangle$ from the statement (iii) of Theorem 8 are called *solutions*. The sentence $@_k(\exists J)(\exists \chi)\rho$ is a *query*. The implication $(i) \Rightarrow (iii)$ reduces the satisfiability of a query by a program (represented here by a hybrid theory) to the search of a pair of substitutions, while the converse implication $(iii) \Rightarrow (i)$ shows that solutions are sound with respect to the given program. We apply Theorem 8 to **IHFOL**. Since $\mathcal{D}^{\mathbf{HFOL}} \subseteq \mathbb{S}ig^{\mathbf{IHFOL}}$ the second hypothesis holds. The functor $\mathbb{S}en_b^{\texttt{CIHI}}$ is $(\mathbb{S}en_b^{\mathbf{IHFOL}} : \mathbb{S}ig^{\mathbf{IHFOL}} \to \mathbb{S}et)$ such that for each $(\Sigma, Nom, \Lambda) \in |\mathbb{S}ig^{\mathbf{IHFOL}}|$ the set $\mathbb{S}en_b^{\mathbf{IHFOL}}(\Sigma, Nom, \Lambda)$ consists of finite conjunctions of sentences in $\mathbb{S}en_0^{\mathbf{HFOL}}(\Sigma, Nom, \Lambda)$. By Corollary 3, any set of sentences in $\mathbf{HFOL}_0$ is epi basic in **HFOL**, which implies that any conjunction of sentences in $\mathbf{HFOL}_0$ is also epi basic in **HFOL**. It follows that condition (2) of Theorem 8 holds. By Corollary 11, any set of hybrid Horn clauses in $\mathbf{IHFOL}_4$ has an initial model. Note that for any nominal k, we have $(\exists J)(\exists X)\rho \models\!\mid^{\mathbf{HFOL}} @_k(\exists J)(\exists X)\rho$, which implies $(\exists J)(\exists X)\rho \models\!\mid^{\mathbf{IHFOL}} @_k(\exists J)(\exists X)\rho$. In **IHFOL**, the queries are sentences of the form $(\exists J)(\exists X)\rho$, where ρ is a finite conjunction of $\mathbf{HFOL}_0$ sentences.

Corollary 12. *For any set of sentences* $\Gamma \subseteq \mathbb{S}en^{\mathbf{IHFOL}_4}((S, F, P), Nom, \Lambda)$, *where* $((S, F, P), Nom, \Lambda) \in |\mathbb{S}ig^{\mathbf{IHFOL}}|$, *and any query* $(\exists J)(\exists X)\rho$, *where* ρ *is a finite conjunction of sentences in* $\mathbb{S}en_0^{\mathbf{HFOL}}((S, F \cup X, P), Nom \cup J, \Lambda)$ *the followings are equivalent:*

(i) $\Gamma \models^{\mathbf{IHFOL}} (\exists J)(\exists X)\rho$,
(ii) $(\mathcal{M}^\Gamma, R^\Gamma) \models^{\mathbf{IHFOL}} (\exists J)(\exists X)\rho$,
(iii) *there exists a hybrid substitution* $\Theta = \{\theta_j : X \to T_{(S,F,P)}(Y)\}_{j \in Nom}$ *and a nominal substitution* $\psi : J \to Nom$ *such that* $\Gamma \models^{\mathbf{IHFOL}} (\forall Y)\Theta^k(\psi(\rho))$ *for some* $k \in Nom$ *and the sorts of variables in* Y *are inhabited, i.e. for any sort* $s \in S$ *and variable* $y \in Y_s$ *there exists a term* $t \in T_{(S,F,P)}$.

The inhabitation requirement for the sorts of the variables in Y means that the inclusion $\iota_y : ((S, F, P), Nom, \Lambda) \to ((S, F \cup Y, P), Nom, \Lambda)$ is conservative. The restriction to injective hybridisations required by Theorem 8 is not needed if the quantification subcategory consists of identities. For example, one can prove a version of Herbrand's theorem for hybrid institutions that can be instantiated to **HPL**.

6 Conclusions

In this paper we have proved the existence of initial models of hybrid Horn clauses. Our initiality results are not based on inclusion systems and quasi-varieties as in [6]. The proof follows the structure of the sentences in the style of [9]. We assume that the atomic sentences of the base institution are epi basic and then the initiality property is proved to be closed under certain sentence building operators. This approach requires less model theoretic infrastructure than [6] and it can be applied to theories for which the corresponding class of models does not form a quasi-variety. We have developed denotational foundations for logic programming in hybrid logics independently of the details of the underlying base institution by employing institutional concepts of quantification, substitution, reachable model and basic set of sentences. In this general setting we have proved Herbrand's theorem. A future direction of research is developing a paramodulation procedure for hybrid logics. The results presented in this paper which do not involve inherited quantification can be applied to hybrid logics with model constraints [6], but much work is needed to cover the rigid quantification [2]. This constitutes another future direction of research.

