Omitting Types Theorem in hybrid dynamic first-order logic with rigid symbols

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Abstract

In the the present contribution, we prove an Omitting Types Theorem (OTT) for an arbitrary fragment of hybrid dynamic first-order logic with rigid symbols (i.e. symbols with fixed interpretations across worlds) closed under *negation* and *retrieve*. The logical framework can be regarded as a parameter and it is instantiated by some well-known hybrid and/or dynamic logics from the literature. We develop a *forcing* technique and then we study a *forcing property* based on local satisfiability, which lead to a refined proof of the OTT. For uncountable signatures, the result requires compactness, while for countable signatures, compactness is not necessary. We apply the OTT to obtain upwards and downwards Löwenheim-Skolem theorems for our logic, as well as a completeness theorem for its *constructor-based* variant.

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1. Introduction

Kripke semantics and hybrid dynamic logics. Modal logics are formalisms for describing and reasoning about multi-graphs. These structures appear naturally in many areas of research. For example, in knowledge representation formalisms, role assertions describe relationships between individuals/objects grouped into classes determined by concepts. Linguistic information can be represented by multi-graphs. Other mathematical entities that can be viewed as multi-graphs are transition systems, derivation trees, semantic networks, etc. Therefore, it is useful to think of a Kripke structure in the following way:

- a frame consisting of a set of nodes together with a family of (typed) edge sets, and
- a mapping from the set of nodes to a class of local models that gives meaning to the nodes.

However, modal logics have no mechanisms for referring to the individual nodes in such structures, which is necessary when they are used as representation formalisms. Hybrid logics increase the expressive power of ordinary modal logics by adding an additional sort of symbols called *nominals* such that each nominal is true relative to exactly one point. The history of hybrid logics goes back to Arthur Prior's work [48]. Further developments can be found in works such as [1, 2, 3, 9]. The research on hybrid logics received an additional boost due to the recent interest in the logical foundations of the *reconfiguration paradigm*. Dynamic logics provide a powerful language for describing programs and reason about their correctness. Logics of programs have the roots in the work in the late 1960s of computer scientists interested in assigning meaning to programming languages and finding a rigorous standard for proofs about the programs. There is a significant body of research on this topic; [46] and [38] are two prominent examples among many others. In the present contribution, we consider a logical system endowed with features from

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both hybrid and dynamic logics, which is built on top of many-sorted first-order logic with equality. Despite its complexity, it displays a certain simplicity due to its modular construction, which is a reminiscent of the hybridization of institutions from [43].

Applications of hybrid dynamic logics. The application domain of the work reported in this contribution refers to a broad range of reconfigurable systems whose states or configurations can be presented explicitly, based on some kind of context-independent data types, and for which we distinguish the computations performed at the local/configuration level from the dynamic evolution of the configurations. This suggests a two-layered approach to the design and analysis of reconfigurable systems, involving:

- *a local view*, which amounts to describing the structural properties of configurations, and
- *a global view*, which corresponds to a specialized language for specifying and reasoning about the way system configurations evolve.

Since configurations can be represented by local models and the dynamic evolution of configurations can be depicted by the accessibility relations of the Kripke structures, hybrid dynamic logics and their fragments are acknowledged as suitable for describing and reasoning about systems with reconfigurable features. In addition, it is well-known (see e.g., [7]) that hybrid logics specialize to temporal logics [37], description logics [5] and feature logics [50]. Therefore, the area of applications of the present work is rather large and it involves knowledge representation, computational linguistics, artificial intelligence, biomedical informatics, semantic networks and ontologies. We recommend [7] for more information on this topic.

Omitting Types Theorem (OTT). Intuitively speaking, in model theory a type is a complete description in the appropriate formal language of a potential element of a model. A model may or may not have elements that satisfy such a description: if it has at least one, we say that it *realizes* the type, if it does not have any, it *omits* the type. Models that realize many types are not difficult to come by in presence of the compactness theorem. But, as Gerald Sacks remarks in [51], it takes a model theorist to *omit* a type. The main tool in this quest is the OTT, which gives sufficient conditions for the existence of models omitting certain types. OTT can be used to construct models in which we have a lot of control over what kind of descriptions the elements of the model satisfy, and such models are typically small (e.g., the standard model of arithmetic, which omits the type $\{x \geq n : n \in \omega\}$). The OTT for countable first-order languages is a result originally from Henkin and Orey [40, 45], and the extension to uncountable languages is due to Chang [10].

In this paper we focus on obtaining an OTT for hybrid dynamic first-order logic with rigid symbols and sufficiently expressive fragments. Observe that an OTT for the full logic would not necessarily have given us the property for its fragments. For this reason, we work within an arbitrary fragment of hybrid dynamic first-order logic with rigid symbols, which can be viewed as a parameter. Thus the generality of our proofs is an important feature, since the parameter is instantiated by many concrete hybrid and/or dynamic logical systems which appear in the literature. We provide a version of OTT for countable languages without any restrictions and a version for uncountable languages provided that the fragment in question is compact. We show that compactness is necessary at least for one fragment of the underlying logic. This situation is similar to that described in a theorem by Lindström for first-order logic with only relational symbols [42]. The OTT for countable first-order languages is a result originally from [22]. The extension of the OTT to uncountable languages is from [10]. One of the best known applications of the OTT is a simple proof of the completeness of ω-logic (a more complex proof without using the OTT can be found in [45]). In the present contribution, we develop this idea further to provide one important application of OTT to computer science, which is described briefly in the following paragraph.

Formal methods practitioners are often interested in properties that are true of a restricted class of models whose elements are reachable by some constructor operations [6, 33, 23]. For this reason, several algebraic specification languages incorporate features to express reachability and to deal with constructors like, for instance, Larch [36], CASL [4] or CITP [35, 31]. This situation is similar to the one in classical model theory, where the models of ω-logic are reachable by the constructors *zero* and *successor*. In the present contribution, the completeness of ω-logic is generalized by replacing the signature of arithmetics with an arbitrary vocabulary for which we distinguish a set of constructor operators. Then we apply OTT to obtained completeness of the logical system resulted from restricting the semantics of the underlying fragment of hybrid dynamic first-order logic with rigid symbols to constructor-based Kripke structures.

In [29], the authors prove a Robinson consistency theorem for a class of many-sorted hybrid logics as a consequence of OTT. An important corollary of this result is an interpolation theorem, which is another rich source of logical results involving composing and decomposing theories.

Institutions. Our approach is rooted in institutional model theory [21], which provides a unifying setting for studying logical systems using category theory. The concept of institution formalizes the intuitive notion of logic, including syntax, semantics and the satisfaction relation between them. The theory of institutions is one major approach in universal logic which promotes the development of logical properties at the most general level of abstraction. However, to make the study available to a broader audience, the authors decided to present the results in a framework given by a concrete logical system, that is, hybrid dynamic first-order logic with rigid symbols. It should be obvious, at least for the experts in institutions, that the main result, OTT, can be easily cast in a more general framework such as the one provided by the definition of stratified institution [19], similarly to the work reported in [28]. Therefore, the area of applications of our results covers a much broader range of hybrid dynamic logics than the one mentioned in the present contribution.