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7 Exiled Proofs

Proof (Lemma 1). Let B be a set of atomic (S, F, P)-sentences in **FOL**. The basic model M^B is the initial model of B and it is constructed as follows: on the quotient $T_{(S,F)}/_{\equiv_B}$ of the term model $T_{(S,F)}$ by the congruence generated by the equational atoms of B, we interpret each relation symbol $\pi \in P$ by $\pi_{M^B} = \{(\widehat{t_1}, \ldots, \widehat{t_n}) \mid \pi(t_1, \ldots, t_n) \in B\}$, where $\widehat{t}$ is the congruence class of t for all terms $t \in T_{(S,F)}$. □

Proof (Proposition 2). We prove the assertion by induction on the structure of sentences.

- **For** $p \in Nom$**:** $(\mathcal{M}^2, R) \models^{k_R} \Theta^k(p)$ iff $(\mathcal{M}^2, R) \models^{k_R} p$ iff $p_R = k_R$ iff $(\mathcal{M}^2, R)\upharpoonright_{\Theta^j} \models^{k_R} p$.
- **For** $\lambda(k_1, \ldots, k_n)$**:** $(\mathcal{M}^2, R) \models^{k_R} \Theta^k(\lambda(k_1, \ldots, k_n))$ iff $(\mathcal{M}^2, R) \models^{k_R} \lambda(k_1, \ldots, k_n)$ iff $(k_R, (k_1)_R, \ldots, (k_n)_R) \in \lambda_R$ iff $(\mathcal{M}^2, R)\upharpoonright_{\Theta^j} \models^{k_R} \lambda(k_1, \ldots, k_n)$.
- **For** $\rho \in \mathbb{S}en^{\mathtt{I}}(\Sigma_1)$**:** $(\mathcal{M}^2, R) \models^{k_R} \Theta^k(\rho)$ iff $(\mathcal{M}^2, R) \models^{k_R} \theta_k(\rho)$ iff $\mathcal{M}^2_{k_R} \models^{\mathtt{I}} \theta_k(\rho)$ iff $\mathcal{M}^2_{k_R}\upharpoonright_{\theta_k} \models^{\mathtt{I}} \rho$ iff $(\mathcal{M}^2, R)\upharpoonright_{\Theta_j} \models^{k_R} \rho$.
- **For** $\neg\rho$**:** $(\mathcal{M}^2, R) \models^{k_R} \Theta^k(\neg\rho)$ iff $(\mathcal{M}^2, R) \models^{k_R} \neg\Theta^k(\rho)$ iff $(\mathcal{M}^2, R) \not\models^{k_R} \Theta^k(\rho)$ iff $(\mathcal{M}^2, R)\upharpoonright_{\Theta_j} \not\models^{k_R} \rho$ iff $(\mathcal{M}^2, R)\upharpoonright_{\Theta_j} \models^{k_R} \neg\rho$.
- **For** $\rho_1 \wedge \rho_2$**:** $(\mathcal{M}^2, R) \models^{k_R} \Theta^k(\rho_1 \wedge \rho_2)$ iff $(\mathcal{M}^2, R) \models^{k_R} \Theta^k(\rho_1) \wedge \Theta^k(\rho_2)$ iff $(\mathcal{M}^2, R) \models^{k_R} \Theta^k(\rho_1)$ and $(\mathcal{M}^2, R) \models^{k_R} \Theta^k(\rho_2)$ iff $(\mathcal{M}^2, R)\upharpoonright_{\Theta_j} \models^{k_R} \rho_1$ and $(\mathcal{M}^2, R)\upharpoonright_{\Theta_j} \models^{k_R} \rho_2$ iff $(\mathcal{M}^2, R)\upharpoonright_{\Theta_j} \models^{k_R} \rho_1 \wedge \rho_2$.
- **For** $@_j\rho$**:** $(\mathcal{M}^2, R) \models^{k_R} \Theta^k(@_j\rho)$ iff $(\mathcal{M}^2, R) \models^{k_R} @_j\Theta^j(\rho)$ iff $(\mathcal{M}^2, R) \models^{j_R} \Theta^j(\rho)$ iff $(\mathcal{M}^2, R)\upharpoonright_{\Theta^j} \models^{j_R} \rho$ iff $(\mathcal{M}^2, R)\upharpoonright_{\Theta^j} \models^{k_R} @_j\rho$.

□

Proof (Corollary 2). Let $(\mathcal{M}, R) \in |\mathbb{M}od^{\mathtt{IHI}}(\Delta)|$ be a model that satisfies $(\forall\varphi_1)\rho$. Assume a φ_2-expansion $(\mathcal{M}^2, R)$ of $(\mathcal{M}, R)$. Let $j \in Nom$ be an arbitrary nominal. By Lemma 3, $(\mathcal{M}^2, R)\upharpoonright_{\Theta^j}$ is a φ_1-expansion of $(\mathcal{M}, R)$. By our assumptions, $(\mathcal{M}^2, R)\upharpoonright_{\Theta^j} \models \rho$. For each $k \in Nom$, we have $(\mathcal{M}^2, R)\upharpoonright_{\Theta^j} \models^{k_R} \rho$, and by Proposition 2, we get $(\mathcal{M}^2, R) \models^{k_R} \Theta^k(\rho)$. □

Proof (Lemma 4).

(1) Let $\varphi : (\Sigma, Nom \cup \{j\}, \Lambda) \to (\Sigma, Nom, \Lambda) \in \mathbb{S}ig^{\mathtt{HI}}$ be the signature morphism which is the identity on all symbols except k and such that $\varphi(j) = k$. Consider a model $(\mathcal{M}, R) \in |\mathbb{M}od^{\mathtt{HI}}(\Sigma, Nom, \Lambda)|$ and a state $s \in |R|$ such that $(\mathcal{M}, R) \models^s (\forall j)\rho$. Since $\iota_j; \varphi = 1_{(\Sigma, Nom, \Lambda)}$, it follows that $(\mathcal{M}, R)\upharpoonright_\varphi$ is a ι_j-expansion of $(\mathcal{M}, R)$. We have $(\mathcal{M}, R)\upharpoonright_\varphi \models^s \rho$, and by the satisfaction condition, $(\mathcal{M}, R) \models^s \varphi(\rho)$. Since $\varphi(\rho) = \rho[j \leftarrow k]$, we get $(\mathcal{M}, R) \models^s \rho[j \leftarrow k]$.

(2) For every $(\mathcal{M}, R) \in |\mathbb{M}od^{\mathtt{HI}}(\Sigma, Nom, \Lambda)|$ such that $(\mathcal{M}, R) \models (\downarrow j)\rho$, we have $(\mathcal{M}, R) \models^{k_R} \rho[j \leftarrow k]$, which implies $(\mathcal{M}, R) \models^{\mathtt{HI}} @_k\rho[j \leftarrow k]$.

□

Proof (Proposition 3). Let $\chi : (\Sigma, Nom, \Lambda) \to (\Sigma', Nom, \Lambda) \in \mathcal{D}^{\mathtt{HI}}$ and assume a χ-expansion $(\mathcal{M}', R) \in |\mathbb{M}od^{\mathtt{IHI}}(\Sigma', Nom, \Lambda)|$ of $(\mathcal{M}, R)$. For any nominal

$k \in Nom$, since $\mathcal{M}'_{k_R}$ is an expansion of $\mathcal{M}_{k_R}$ along $\Sigma \xrightarrow{\chi} \Sigma' \in \mathcal{D}$, there exists $\theta_k : \chi \to 1_\Sigma$ such that $\mathcal{M}_{k_R} \upharpoonright_{\theta_k} = \mathcal{M}'_{k_R}$. Hence, $\Theta = \{\chi \xrightarrow{\theta_k} 1_\Sigma\}_{k \in Nom}$ is a hybrid substitution such that $(\mathcal{M}, R)\upharpoonright_{\Theta^j} = (\mathcal{M}', R)$ for all nominals $j \in Nom$.
□

Proof (Theorem 3). Given a set of sentences $\Gamma \subseteq \mathbb{S}en_0^{\mathtt{HI}}(\Sigma, Nom, \Lambda)$, where $(\Sigma, Nom, \Lambda) \in |\mathbb{S}ig^{\mathtt{HI}}|$, we construct a basic model $\mathcal{M}^\Gamma \in |\mathbb{M}od^{\mathtt{HI}}(\Sigma, Nom, \Lambda)|$ for Γ as follows:

- $|R^\Gamma|$ is $Nom/_\equiv$, where $\equiv$ is the congruence relation on nominals defined by $\{j \equiv k \mid \Gamma \models @_j k\}$,
- $R^\Gamma_\lambda = \{(\widehat{j}, \widehat{k_1}, \ldots, \widehat{k_n}) \mid \Gamma \models^{\mathtt{HI}} @_j \lambda(k_1, \ldots, k_n)\}$ for all $\lambda \in \Lambda_{n+1}$, where for all $k \in Nom$ we denote by $\widehat{k}$ the congruence class of k,
- for all $k \in Nom$ let $\mathcal{M}^\Gamma_{\widehat{j}} = M^{B_{\widehat{j}}}$, where
  - $B_{\widehat{j}} = \{\rho \in \mathbb{S}en_0(\Sigma) \mid \Gamma \models^{\mathtt{HI}} @_j \rho\}$ and
  - $M^{B_{\widehat{j}}}$ is a basic model of $B_{\widehat{j}}$ in $\mathtt{I}$.

We prove that for all models $(\mathcal{M}, R) \in |\mathbb{M}od^{\mathtt{HI}}(\Sigma, Nom, \Lambda)|$ we have

$$(\mathcal{M}, R) \models^{\mathtt{HI}} \Gamma \text{ iff there exists a unique arrow } (\mathcal{M}^\Gamma, R^\Gamma) \to (\mathcal{M}, R)$$

"⇒" For the direct implication assume that $(\mathcal{M}, R) \models^{\mathtt{HI}} \Gamma$.

(1) If $\Gamma \models^{\mathtt{HI}} @_j k$ then $(\mathcal{M}, R) \models^{\mathtt{HI}} @_j k$; it follows that there exists a unique function $h^{st} : |R^\Gamma| \to |R|$ defined by $h^{st}(\widehat{k}) = k_R$ for all $k \in Nom$.
(2) If $(\widehat{j}, \widehat{k_1}, \ldots, \widehat{k_n}) \in \lambda_{R^\Gamma}$ then by the definition of the **REL**-model R^Γ, we get $\Gamma \models^{\mathtt{HI}} @_j \lambda(k_1, \ldots, k_n)$; it follows that $(\mathcal{M}, R) \models^{\mathtt{HI}} @_j \lambda(k_1, \ldots, k_n)$, which implies $(\mathcal{M}, R) \models^{j_R} \lambda(k_1, \ldots, k_n)$, and we get $(j_R, (k_1)_R, \ldots (k_n)_R) \in \lambda_R$. Hence, h^{st} is a **REL** homomorphism.
(3) Let $j \in Nom$; since $\Gamma \models^{\mathtt{HI}} @_j \rho$ implies $(\mathcal{M}, R) \models^{\mathtt{HI}} @_j \rho$ for all $\rho \in \mathbb{S}en_0^{\mathtt{I}}(\Sigma)$, we have that $\mathcal{M}_{j_R} \models^{\mathtt{I}} B_{\widehat{j}}$; it follows that there exists a unique arrow $h^{mod}_{\widehat{j}} : M^{B_{\widehat{j}}} \to M_{j_R}$. We define $h^{mod} = \{h^{mod}_{\widehat{j}} : M^{B_{\widehat{j}}} \to M_{j_R}\}_{\widehat{j} \in |R^\Gamma|}$.

"⇐" For the converse implication assume there exists a unique arrow $h : (\mathcal{M}^\Gamma, R^\Gamma) \to (\mathcal{M}, R)$. We prove the following statements:

(1) *$\Gamma \models^{\mathtt{HI}} @_j k$ implies $(\mathcal{M}, R) \models^{\mathtt{HI}} @_j k$ for all $k \in Nom$*: if $\Gamma \models^{\mathtt{HI}} @_j k$ then $j \equiv k$, and we have $h^{st}(\widehat{j}) = h^{st}(\widehat{k})$, which implies $j_R = k_R$; it follows that $(\mathcal{M}, R) \models^{j_R} k$, meaning that $(\mathcal{M}, R) \models^{\mathtt{HI}} @_j k$.
(2) *$\Gamma \models^{\mathtt{HI}} @_j \lambda(k_1, \ldots, k_n)$ implies $(\mathcal{M}, R) \models^{\mathtt{HI}} @_j \lambda(k_1, \ldots, k_n)$ for all $\lambda \in \Lambda_{n+1}$, $j \in Nom$, $k_i \in Nom$ and $i \in \{1, \ldots, n\}$*: if $\Gamma \models^{\mathtt{HI}} @_j \lambda(k_1, \ldots, k_n)$ then $(\widehat{j}, \widehat{k_1}, \ldots, \widehat{k_n}) \in \lambda_{R^\Gamma}$ and $(h^{st}(\widehat{j}), h^{st}(\widehat{k_1}), \ldots, h^{st}(\widehat{k_n})) \in \lambda_R$, which implies $(j_R, (k_1)_R, \ldots, (k_n)_R) \in \lambda_R$; it follows that $(\mathcal{M}, R) \models^{j_R} \lambda(k_1, \ldots, k_n)$, meaning that $(\mathcal{M}, R) \models^{\mathtt{HI}} @_j \lambda(k_1, \ldots, k_n)$.