Forcing. OTT is proved in this paper by means of a forcing technique. The basic intuition of the method of forcing is that we build a model for a set T of sentences by considering larger and larger descriptions of the model (in terms of the sentences that it satisfies) which are consistent with T. Forcing was invented by Paul Cohen [12, 13] in set theory to prove the independence of the continuum hypothesis from the other axioms of Zermelo-Fraenkel set theory. Abraham Robinson [49] developed a generalization of the forcing method in model theory which proved to be extremely useful in particular in the context of the model theory of infinitary logic where the central tool of first-order model theory (compactness) failed. It turns out that, for example, in the model theory of first-order logic with countably infinite conjunctions and disjunctions, the central theorem of forcing (the Generic Model Theorem) can be used as fruitful replacement of the compactness theorem, providing proofs of preservation theorems, Craig interpolation theorems, two cardinal theorems and, of course, OTT. With this knowledge at hand, it is natural to use the forcing method in contexts where compactness is not necessarily in the picture such as ours. In institutional model theory, forcing was introduced in [34] to prove a Gödel Completeness Theorem. It was developed further for stratified institutions [28] to prove the completeness of a large class of hybrid logics. The present contribution extends the forcing introduced in [28] to cover logics with both hybrid and dynamic features and studies a forcing property based on local satisfiability to deliver an Omitting Types Theorem.

Structure of the paper. The framework of many-sorted first-order logic in the institutional setting is reviewed in §2. In §3 we introduce all the necessary preliminaries about hybrid dynamic first-order logic with rigid symbols, which expands the base system from §2. Some technical notions necessary for developing our arguments, such as a *reachable model* and a language *fragment*, are dealt with in §4. In §5 we develop the basics of the forcing technique in our present context, and in §6 we present a semantic forcing property, crucial for proving the main result. The main result, an Omitting Types Theorem for both countable and uncountable signatures is given in §7. Next, two applications are given: in §8 we apply the main result to obtain a completeness theorem for the constructor-based variant of the logic, and in §9 we obtain Löwenheim-Skolem theorems (upwards and downwards) as consequences of the OTT. Finally, in §10 we show that for a certain fragment of the logic we consider, compactness is a necessary condition for the OTT for uncountable signatures to hold.

2. Many-sorted first-order logic (FOL)

In this section, we recall the definition of first-order logic as presented in institutional model theory [21].

Signatures. Signatures are of the form (S, F, P), where S is a set of sorts, $F = \{F_{\mathsf{ar}\to s}\}_{(\mathsf{ar},s)\in S^*\times S}$ is a ($S^*\times S$ -indexed) set of operation symbols, and $P = \{P_{\mathsf{ar}}\}_{\mathsf{ar}\in S^*}$ is a (S^*-indexed) set of relation symbols. If $\mathsf{ar} = \varepsilon$ then an element of $F_{\mathsf{ar}\to s}$ is called a *constant symbol*. Generally, ar ranges over arities, which are understood here as strings of sorts; in other words an arity gives the number of arguments together with their sorts. We overload the notation and let F and

P also denote $\biguplus_{(\mathsf{ar},s)\in S^*\times S} F_{\mathsf{ar}\to s}$ and $\biguplus_{\mathsf{ar}\in S^*} P_{\mathsf{ar}}$, respectively. Therefore, we may write $\sigma \in F_{\mathsf{ar}\to s}$ or $(\sigma\colon \mathsf{ar} \to s) \in F$; both have the same meaning, which is: σ is an operation symbol of type $\mathsf{ar} \to s$. Throughout this paper, we let Σ, Σ' and Σ_i to range over first-order signatures of the form (S, F, P), (S', F', P') and (S_i, F_i, P_i), respectively.

Signature morphisms. A number of usual tricks, such as adding constants, but also, importantly, quantification, are viewed as signature expansions, so moving between signatures is common. To make such transitions smooth, a notion of a *signature morphism* is introduced. A signature morphism $\varphi\colon \Sigma \to \Sigma'$ is a triple $\chi = (\chi^{st}, \chi^{op}, \chi^{rl})$ of maps: (a) $\chi^{st}\colon S \to S'$, (b) $\chi^{op} = \{\chi^{op}_{\mathsf{ar}\to s}\colon F_{\mathsf{ar}\to s} \to F'_{\chi^{st}(\mathsf{ar})\to\chi^{st}(s)} \mid \mathsf{ar} \in S^*, s \in S\}$, and (c) $\chi^{rl} = \{\chi^{rl}_{\mathsf{ar}}\colon P_{\mathsf{ar}} \to P'_{\chi^{st}(\mathsf{ar})} \mid \mathsf{ar} \in S^*\}$. When there is no danger of confusion, we may let χ denote either of $\chi^{st}, \chi^{op}_{\mathsf{ar}\to s}, \chi^{rl}_{\mathsf{ar}}$.

Fact 1. First-order signature morphisms form a category $\mathsf{Sig}^{\mathsf{FOL}}$ under the componentwise composition as functions.

Models. Given a signature Σ, a Σ-model is a triple

$$\mathfrak{A} = (\{\mathfrak{A}_s\}_{s\in S}, \{\mathfrak{A}_\sigma\}_{(\mathsf{ar},s)\in S^*\times S, \sigma\in F_{\mathsf{ar}\to s}}, \{\mathfrak{A}_\pi\}_{\mathsf{ar}\in S^*, \pi\in P_{\mathsf{ar}}})$$

interpreting each sort s as a non-empty set $\mathfrak{A}_s$, each operation symbol $\sigma \in F_{\mathsf{ar}\to s}$ as a function $\mathfrak{A}_\sigma\colon \mathfrak{A}_{\mathsf{ar}} \to \mathfrak{A}_s$ (where $\mathfrak{A}_{\mathsf{ar}}$ stands for $\mathfrak{A}_{s_1} \times \ldots \times \mathfrak{A}_{s_n}$ if $\mathsf{ar} = s_1 \ldots s_n$), and each relation symbol $\pi \in P_{\mathsf{ar}}$ as a relation $\mathfrak{A}_\pi \subseteq \mathfrak{A}_{\mathsf{ar}}$. Morphisms between models are the usual Σ-homomorphisms, i.e., S-sorted functions that preserve the structure.

Fact 2. For any signature Σ, the Σ-homomorphisms form a category $\mathsf{Mod}^{\mathsf{FOL}}(\Sigma)$ under the obvious composition as many-sorted functions.

For any signature morphism $\chi\colon \Sigma \to \Sigma'$, the reduct functor $\_\!\upharpoonright_\chi\colon \mathsf{Mod}(\Sigma') \to \mathsf{Mod}(\Sigma)$ is defined as follows:

1. The reduct $\mathfrak{A}'\!\upharpoonright_\chi$ of a Σ'-model $\mathfrak{A}'$ is a defined by $(\mathfrak{A}'\!\upharpoonright_\chi)_x = \mathfrak{A}'_{\chi(x)}$ for each sort $s \in S$, operation symbol $x \in F$ or relation symbol $x \in P$. Note that, unlike the single-sorted case, the reduct functor modifies the universes of models. For the universe of $\mathfrak{A}'\!\upharpoonright_\chi$ is $\{\mathfrak{A}'_{\chi(s)}\}_{s\in S}$, which means that the sorts outside the image of S are discarded. Otherwise, the notion of reduct is standard.
2. The reduct $h'\!\upharpoonright_\chi$ of a homomorphism h' is defined by $(h'\!\upharpoonright_\chi)_s = h'_{\chi(s)}$ for all sorts $s \in S$.

Fact 3. $\mathsf{Mod}^{\mathsf{FOL}}$ becomes a functor $\mathsf{Sig}^{\mathsf{FOL}} \to \mathbb{C}\mathsf{at}^{op}$, with $\mathsf{Mod}^{\mathsf{FOL}}(\chi)(h') = h'\!\upharpoonright_\chi$ for each signature morphism $\chi\colon \Sigma \to \Sigma'$ and each Σ'-homomorphism h'.