(3) $\Gamma \models^{\mathtt{HI}} @_j\rho$ *implies* $(\mathcal{M}, R) \models^{\mathtt{HI}} @_j\rho$ *for all* $j \in Nom$ *and* $\rho \in \mathbb{S}en_0^{\mathtt{I}}(\Sigma)$: if $\Gamma \models^{\mathtt{HI}} @_j\rho$ then $\rho \in B_{\widehat{j}}$, and since there exists a unique arrow $h_{\widehat{j}} : M^{B_{\widehat{j}}} \to M_{j_R}$ (in I), it follows that $M_{j_R} \models^{\mathtt{I}} \rho$, which implies $(\mathcal{M}, R) \models^{j_R} \rho$; hence $(\mathcal{M}, R) \models^{\mathtt{HI}} @_j\rho$.

For any $\rho \in \Gamma$, we have $\Gamma \models^{\mathtt{HI}} \rho$, and by the above statements, $(\mathcal{M}, R) \models^{\mathtt{HI}} \rho$.

If each set of sentences in $\mathtt{I}_0$ has a basic model that is $\mathbb{S}ub$-reachable, then we may assume that $\mathcal{M}^{\Gamma}_{\widehat{j}} = M^{B_{\widehat{j}}}$ is $\mathbb{S}ub$-reachable for all $j \in Nom$. It follows that $(\mathcal{M}^{\Gamma}, R^{\Gamma})$ is $\mathbb{S}ub$-reachable too. □

Proof (Theorem 4). We define the basic model $(\mathcal{M}^{\Gamma}, R^{\Gamma})$ as in the proof of Theorem 3. Note that $\equiv$ consists of identities, i.e. $|R^{\Gamma}| = Nom$, which implies $(\mathcal{M}^{\Gamma}, R^{\Gamma}) \in |\mathbb{M}od^{\mathtt{IHI}}(\Sigma, Nom, \Lambda)|$. It follows that Γ is epi basic in the injective hybridisation IHI. If each set of sentences in $\mathtt{I}_0$ has a basic model that is $\mathbb{S}ub$-reachable, then we may assume that $\mathcal{M}^{\Gamma}_{\widehat{j}} = M^{B_{\widehat{j}}}$ is $\mathbb{S}ub$-reachable for all $j \in Nom$; it follows that $(\mathcal{M}^{\Gamma}, R^{\Gamma})$ is $\mathbb{S}ub$-reachable. □

Proof (Theorem 5). Consider a signature $\Delta \in |\mathbb{S}ig^{\mathtt{CHI}}|$ and a set of sentences $\Gamma \subseteq \mathbb{S}en_1^{\mathtt{CHI}}(\Delta)$. We define the set of sentences $\Gamma_{\bullet} = \{\rho \in \mathbb{S}en_{\bullet}^{\mathtt{CHI}}(\Delta) \mid \Gamma \models^{\mathtt{CHI}} \rho\}$. Let $(\mathcal{M}^{\Gamma}, R^{\Gamma}) \in |\mathbb{M}od^{\mathtt{CHI}}(\Delta)|$ be the initial model of $\Gamma_{\bullet}$ in the institution $\mathtt{CHI}_{\bullet} = (\mathbb{S}ig^{\mathtt{CHI}}, \mathbb{S}en_{\bullet}^{\mathtt{CHI}}, \mathbb{M}od^{\mathtt{CHI}}, \models^{\mathtt{CHI}})$. Note that for all $(\mathcal{M}, R) \in |\mathbb{M}od^{\mathtt{CHI}}(\Delta)|$ which satisfies Γ there exists a unique homomorphisms $(\mathcal{M}^{\Gamma}, R^{\Gamma}) \to (\mathcal{M}, R)$ as $(\mathcal{M}, R) \models^{\mathtt{CHI}} \Gamma_{\bullet}$. If we show that $(\mathcal{M}^{\Gamma}, R^{\Gamma}) \models^{\mathtt{CHI}} \Gamma$ then it follows that $(\mathcal{M}^{\Gamma}, R^{\Gamma})$ is the initial model of Γ in $\mathtt{CHI}_1$.