Sentences. We assume a countably infinite set of variable names $\{v_i \mid i < \omega\}$. A variable for a signature Σ is a triple $\langle v_i, s, \Sigma\rangle$, where v_i is a variable name, and s is a sort in Σ. Given a signature Σ, the S-sorted set of Σ-terms is denoted by T_Σ. The set $\mathsf{Sen}^{\mathsf{FOL}}(\Sigma)$ of sentences over Σ is given by the following grammar:

$$\gamma ::= t = t' \mid \pi(t_1, \ldots, t_n) \mid \neg\gamma \mid \vee\Gamma \mid \exists X \cdot \gamma'$$

where (a) $t = t'$ is an equation with $t, t' \in T_{\Sigma,s}$ and $s \in S$, (b) $\pi(t_1, \ldots, t_n)$ is a relational atom with $\pi \in P_{s_1\ldots s_n}$, $t_i \in T_{\Sigma,s_i}$ and $s_i \in S$, (c) Γ is a finite set of Σ-sentences, (d) X is a finite set of variables for Σ, (e) γ' is a $\Sigma[X]$-sentence, where $\Sigma[X] = (S, F[X], P)$, and $F[X]$ is the set of function symbols obtained by adding the variables in X as constants to F.

Sentence translations. Quantification comes with some subtle issues related to the translation of quantified sentences along signature morphisms that require a closer look. The translation of a variable $\langle v_i, s, \Sigma\rangle$ along a signature morphism $\chi\colon \Sigma \to \Sigma'$ is $\langle v_i, \chi(s), \Sigma'\rangle$. The sentence translations are defined by induction on the structure of sentences simultaneously for all signature morphisms $\chi\colon \Sigma \to \Sigma'$:

- $\chi(t = t') := \chi(t) = \chi(t')$, where the function $\chi : T_\Sigma \to T_{\Sigma'}$ is formally defined by $\chi(\sigma(t_1, \ldots, t_n)) = \chi(\sigma)(\chi(t_1), \ldots, \chi(t_n))$ for all function symbols $\sigma : s_1 \ldots s_n \to s \in F$ and all terms $t_i \in T_{\Sigma,s_i}$, where $i \in \{1, \ldots, n\}$.
- $\chi(\pi(t_1, \ldots, t_n)) := \chi(\pi)(\chi(t_1), \ldots, \chi(t_n))$.
- $\chi(\neg\gamma) := \neg\chi(\gamma)$.

- $\chi(\vee\Gamma) \coloneqq \vee\chi(\Gamma)$.

$$
\begin{tikzcd}
\gamma' \arrow[r, dotted, no head] & \Sigma[X] \arrow[r, "\chi'"] & \Sigma'[X'] \arrow[r, dotted, no head] & \chi'(\gamma') \\
& \Sigma \arrow[u, hook] \arrow[r, "\chi"'] & \Sigma' \arrow[u, hook] &
\end{tikzcd}
$$

- $\chi(\exists X \cdot \gamma') \coloneqq \exists X' \cdot \chi'(\gamma')$, where $X' = \{\langle v_i, \chi(s), \Sigma'\rangle \mid \langle v_i, s, \Sigma\rangle \in X\}$ and $\chi' : \Sigma[X] \to \Sigma'[X']$ is the extension of χ which maps each variable $\langle v_i, s, \Sigma\rangle \in X$ to $\langle v_i, \chi(s), \Sigma'\rangle \in X'$. Notice that $\chi'(\gamma')$ is well-defined, as γ' has a simpler structure than $\exists X \cdot \gamma'$.

Fact 4. $\mathsf{Sen}^{\mathsf{FOL}}$ is a functor $\mathsf{Sig}^{\mathsf{FOL}} \to \mathbb{S}\mathrm{et}$, which commutes with the sentence building operators.

For the sake of simplicity, we will identify a variable only by its name and sort provided that there is no danger of confusion. Using this convention, each inclusion $\iota \colon \Sigma \hookrightarrow \Sigma'$ is canonically extended to an inclusion of sentences $\iota \colon \mathsf{Sen}^{\mathsf{FOL}}(\Sigma) \hookrightarrow \mathsf{Sen}^{\mathsf{FOL}}(\Sigma')$, which corresponds to the approach of classical model theory.

Satisfaction relation. Satisfaction is the usual first-order satisfaction and it is defined using the natural interpretations of ground terms t as elements $\mathfrak{A}_t$ in models $\mathfrak{A}$:

- $\mathfrak{A} \models_\Sigma t_1 = t_2$ iff $\mathfrak{A}_{t_1} = \mathfrak{A}_{t_2}$.
- $\mathfrak{A} \models_\Sigma \pi(t_1, \dots, t_n)$ iff $(\mathfrak{A}_{t_1}, \dots, \mathfrak{A}_{t_n}) \in \mathfrak{A}_\pi$.
- $\mathfrak{A} \models_\Sigma \neg\gamma$ iff $\mathfrak{A} \not\models_\Sigma \gamma$.
- $\mathfrak{A} \models_\Sigma \exists X \cdot \gamma'$ iff $\mathfrak{A}' \models_{\Sigma'} \gamma'$ for some expansion $\mathfrak{A}'$ of $\mathfrak{A}$ along the inclusion $\iota : \Sigma \hookrightarrow \Sigma[X]$, that is, $\mathfrak{A}' \upharpoonright_\Sigma = \mathfrak{A}$. [1]

When there is no danger of confusion we may drop the subscript Σ from the notation $\models_\Sigma$. An expansion of $\mathfrak{A}$ to the signature $\Sigma[X]$ consists of a pair $(\mathfrak{A}, f)$, where $f : X \to \{\mathfrak{A}_s\}_{s \in S}$ is a many-sorted function called *valuation*. If $X = \{x_1, \dots, x_n\}$ then $\exists\{x_1, \dots, x_n\} \cdot \gamma'$ is, simply, denoted by $\exists x_1, \dots, x_n \cdot \gamma'$. Moreover, if $f(x_i) = a_i$ for all $i \in \{1, \dots, n\}$ then, classically, $(\mathfrak{A}, f) \models \gamma'$ is denoted by $\mathfrak{A} \models \gamma'(a_1, \dots, a_n)$. Hence, $\mathfrak{A} \models \exists x_1, \dots, x_n \cdot \gamma'$ iff $\mathfrak{A} \models \gamma'(a_1, \dots, a_n)$ for some tuple of elements $(a_1, \dots, a_n) \in \mathfrak{A}_{s_1} \times \cdots \times \mathfrak{A}_{s_n}$, where s_i is the sort of x_i for each $i \in \{1, \dots, n\}$.

Non-void signatures. A first-order signature Σ is called *non-void* if all sorts in Σ are inhabited by terms, that is $T_{\Sigma,s} \neq \emptyset$ for all sorts s in Σ. If Σ is a *non-void* signature then the set of Σ-terms T_Σ can be regarded as a first-order model which interprets (a) any function symbol $(\sigma \colon \mathsf{ar} \to s) \in F$ as a function $T_{\Sigma,\sigma} \colon T_{\Sigma,\mathsf{ar}} \to T_{\Sigma,s}$ defined by $T_{\Sigma,\sigma}(t) = \sigma(t)$ for all $t \in T_{\Sigma,\mathsf{ar}}$, and (b) any relation symbol as the empty set.

Notations. For each first-order signature Σ, we denote by $\bot$ the Σ-sentence $\vee\emptyset$. Obviously, $\bot$ is not satisfiable and $\chi(\bot) = \bot$ for all signature morphisms $\chi \colon \Sigma \to \Sigma'$. Let T and Γ be two theories over Σ.

- $\mathfrak{A} \models T$ if $\mathfrak{A} \models \varphi$ for all $\varphi \in T$, where $\mathfrak{A}$ is any first-order Σ-structure.
- $T \models \Gamma$ if for all first-order structures $\mathfrak{A}$ over Σ, we have $\mathfrak{A} \models T$ implies $\mathfrak{A} \models \Gamma$.
- $T \models\!\!\mid \Gamma$ if $T \models \Gamma$ and $\Gamma \models T$. In this case, we say that T and Γ are semantically equivalent.