Let $\gamma \in \Gamma$. By our assumptions, there exists $\bigwedge H \Rightarrow C \in \mathbb{S}en^{\mathtt{CHI}}(\Delta)$, with $H \subseteq \mathbb{S}en_*^{\mathtt{CHI}}(\Delta)$ and $C \in \mathbb{S}en_{\bullet}^{\mathtt{CHI}}(\Delta)$, such that $\gamma \models\!\mid^{\mathtt{CHI}} \bigwedge H \Rightarrow C$. We assume $(\mathcal{M}^{\Gamma}, R^{\Gamma}) \models^{\mathtt{CHI}} H$ and we focus on proving $(\mathcal{M}^{\Gamma}, R^{\Gamma}) \models^{\mathtt{CHI}} C$. Since H is basic in HI, $\Gamma \models^{\mathtt{CHI}} H$. (Indeed, for any $(\mathcal{M}, R) \in |\mathbb{M}od^{\mathtt{CHI}}(\Delta)|$ that satisfy Γ, by the definition of $(\mathcal{M}^{\Gamma}, R^{\Gamma})$, there exists a unique arrow $(\mathcal{M}^{\Gamma}, R^{\Gamma}) \xrightarrow{h} (\mathcal{M}, R) \in \mathbb{M}od^{\mathtt{CHI}}(\Delta)$; since $(\mathcal{M}^{\Gamma}, R^{\Gamma}) \models^{\mathtt{HI}} H$ and H is basic in HI, there exists an arrow $(\mathcal{M}^{H}, R^{H}) \xrightarrow{g} (\mathcal{M}^{\Gamma}, R^{\Gamma}) \in \mathbb{M}od^{\mathtt{HI}}(\Delta)$, where $(\mathcal{M}^{H}, R^{H}) \in |\mathbb{M}od^{\mathtt{HI}}(\Delta)|$ is a basic model of H; we have $(\mathcal{M}^{H}, R^{H}) \xrightarrow{g;h} (\mathcal{M}, R) \in \mathbb{M}od^{\mathtt{HI}}(\Delta)$, and since H is basic in HI, $(\mathcal{M}, R) \models^{\mathtt{HI}} H$, which implies $(\mathcal{M}, R) \models^{\mathtt{CHI}} H$.) We have $\Gamma \models^{\mathtt{CHI}} H$ and $\bigwedge H \Rightarrow C \in \Gamma$, which implies $\Gamma \models^{\mathtt{CHI}} C$. It follows that $C \in \mathbb{S}en_{\bullet}^{\mathtt{CHI}}(\Delta)$, and we get $(\mathcal{M}^{\Gamma}, R^{\Gamma}) \models^{\mathtt{CHI}} C$. □

Proof (Theorem 6). Let $\Delta = (\Sigma, Nom, \Lambda) \in |\mathbb{S}ig^{\mathtt{CHI}}|$ be a signature, and $\Gamma \subseteq \mathbb{S}en_2^{\mathtt{CHI}}(\Delta)$ a set of sentences. We define $\Gamma_1 = \{\rho \in \mathbb{S}en_1^{\mathtt{CHI}}(\Delta) \mid \Gamma \models^{\mathtt{CHI}} \rho\}$. Let $(\mathcal{M}^{\Gamma}, R^{\Gamma})$ be the initial model of Γ_1 in $\mathtt{CHI}_1$. Note that for all $(\mathcal{M}, R) \in |\mathbb{M}od^{\mathtt{CHI}}(\Delta)|$ which satisfies Γ there exists a unique homomorphisms $(\mathcal{M}^{\Gamma}, R^{\Gamma}) \to (\mathcal{M}, R) \in \mathbb{M}od^{\mathtt{CHI}}(\Delta)$. If we show that $(\mathcal{M}^{\Gamma}, R^{\Gamma}) \models^{\mathtt{CHI}} \Gamma$ then we have proved that $(\mathcal{M}^{\Gamma}, R^{\Gamma})$ is the initial model of Γ.