3. hybrid dynamic first-order logic with rigid symbols (HDFOLR)

In this section, we present hybrid dynamic first-order logic with rigid symbols, which is an extension of hybrid first-order logic with rigid symbols [28] with features of dynamic logics. Some preliminary attempts to the presentation of this logic framework can be found in [30].

[1] In the case of inclusions $\iota : \Sigma \hookrightarrow \Sigma[X]$, the corresponding reduct functor $\upharpoonright_\iota$ is denoted by $\upharpoonright_\Sigma$.

Signatures. The signatures are of the form $\Delta = (\Sigma^{\mathsf{n}}, \Sigma^{\mathsf{r}} \subseteq \Sigma)$, where

1. $\Sigma^{\mathsf{n}} = (S^{\mathsf{n}}, F^{\mathsf{n}}, P^{\mathsf{n}})$ is a single-sorted first-order signature such that $S^{\mathsf{n}} = \{\mathsf{any}\}$ is a singleton, F^{n} is a set of constants called *nominals*, and P^{n} is a set of binary relation symbols called *modalities*,
2. $\Sigma = (S, F, P)$ is a many-sorted first-order signature such that S is a set of sorts, F is a $(S^* \times S)$-indexed set of function symbols, and P is a S^*-indexed set of relation symbols, and
3. $\Sigma^{\mathsf{r}} = (S^{\mathsf{r}}, F^{\mathsf{r}}, P^{\mathsf{r}})$ is a many-sorted first-order subsignature of *rigid* symbols.

Throughout this paper, we let Δ and Δ_i range over HDFOLR signatures of the form $(\Sigma^{\mathsf{n}}, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ and $(\Sigma^{\mathsf{n}}_i, \Sigma^{\mathsf{r}}_i \subseteq \Sigma_i)$, respectively.

Signature morphisms. A *signature morphism* $\chi \colon \Delta \to \Delta_1$ consists of a pair of first-order signature morphisms $\chi^{\mathsf{n}} \colon \Sigma^{\mathsf{n}} \to \Sigma^{\mathsf{n}}_1$ and $\chi \colon \Sigma \to \Sigma_1$ such that $\chi(\Sigma^{\mathsf{r}}) \subseteq \Sigma^{\mathsf{r}}_1$.

Fact 5. HDFOLR signature morphisms form a category $\mathsf{Sig}^{\mathsf{HDFOLR}}$ under the component-wise composition as first-order signature morphisms.

Kripke structures. For every signature Δ, the class of Kripke structures over Δ consists of pairs (W, M), where

1. W is a first-order structure over Σ^{n}, called a frame, with the universe $|W|$ consisting of a non-empty set of possible worlds, and
2. $M \colon |W| \to |\mathsf{Mod}^{\mathsf{FOL}}(\Sigma)|$ is a mapping from the universe of W to the class of first-order Σ-structures such that the following sharing condition holds: $M_{w_1} \upharpoonright_{\Sigma^{\mathsf{r}}} = M_{w_2} \upharpoonright_{\Sigma^{\mathsf{r}}}$ for all possible worlds $w_1, w_2 \in |W|$.

Kripke homomorphisms. A *morphism* $h \colon (W, M) \to (W', M')$ is also a pair $(W \xrightarrow{h} W', \{M_w \xrightarrow{h_w} M'_{h(w)}\}_{w \in |W|})$ consisting of first-order homomorphisms such that $h_{w_1,s} = h_{w_2,s}$ for all possible worlds $w_1, w_2 \in |W|$ and all rigid sorts $s \in S^{\mathsf{r}}$.

Fact 6. For any signature Δ, the Δ-homomorphisms form a category $\mathsf{Mod}^{\mathsf{HDFOLR}}(\Delta)$ under the component-wise composition as first-order homomorphisms.

Every signature morphism $\chi \colon \Delta \to \Delta'$ induces appropriate *reductions of models*, as follows: every Δ'-model (W', M') is reduced to a Δ-model $(W', M') \upharpoonright_\chi$ that interprets every symbol x in Δ as $(W', M')_{\chi(x)}$. When χ is an inclusion, we usually denote $(W', M') \upharpoonright_\chi$ by $(W', M') \upharpoonright_\Delta$ – in this case, the model reduct simply forgets the interpretation of those symbols in Δ' that do not belong to Δ.

Fact 7. $\mathsf{Mod}^{\mathsf{HDFOLR}}$ becomes a functor $\mathsf{Sig}^{\mathsf{HDFOLR}} \to \mathbb{C}\mathsf{at}^{op}$, with $\mathsf{Mod}^{\mathsf{HDFOLR}}(\chi)(W, M) = (W, M) \upharpoonright_\chi$ for each signature morphism $\chi \colon \Delta \to \Delta'$ and each Kripke structure (W, M) over Δ'.

Actions. As in dynamic logic, HDFOLR supports structured actions obtained from atoms using sequential composition, union, and iteration. The set A^{n} of *actions* over Σ^{n} is defined in an inductive fashion, according to the grammar:

$$\mathfrak{a} ::= \lambda \mid \mathfrak{a} \mathbin{⨾} \mathfrak{a} \mid \mathfrak{a} \cup \mathfrak{a} \mid \mathfrak{a}^*$$

where $\lambda \in P^{\mathsf{n}}$ is a binary relation symbol (to be interpreted as a binary relation on states by the Kripke structures). Given a natural number $m > 0$, we denote by $\mathfrak{a}^m$ the composition $\mathfrak{a} \mathbin{⨾} \cdots \mathbin{⨾} \mathfrak{a}$ (where the action $\mathfrak{a}$ occurs m times). Actions are interpreted in Kripke structures as *accessibility relations* between possible worlds. This is done by extending the interpretation of binary relation symbols from P^{n}: $W_{\mathfrak{a}_1 ⨾ \mathfrak{a}_2} = W_{\mathfrak{a}_1} \mathbin{⨾} W_{\mathfrak{a}_2}$ (diagrammatic composition of relations), $W_{\mathfrak{a}_1 \cup \mathfrak{a}_2} = W_{\mathfrak{a}_1} \cup W_{\mathfrak{a}_2}$ (union), and $W_{\mathfrak{a}^*} = (W_{\mathfrak{a}})^*$ (reflexive & transitive closure).

Hybrid terms. For any signature Δ, we make the following notational conventions:

1. $S^{\mathtt{e}} := S^{\mathtt{r}} \cup \{\mathsf{any}\}$ the extended set of rigid sorts, where any is the sort of nominals,
2. $S^{\mathtt{f}} := S \setminus S^{\mathtt{r}}$ the subset of flexible sorts,
3. $F^{\mathtt{f}} := F \setminus F^{\mathtt{r}}$ the subset of flexible function symbols, where $F \setminus F^{\mathtt{r}} = \{F_{\mathsf{ar}\to s} \setminus F^{\mathtt{r}}_{\mathsf{ar}\to s}\}_{(\mathsf{ar},s)\in S^*\times S}$,
4. $P^{\mathtt{f}} := P \setminus P^{\mathtt{r}}$ the subset of flexible relation symbols, where $P \setminus P^{\mathtt{r}} = \{P_{\mathsf{ar}} \setminus P^{\mathtt{r}}_{\mathsf{ar}}\}_{\mathsf{ar}\in S^*}$.

The *rigidification* of Σ with respect to F^{n} is the signature $@\Sigma = (@S, @F, @P)$, where

1. $@S := \{@_k\, s \mid k \in F^{\mathsf{n}} \text{ and } s \in S\}$,
2. $@F := \{@_k\, \sigma \colon @_k\, \mathsf{ar} \to @_k\, s \mid k \in F^{\mathsf{n}} \text{ and } (\sigma \colon \mathsf{ar} \to s) \in F\}$, [2] and
3. $@P := \{@_k\, \pi \colon @_k\, \mathsf{ar} \mid k \in F^{\mathsf{n}} \text{ and } (\pi \colon \mathsf{ar}) \in P\}$.