Let $\gamma \in \Gamma$. By our assumptions there exists $(\forall j)\rho \in \Gamma$ (where j is a nominal variable, $\rho \in \mathbb{S}en_1^{\mathtt{CHI}}(\Delta[j])$ and $\Delta[j] = (\Sigma, Nom \cup \{j\}, \Lambda)$) such that $\gamma \models\!\mid^{\mathtt{CHI}} (\forall j)\rho$. We focus on proving $(\mathcal{M}^{\Gamma}, R^{\Gamma}) \models^{\mathtt{CHI}} (\forall j)\rho$. Since $(\mathcal{M}^{\Gamma}, R^{\Gamma})$ is $\mathbb{S}ub$-reachable, it suffices to prove that for any $k \in Nom$ we have $(\mathcal{M}^\Gamma, R^\Gamma) \models^{\texttt{CHI}} \rho[j \to k]$. Consider an arbitrary nominal $k \in Nom$. By Lemma 4, $\Gamma \models^{\texttt{HI}} \rho[j \to k]$, which implies $\Gamma \models^{\texttt{CHI}} \rho[j \to k]$, and we get $\rho[j \to k] \in \Gamma_1$. It follows that $(\mathcal{M}^\Gamma, R^\Gamma) \models \rho[j \to k]$. □

Proof (Corollary 8). By induction on the number of variables used for quantification. By Corollary 8, the induction base holds. For the induction step, for any natural number n, let $\mathbf{HFOL}_3^n$ be the restriction of $\mathbf{HFOL}_3$ to sentences of the form $(\forall J)\rho$, where J consists of at most n nominal variables and ρ is a sentence of $\mathbf{HFOL}_2$; then assuming that any set of $\mathbf{HFOL}_3^n$ has an initial $\mathbb{S}ub^{\mathbf{FOL}}$-reachable model, by Theorem 7, any set of $\mathbf{HFOL}_3^{n+1}$ sentences has an initial $\mathbb{S}ub^{\mathbf{FOL}}$-reachable model. □

Proof (Theorem 7). Let $\Delta = (\Sigma, Nom, \Lambda) \in |\mathbb{S}ig^{\texttt{CIHI}}|$ and $\Gamma \subseteq \mathbb{S}en_2^{\texttt{CIHI}}(\Delta)$. We define $\Gamma_1 = \{\rho \in \mathbb{S}en_1^{\texttt{CIHI}}(\Delta) \mid \Gamma \models^{\texttt{CIHI}} \rho\}$. Let $(\mathcal{M}^\Gamma, R^\Gamma) \in |\mathbb{M}od^{\texttt{CIHI}}(\Delta)|$ be the initial model of Γ_1 in $\texttt{CIHI}_1$. Note that for all $(\mathcal{M}, R) \in |\mathbb{M}od^{\texttt{CIHI}}(\Delta)|$ which satisfies Γ there exists a unique homomorphisms $(\mathcal{M}^\Gamma, R^\Gamma) \to (\mathcal{M}, R)$. If we show $(\mathcal{M}^\Gamma, R^\Gamma) \models^{\texttt{CIHI}} \Gamma$ then we have proved that $(\mathcal{M}^\Gamma, R^\Gamma)$ is the initial model of Γ in $\texttt{CHI}_1$.

Let $\gamma \in \Gamma$. By our assumptions there exists $(\forall \chi)\rho \in \Gamma$, where $\Delta \overset{\chi}{\to} \Delta' \in \mathcal{Q}^{\texttt{HI}}$, $\Delta' = (\Sigma', Nom, \Lambda)$ and $\rho \in \mathbb{S}en_2^{\texttt{CIHI}}(\Delta')$ such that $\gamma \models\!\!\mid^{\texttt{CIHI}} (\forall \chi)\rho$. We focus on proving $(\mathcal{M}^\Gamma, R^\Gamma) \models^{\texttt{CIHI}} (\forall \chi)\rho$. Let $(\mathcal{M}', R^\Gamma)$ be a χ-expansion of $(\mathcal{M}^\Gamma, R^\Gamma)$ and $s \in |\mathcal{R}^\Gamma|$. Since $(\mathcal{M}^\Gamma, R^\Gamma)$ is $\mathbb{S}ub$-reachable, there exists $k \in Nom$ such that $s = k_{(R^\Gamma)}$. By Proposition 3, there exists a hybrid substitution $\Theta = \{(\Sigma \overset{\chi}{\to} \Sigma') \overset{\theta_j}{\to} 1_\Sigma\}_{j \in Nom}$ such that $(\mathcal{M}^\Gamma, R^\Gamma) \upharpoonright_{\Theta^j} = (\mathcal{M}', R^\Gamma)$ for all $j \in Nom$. By Corollary 2, $\Gamma \models^{\texttt{IHI}} @_k \Theta^k(\rho)$, which implies $\Gamma \models^{\texttt{CIHI}} @_k \Theta^k(\rho)$. By the second assumption made at the beginning of this subsection, $@_k \Theta^k(\rho) \in \Gamma_1$. We have $(\mathcal{M}^\Gamma, R^\Gamma) \models^{k_{(R^\Gamma)}} \Theta^k(\rho)$, and by Proposition 2, $(\mathcal{M}', R^\Gamma) \models^{k_{(R^\Gamma)}} \rho$. □