It should be noted that $@_k$ is used polymorphically. Here it is a device from metalanguage that creates new symbols out of existing ones. Later on $@_k$ will also be used as a sentence-building operator. The context always decides which of these uses are intended. Since the rigid symbols have the same interpretation across the worlds, we define $@_k\, x := x$ for all nominals $k \in F^{\mathsf{n}}$ and all symbols x in $\Sigma^{\mathtt{r}}$. The set of *rigid* Δ*-terms* is $T_{@\Sigma}$, while the set of *open* Δ*-terms* is T_Σ. The set of *hybrid* Δ*-terms* is $T_{\overline{\Sigma}}$, where $\overline{\Sigma} = (\overline{S}, \overline{F}, \overline{P})$, $\overline{S} = S \cup @S^{\mathtt{f}}$, $\overline{F} = F \cup @F^{\mathtt{f}}$, and $\overline{P} = P \cup @P^{\mathtt{f}}$.

Remark 8. The set of hybrid terms include both open and rigid terms, that is, $T_\Sigma \subseteq T_{\overline{\Sigma}}$ and $T_{@\Sigma} \subseteq T_{\overline{\Sigma}}$.

The interpretation of the hybrid terms into Kripke structures is defined as follows: for any Δ-model (W, M), and any possible world $w \in |W|$,

1. $M_{w,\sigma(t)} = (M_{w,\sigma})(M_{w,t})$, where $(\sigma \colon \mathsf{ar} \to s) \in F$, and $t \in T_{\overline{\Sigma},\mathsf{ar}}$, [3]
2. $M_{w,(@_k\,\sigma)(t)} = (M_{w',\sigma})(M_{w,t})$, where $(@_k\,\sigma \colon @_k\,\mathsf{ar} \to @_k\, s) \in @F^{\mathtt{f}}$, $t \in T_{\overline{\Sigma},@_k\,\mathsf{ar}}$ and $w' = W_k$.

Sentences. The simplest sentences defined over a signature Δ, usually referred to as atomic, are given by:

$$\rho ::= k \mid t_1 = t_2 \mid \pi(t)$$

where (a) $k, k' \in F^{\mathsf{n}}$ are nominals, (b) $t_i \in T_{\overline{\Sigma},s}$ are hybrid terms, $s \in \overline{S}$ is a hybrid sort, (c) $\pi \in \overline{P}_{\mathsf{ar}}$, $\mathsf{ar} \in (\overline{S})^*$ and $t \in T_{\overline{\Sigma},\mathsf{ar}}$. We refer to these sentences, in order, as *nominal sentences*, *hybrid equations* and *hybrid relations*, respectively. The set $\mathsf{Sen}^{\mathsf{HDFOLR}}(\Delta)$ of *full sentences* over Δ is given by the following grammar:

$$\gamma ::= \rho \mid @_k\,\gamma \mid \neg\gamma \mid \vee\Gamma \mid {\downarrow} z \cdot \gamma' \mid \exists X \cdot \gamma'' \mid \langle \mathfrak{a} \rangle \gamma$$

where (a) ρ is an atomic sentence, (b) $k \in F^{\mathsf{n}}$ is a nominal, (c) $\mathfrak{a} \in A^{\mathsf{n}}$ is an action, (d) Γ is a finite set of Δ-sentences, (e) z is a nominal variable for Δ, (f) γ' is a sentence over the signature $\Delta[z]$ obtained by adding z as a new constant to F^{n}, (g) X is a set of variables for Δ of sorts from the extended set $S^{\mathtt{e}}$ of rigid sorts, and (h) γ'' is a a sentence over the signature $\Delta[X]$ obtained by adding the variables in X as new constants to F^{n} and $F^{\mathtt{r}}$. Other than the first kind of sentences (*atoms*), we refer to the sentence-building operators, in order, as *retrieve*, *negation*, *disjunction*, *store*, *existential quantification* and *possibility*, respectively. Notice that *possibility* is parameterized by actions. Other sentence building operators can be introduced using the classical definitions. For example, $[\mathfrak{a}]\gamma$ is defined as $\neg\langle\mathfrak{a}\rangle\neg\gamma$ and $\forall X \cdot \gamma'$ is defined as $\neg\exists X \cdot \neg\gamma'$.

[2] $@_k\,(s_1 \dots s_n) := @_k\, s_1 \dots @_k\, s_n$ for all arities $s_1 \dots s_n$.

[3] $M_{w,(t_1,\dots,t_2)} := M_{w,t_1}, \dots, M_{w,t_n}$ for all tuples of hybrid terms $(t_1, \dots, t_n)$.

Sentence translations. Every signature morphism $\chi\colon \Delta \to \Delta'$ induces *translations of sentences*, as follows: each Δ-sentence γ is translated to a Δ'-sentence $\chi(\gamma)$ by replacing (in an inductive manner) the symbols in Δ with symbols from Δ' according to χ.

Fact 9. $\mathsf{Sen}^{\mathsf{HDFOLR}}$ is a functor $\mathsf{Sig}^{\mathsf{HDFOLR}} \to \mathbb{S}\mathsf{et}$.

Local satisfaction relation. Given a Δ-model (W, M) and a world $w \in |W|$, we define the *satisfaction of* Δ*-sentences at* w by structural induction as follows:

1. *For atomic sentences*:
   - $(W, M) \models^w k$ iff $W_k = w$ for all nominals k;
   - $(W, M) \models^w t_1 = t_2$ iff $M_{w,t_1} = M_{w,t_2}$ for all hybrid equations $t_1 = t_2$;
   - $(W, M) \models^w \pi(t)$ iff $M_{w,t} \in M_{w,\pi}$ for all hybrid relations $\pi(t)$.
2. *For full sentences*:
   - $(W, M) \models^w @_k\, \gamma$ iff $(W, M) \models^{w'} \gamma$, where $w' = W_k$;
   - $(W, M) \models^w \neg\gamma$ iff $(W, M) \not\models^w \gamma$;
   - $(W, M) \models^w \vee\Gamma$ iff $(W, M) \models^w \gamma$ for some $\gamma \in \Gamma$;
   - $(W, M) \models^w \downarrow z \cdot \gamma$ iff $(W^{z\leftarrow w}, M) \models^w \gamma$,
     where $(W^{z\leftarrow w}, M)$ is the unique $\Delta[z]$-expansion of (W, M) that interprets the variable z as w; [4]
   - $(W, M) \models^w \exists X \cdot \gamma$ iff $(W', M') \models^w \gamma$ for some expansion (W', M') of (W, M) to the signature $\Delta[X]$; [4]
   - $(W, M) \models^w \langle \mathfrak{a} \rangle \gamma$ iff $(W, M) \models^{w'} \gamma$ for some $w' \in |W|$ such that $(w, w') \in W_{\mathfrak{a}}$.

Notice that any sentence of the form $\downarrow z \cdot \gamma$ is semantically equivalent to $\forall z \cdot z \Rightarrow \gamma$. However, since we are going to prove logical properties for fragments of HDFOLR which may not have quantification, we introduced the operator store $\downarrow$ independently. The following *satisfaction condition* can be proved by induction on the structure of Δ-sentences. The proof is essentially identical to those developed for several other variants of hybrid logic presented in the literature (see, e.g. [18]).

Proposition 10 (Local satisfaction condition for signature morphisms)**.** *For every signature morphism* $\chi\colon \Delta \to \Delta'$*,* Δ'*-model* (W', M')*, possible world* $w' \in |W'|$*, and* Δ*-sentence* γ*, we have* $(W', M') \models^w \chi(\gamma)$ *iff* $(W', M')\upharpoonright_\chi \models^w \gamma$*.* [5]

Non-void signatures. A signature $\Delta = (\Sigma^{\mathsf{n}}, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ is called *non-void* if both Σ^{n} and Σ are non-void first-order signatures. Notice that for any non-void signature, the set of nominals is not empty, that is, $F^{\mathsf{n}} \neq \emptyset$, and the set of hybrid terms of any sort is not empty, that is, $T_{\overline{\Sigma},s} \neq \emptyset$ for all sorts $s \in S$.

Lemma 11. *If* $\Delta = (\Sigma^{\mathsf{n}}, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ *is non-void then there exists an initial model of terms* (W^Δ, M^Δ) *defined as follows: (1)* $W^\Delta = F^{\mathsf{n}}$*, and (2)* $M^\Delta\colon F^{\mathsf{n}} \to |\mathsf{Mod}^{\mathsf{FOL}}(\Sigma)|$*, where for all* $k \in F^{\mathsf{n}}$*,* M^Δ_k *is a first-order structure such that*

(a) $M^\Delta_{k,s} = T_{@\Sigma,@_k s}$ *for all sorts* $s \in S$*,*

(b) $M^\Delta_{k,\sigma}\colon T_{@\Sigma,@_k \mathsf{ar}} \to T_{@\Sigma,@_k s}$ *is defined by* $M^\Delta_{k,\sigma}(t) = (@_k\, \sigma)(t)$ *for all function symbols* $(\sigma\colon \mathsf{ar} \to s) \in F$ *and all tuples of hybrid terms* $t \in T_{@\Sigma,@_k \mathsf{ar}}$*, and*

(c) $M^\Delta_{k,\pi}$ *is the empty set for all relation symbols* $(\pi\colon \mathsf{ar}) \in P$*.*

The proof of Lemma 11 is based on the unique interpretation of terms into models, and it is straightforward. We leave it as an exercise for the reader.

[4] An expansion of (W, M) to $\Delta[X]$ is a Kripke structure (W', M') over $\Delta[X]$ that interprets all symbols in Δ in the same way as (W, M).

[5] By the definition of reducts, (W', M') and $(W', M')\upharpoonright_\chi$ have the same possible worlds.

Notations. Take a signature Δ, a Kripke structure $(W, M) \in |\mathsf{Mod}^{\mathsf{HDFOLR}}(\Delta)|$, a sentence $\varphi \in \mathsf{Sen}^{\mathsf{HDFOLR}}(\Delta)$, and two theories $T, \Gamma \subseteq \mathsf{Sen}^{\mathsf{HDFOLR}}(\Delta)$.

- We say that (W, M) (globally) satisfies φ, in symbols, $(W, M) \models \varphi$, if $(W, M) \models^w \varphi$ for all $w \in |W|$.
- We say that (W, M) satisfies Γ, in symbols, $(W, M) \models \Gamma$, if $(W, M) \models \gamma$ for all $\gamma \in \Gamma$.
- We say that T (globally) satisfies Γ, in symbols, $T \models \Gamma$, if $(V, N) \models T$ implies $(V, N) \models \Gamma$ for all $(V, N) \in |\mathsf{Mod}^{\mathsf{HDFOLR}}(\Delta)|$. [6]
- We say that T is semantically equivalent to Γ, in symbols, $T \models\!\mid \Gamma$, if $T \models \Gamma$ and $\Gamma \models T$.
- We let $+\varphi$ denote the sentence $\forall z^\circ \cdot @_{z^\circ}\, \varphi$, and $-\varphi$ denote the sentence $\exists z^\circ \cdot @_{z^\circ} \neg\varphi$, where z° is a distinguished nominal variable for Δ.

Lemma 12. *Assume a signature Δ, a nominal k in Δ, a nominal variable x for Δ, two sentences φ and γ over Δ, a theory T over Δ, a sentence ψ over $\Delta[x]$, a Kripke structure (W, M) over Δ, and a possible world $w \in |W|$.*

1. $(W, M) \models^w +\varphi$ *iff* $(W, M) \models +\varphi$ *iff* $(W, M) \models \varphi$.
2. $\varphi \models\!\mid +\varphi \models\!\mid @_k + \varphi$, *while* $\varphi \Rightarrow \gamma \models\!\mid +\varphi \Rightarrow \gamma$ *does not hold, in general.*
3. $T \models @_k\, (\varphi \Rightarrow \gamma)$ *iff* $T \cup \{@_k\, \varphi\} \models @_k\, \gamma$.
4. $T \models @_k\, \neg\varphi$ *iff* $T \cup \{@_k\, \varphi\} \models \bot$.
5. $T \cup \{\psi\}$ *is satisfiable over* $\Delta[x]$ *iff* $T \cup \{\exists x \cdot +\psi\}$ *is satisfiable over* Δ.

The proof of this lemma is straightforward and we leave it as an exercise for the interested reader. Informally, the key is that in the sentence $+\varphi = \forall z^\circ \cdot @_{z^\circ}\, \varphi$ the quantifier $\forall z^\circ$ binds the free variable z° in $@_{z^\circ}$, so $\forall z^\circ \cdot @_{z^\circ}\, \varphi$ means 'φ holds at all worlds w'.

By using the 'storing and retrieving' intuition it is easy to define complex properties. For example, consider any signature with only one binary relation symbol λ for nominals and work in a non-dynamic setting, that is, the setting of Hybrid First-Order Logic with Rigid symbols, where the only action allowed is λ itself. In this context, we can let $\Diamond := \langle\lambda\rangle$ (with $\Box$ just being the dual $\neg\Diamond\neg$). Then the temporal operator 'until' U – with the following semantics: $U(\varphi, \psi)$ is true at a state w if there is a future state w' where φ holds, such that ψ holds in all states between w and w' – can be defined as follows:

$$U(\varphi, \psi) := \downarrow x \cdot \Diamond \downarrow y \cdot (\varphi \wedge @_x \Box(\Diamond y \Rightarrow \psi)).$$

The idea is to name the current state x using $\downarrow$, and then by $\Diamond$, we identify a successor state, which we call y, where φ holds. Using @, the point of evaluation is changed to x, and then at all successors of x connected to y, ψ holds.

Example 13. Let $\Delta = (\Sigma^{\mathsf{n}}, \Sigma^{\mathsf{r}} \subseteq \Sigma^{\mathsf{f}})$ be a signature defined as follows:

- $\Sigma^{\mathsf{n}} = (F^{\mathsf{n}}, P^{\mathsf{n}})$ such that F^{n} consists of all natural numbers, and P^{n} has one elements $\lambda : 2$. [7]
- $\Sigma = (S, F, P)$, where $S = \emptyset$ and $P = \{\mathtt{green}, \mathtt{red}\}$. [8]
- $\Sigma^{\mathsf{r}} = \emptyset$.

Let (W, M) the Kripke structure over Δ defined as follows:

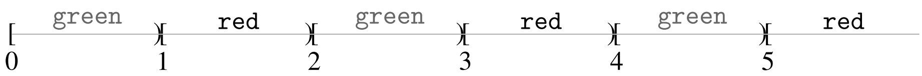
- $|W|$ is the set of all real numbers greater or equal than 0, W interprets each natural number as itself, that is, $W_k = k$ for all $k \in F^{\mathsf{n}}$, and W interprets λ as the usual strict order on real numbers.

[6] Notice that the semantics of $\varphi \models \gamma$ is different from the standard one, where $\varphi \models \gamma$ is interpreted locally, that is, $(V, N) \models^w \varphi$ implies $(V, N) \models^w \gamma$ for all Kripke structures (V, N) and all possible worlds w in V.