Proof (Corollary 11). By induction on the number of variables used for quantification. By Corollary 10, the induction base holds. For the induction step, for any natural number n, let $\mathbf{IHFOL}_4^n$ be the institution obtained from $\mathbf{HFOL}_4$ by restricting the syntax to sentences of the form $(\forall J)\rho$, where J consists of at most n nominal variables and ρ is a sentence of $\mathbf{IHFOL}_3$; then assuming that any set of $\mathbf{IHFOL}_4^n$ sentences has an initial $\mathbb{S}ub^{\mathbf{FOL}}$-reachable model, by Theorem 6, any set of $\mathbf{IHFOL}_4^{n+1}$ sentences has an initial $\mathbb{S}ub^{\mathbf{FOL}}$-reachable model. □

Proof (Theorem 8).

$(i) \Rightarrow (ii)$ Obvious since $(\mathcal{M}^\Gamma, R^\Gamma) \models^{\texttt{CIHI}} \Gamma$.

$(ii) \Rightarrow (iii)$ Since $(\mathcal{M}^\Gamma, R^\Gamma) \models^{k_{(R^\Gamma)}} (\exists J)(\exists \chi)\rho$, there exists a χ-expansion $(\mathcal{M}', R^\Gamma)$ of $(\mathcal{M}^\Gamma, R^\Gamma)$ and a function $\psi : J \to Nom$ such that $(\mathcal{M}', R^\Gamma) \models^{k_{(R^\Gamma)}} \psi(\rho)$. By the reachability of $(\mathcal{M}^\Gamma, R^\Gamma)$, there exists a hybrid substitution $\Theta = \{\theta_j : \chi \to 1_\Sigma\}_{j \in Nom}$ such that $(\mathcal{M}^\Gamma, R^\Gamma) \upharpoonright_{\Theta^j} = (\mathcal{M}', R^\Gamma)$ for all nominals $j \in Nom$. By Proposition 2, $(\mathcal{M}^\Gamma, R^\Gamma) \models^{k_{(R^\Gamma)}} \Theta^k(\psi(\rho))$. Since $(\mathcal{M}^\Gamma, R^\Gamma)$ is the initial model of Γ in $\texttt{CIHI}$, for any $(\mathcal{M}, R) \in |\mathbb{M}od^{\texttt{CIHI}}(\Delta)|$ that satisfies Γ

there exists a unique arrow $(\mathcal{M}^{\Gamma}, R^{\Gamma}) \to (\mathcal{M}, R)$, and since $\Theta^k(\psi(\rho))$ is basic, $(\mathcal{M}, R) \models^{k_R} \Theta^k(\psi(\rho))$. It follows that $\Gamma \models^{\texttt{CIHI}} @_k\Theta^k(\psi(\rho))$.

$(iii) \Rightarrow (i)$ Let $(\mathcal{M}, R) \in |\mathbb{M}od^{\texttt{CIHI}}(\Delta)|$ that satisfies Γ. By the conservativity of φ, there exists a φ-expansion $(\mathcal{N}, R) \in |\mathbb{M}od^{\texttt{CIHI}}(\Delta'')|$ of $(\mathcal{M}, R)$, where $\Delta'' = (\Sigma'', Nom, \Lambda)$. Since $\Gamma \models^{\texttt{CIHI}} @_k(\forall\varphi)\Theta^k(\psi(\rho))$, $(\mathcal{N}, R) \models^{k_R} \Theta^k(\psi(\rho))$. By Proposition 2, $(\mathcal{N}, R)\upharpoonright_{\Theta_j} \models^{k_R} \psi(\rho)$ for all nominals $j \in Nom$. Note that $(\mathcal{N}, R)\upharpoonright_{\Theta_j}$ is a χ-expansion of $(\mathcal{M}, R)$ for all nominals $j \in Nom$. It follows that $(\mathcal{M}, R) \models^{k_R} (\exists J)(\exists\chi)\rho$. Since $(\mathcal{M}, R) \in |\mathbb{M}od^{\texttt{CIHI}}(\Delta)|$ is an arbitrary model of Γ, we get $\Gamma \models^{\texttt{CIHI}} @_k(\exists J)(\exists\chi)\rho$. □