[7] In case of single-sorted signatures, arities are represented by natural numbers.

[8] In case of signatures with the empty set of sorts, all arities are empty, and therefore, are disregarded.

- $(W, M) \models^w$ `green` for all $w \in [2k, 2k + 1)$, where $k \in F^{\mathsf{n}}$ is a natural number.
- $(W, M) \models^w$ `red` for all $w \in [2k + 1, 2k + 2)$, where $k \in F^{\mathsf{n}}$ is a natural number.



Notice that $(W, M) \models^{2k} U(\texttt{red}, \texttt{green})$ and $(W, M) \models^{2k+1} U(\texttt{green}, \texttt{red})$ for all natural numbers k.

Example 14. Let Δ be a signature defined as follows:

- $\Sigma^{\mathsf{n}} = (F^{\mathsf{n}}, P^{\mathsf{n}})$ such that F^{n} consists of all natural numbers, and P^{n} has one element λ.
- $\Sigma = (S, F, P)$, $S = \{Elt, List\}$, $F = \{empty :\to List, cons : Elt\ List \to List, delete : List \to List\}$ and $P = \emptyset$.
- $S^{\mathtt{r}} = S$ and $F^{\mathtt{r}} = \{empty :\to List, cons : Elt\ List \to List\}$.

Let (W, M) be a Kripke structure over Δ defined as follows:

- Let $|W| = F^{\mathsf{n}}$ the set of natural numbers, and let W_λ be the usual strict order among the natural numbers.

For all possible worlds $n \in |W|$, the first order structure M_n is define as follows:

- M_n interprets Elt as a set, and $List$ as the set of lists with elements from $M_{n,Elt}$;
- the function $M_{n,delete} : M_{n,List} \to M_{n,List}$ delete the first n elements from the list given as input.

For all possible worlds $n \in |W|$, the following local satisfaction relations hold:

- $(W, M) \models^n \Diamond m$ for all $m > n$ and $(W, M) \models^n \neg m$ for all $m \neq n$;
- $(W, M) \models^n \forall L : List \cdot delete(L) = L$ if $n = 0$;
- $(W, M) \models^n \forall E : Elt, L : List \cdot (@_{n+1}\, delete) cons(E, L) = delete(L)$.

Notice that the first *delete* has a fixed interpretation in the state $n + 1$, while the second *delete* is interpreted in the current state, which is n.

4. Logical concepts

In this section, we recall some concepts necessary to prove our results.

4.1. Substitutions

Let Δ be a signature, C_1 and C_2 two sets of new constants for Δ of sorts in $S^{\mathtt{e}}$, the extended set of rigid sorts. A substitution $\theta : C_1 \to C_2$ over Δ is a mapping from C_1 to $|(W^{\Delta[C_2]}, M^{\Delta[C_2]})|$, the carrier sets of the initial Kripke structure $(W^{\Delta[C_2]}, M^{\Delta[C_2]})$ over $\Delta[C_2]$ defined in Lemma 11. This notion of substitution was introduced in [27] for setting the foundations of logic programming in hybrid logics. The following result is a straightforward generalization of [27, Corollary 39] to the hybrid dynamic framework.

Proposition 15 (Local satisfaction condition for substitutions)**.** *A substitution $\theta : C_1 \to C_2$ over Δ uniquely determines:*

1. *a sentence function θ:* $\mathsf{Sen}^{\mathsf{HDFOLR}}(\Delta[C_1]) \to \mathsf{Sen}^{\mathsf{HDFOLR}}(\Delta[C_2])$*, which preserves Δ and maps each constant $c \in C_1$ to a rigid term $\theta(c)$ over $\Delta[C_2]$, and*
2. *a reduct functor $\upharpoonright_\theta$:* $\mathsf{Mod}^{\mathsf{HDFOLR}}(\Delta[C_2]) \to \mathsf{Mod}^{\mathsf{HDFOLR}}(\Delta[C_1])$*, which preserves the interpretation of Δ and interprets each $c \in C_1$ as $\theta(c)$,*

such that the following local satisfaction condition holds:

$$(W, M) \models^w \theta(\gamma) \text{ iff } (W, M)\upharpoonright_\theta \models^w \gamma$$

for all $\Delta[C_1]$-sentences γ, all Kripke structures (W, M) over $\Delta[C_2]$ and all possible worlds $w \in |W|$.

4.2. Fragments

By restricting the signatures and/or the sentences of HDFOLR, one can obtain well-known hybrid logics studied in the literature.

Definition 16 (Fragment). A fragment $\mathcal{L}$ of HDFOLR is obtained by restricting the syntax of HDFOLR, that is, $\mathsf{Sig}^{\mathcal{L}}$ is a subcategory of $\mathsf{Sig}^{\mathsf{HDFOLR}}$ and $\mathsf{Sen}^{\mathcal{L}}\colon \mathsf{Sig}^{\mathcal{L}} \to \mathbb{S}\mathsf{et}$ is a subfunctor of $\mathsf{Sen}^{\mathsf{HDFOLR}}\colon \mathsf{Sig}^{\mathsf{HDFOLR}} \to \mathbb{S}\mathsf{et}$, such that

1. for any signature $\Delta \in |\mathsf{Sig}^{\mathcal{L}}|$, any set C of new nominals and any set D of new rigid constants, we have $\Delta \hookrightarrow \Delta[D, C] \in \mathsf{Sig}^{\mathcal{L}}$,
2. for any substitution $\theta\colon \langle C_1, D_1\rangle \to \langle C_2, D_2\rangle$ over a signature $\Delta \in |\mathsf{Sig}^{\mathcal{L}}|$ and any sentence $\gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta[C_1, D_1])$, we have $\theta(\gamma) \in \mathsf{Sen}^{\mathcal{L}}(\Delta[C_2, D_2])$, and
3. $\mathcal{L}$ is closed under subsentence relation, that is,
   - if $\langle \mathfrak{a}_1 \mathbin{;} \mathfrak{a}_2\rangle\gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta)$ then $\langle \mathfrak{a}_1\rangle\gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta)$ and $\langle \mathfrak{a}_2\rangle\gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta)$,
   - if $\langle \mathfrak{a}_1 \cup \mathfrak{a}_2\rangle\gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta)$ then $\langle \mathfrak{a}_1\rangle\gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta)$ and $\langle \mathfrak{a}_2\rangle\gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta)$,
   - if $\langle \mathfrak{a}^*\rangle\gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta)$ then $\langle \mathfrak{a}^n\rangle\gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta)$ for some $n \in \omega$,
   - if $\langle \mathfrak{a}\rangle\gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta)$ then $\gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta)$,
   - if $\neg\gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta)$ then $\gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta)$,
   - if $\vee\Gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta)$ then $\gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta)$ for all $\gamma \in \Gamma$,
   - if $@_k\, \gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta)$ then $\gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta)$,
   - if $\downarrow z \cdot \gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta)$ then $\gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta[z])$, and
   - if $\exists X \cdot \gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta)$ then $\gamma \in \mathsf{Sen}^{\mathcal{L}}(\Delta[X])$.

According to Definition 16, a fragment $\mathcal{L}$ of HDFOLR has the same models as HDFOLR. By the closure under the subsentence relation, the sentences of $\mathcal{L}$ are constructed from some atomic sentences by applying Boolean connectives, possibility over action relations, retrieve, store or existential quantifiers, if these sentence building operators are available in $\mathcal{L}$. It does not imply that $\mathcal{L}$ is closed under any of these operators.

Example 17 (Hybrid First-Order Logic with Rigid symbols (HFOLR) [28]). This is the hybrid variant of HDFOLR obtained by discarding structured actions and allowing possibility over binary modalities. According to [28], HFOLR is compact.

Example 18 (hybrid dynamic Propositional Logic (HDPL)). This is the dynamic variant of the most common form of multi-modal hybrid logic (e.g. [1]). HDPL is obtained from HDFOLR by restricting the signatures $\Delta = (\Sigma^{\mathsf{n}}, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ such that the set of sorts in Σ is empty, and the set of sentences is given by the following grammar:

$$\gamma ::= \rho \;\mid\; k \;\mid\; @_k\, \gamma \;\mid\; \neg\gamma \;\mid\; \vee\,\Gamma \;\mid\; \langle \mathfrak{a}\rangle\gamma$$

where (a) ρ is a propositional symbol, (b) $k \in F^{\mathsf{n}}$ is a nominal, (c) $\mathfrak{a} \in A^{\mathsf{n}}$ is an action, and (d) Γ is a finite set of sentences over Δ. Notice that if $\Sigma = (S, F, P)$ and $S = \emptyset$ then P contains only propositional symbols. HPL is the fragment of HDPL obtained by discarding structured actions.

Example 19 (Rigid First-Order Hybrid Logic (RFOHL) [8]). This logic is obtained from HFOLR by restricting the signatures $\Delta = (\Sigma^{\mathsf{n}}, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ such that (a) Σ^{n} has only one binary modality, (b) Σ is single-sorted, (c) there are no rigid function symbols except variables (regarded here as special constants), and (d) there are no rigid relation symbols.

All examples of logics given above are fragments of HDFOLR. In the following, we give an example of logic which is obtained from HDFOLR by some syntactic restrictions and it is not a fragment according to Definition 16.

Example 20 (Hybrid First-Order Logic with user-defined Sharing (HFOLS)). This logic has the same signatures and Kripke structure as HFOLR. The sentences are obtained from atoms constructed with open terms only, that is, if $\Delta = (\Sigma^{\mathsf{n}}, \Sigma^{\mathsf{r}} \subseteq \Sigma)$, all (ground) equations over Δ are of the form $t_1 = t_2$, where $t_1, t_2 \in T_\Sigma$, and all (ground) relation over Δ are of the form $\pi(t)$, where $(\pi : \mathsf{ar}) \in P$ and $t \in T_{\Sigma,\mathsf{ar}}$. Variants of HFOLS have been used in works such as [43, 20, 18].

HFOLS is not a fragment of HDFOLR in the sense of Definition 16, as it is not closed under substitutions. Retrieve is applied only to sentences and not to function or relation symbols. However, according to [28], that is no loss of expressivity as HFOLS has the same expressive power as HFOLR.

Lemma 21. *For each signature Δ and each sentence $\gamma \in \mathsf{Sen}^{\mathsf{HFOLR}}(\Delta)$ there exists a sentence $\gamma' \in \mathsf{Sen}^{\mathsf{HFOLS}}(\Delta)$ such that $(W, M) \models^w \gamma$ iff $(W, M) \models^w \gamma'$ for all Kripke structures (W, M) over Δ and all possible worlds in W.*

Proof. By using [28, Lemma 2.20] which shows that for any atomic sentence in HFOLR there exists a sentence in HFOLS which is satisfied by the same class of Kripke structures. □

The forcing technique and the Omitting Types Theorem are not applicable to HFOLS even if it has the same expressivity power as HFOLR. This is due to the absence of a proper support for the substitutions described in Section 4.1. By Lemma 21, the results can be borrowed from HFOLR to HFOLS. It is worth noting that HFOLS can be extended with features of dynamic logics such that the dynamic variant of HFOLS matches the expressivity of HDFOLR by the same arguments used in the proof of Lemma 21 .

4.3. Reachable models

In this section, we give a category-based description of the models which consist of elements that are denotations of terms. The concept of reachable model appeared in institutional model-theory in [47], and it has been used successfully in several abstract developments such as proof-theoretic results [34, 33, 25] as well as model-theoretic results [23, 24, 32, 26, 27, 14]. The following definition is an instance of an abstract notion of reachable Kripke structure given in [27, Definition 44].

Definition 22. A Kripke structure (W, M) over a signature $\Delta = (\Sigma^{\mathsf{n}}, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ is *reachable* if for each set of new constants C of sorts in the extended set of rigid sorts, and any expansion (W', M') of (W, M) to $\Delta[C]$, there exists a substitution $\theta \colon C \to \emptyset$ over Δ such that $(W, M)\upharpoonright_\theta = (W', M')$.

Proposition 23 (Reachable Kripke structures). *A Kripke structure is reachable iff*

1. *its set of states consists of denotations of nominals, and*
2. *its carrier sets for the rigid sorts consist of denotations of rigid terms.*

See [27, Proposition 49] for a proof of the above proposition. It follows that a model (W, M) is reachable iff the unique homomorphism from the initial Kripke structure $h \colon (W^\Delta, M^\Delta) \to (W, M)$ is surjective, that is, $h \colon W^\Delta \to W$ is surjective and $h_w \colon M^\Delta_w \to M_{h(w)}$ is surjective for all possible worlds $w \in |W^\Delta|$.

4.4. Basic sentences

In this section, we recall an important property of certain simple sentences of hybrid logics, which play the role analogous to atomic sentences of first-order logic.

Definition 24 (Basic set of sentences). A set of sentences B over a signature $\Delta = (\Sigma^{\mathsf{n}}, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ is *basic* if there exists a Kripke structure (W^B, M^B) such that

$$(W, M) \models B \text{ iff there exists a homomorphism } h : (W^B, M^B) \to (W, M)$$

for all Kripke structures (W, M). We say that (W^B, M^B) is a *basic model* of B. If in addition the homomorphism h is unique then the set B is called *epi-basic*.

The notion of basic set of sentences is from [15] where it was used to develop an institution-independent technique of the ultraproduct method for proving two important results for first-order logics cast as institutions: compactness and axiomatizability. According to [15] and [17], in first-order logic, any set of atomic sentences is basic. One important property of basic sentences is the preservation of their satisfaction along homomorphisms: given a set of basic sentences B and a homomorphism $h\colon M \to N$, if $M \models B$ then $N \models B$. In hybrid logics, this property does not hold, in general. The following example is from [28].

Example 25. Consider the following HPL signature $\Delta = (\Sigma^{\mathsf{n}}, \mathtt{Prop})$ such that $F^{\mathsf{n}} = \{k\}$, $P^{\mathsf{n}} = \{\lambda\colon \mathsf{any}\ \mathsf{any}\}$ and $\mathtt{Prop} = \{\rho\}$. Let $h\colon (W, M) \hookrightarrow (W', M')$ be the inclusion homomorphism defined by:

1. $|W| = \{k\}$, $W_\lambda = \{(k, k)\}$, ρ is true in M_k, and
2. $|W'| = \{k, w\}$, $W'_\lambda = \{(k, k)\}$, ρ is true in M'_k, ρ is not true in M'_w.

Example 25 points out a significant difference between ordinary logics and hybrid (or, more generally, modal) logics. Note that $(W, M) \models^{\mathsf{HPL}} k$, $(W, M) \models^{\mathsf{HPL}} \langle\lambda\rangle k$ and $(W, M) \models^{\mathsf{HPL}} \rho$. Since $(W', M') \not\models^{w} k$, $(W', M') \not\models^{w} \langle\lambda\rangle k$ and $(W', M') \not\models^{w} \rho$ we have $(W', M') \not\models^{\mathsf{HPL}} k$, $(W', M') \not\models^{\mathsf{HPL}} \langle\lambda\rangle k$ and $(W', M') \not\models^{\mathsf{HPL}} \rho$. Thus, homomorphisms do not preserve satisfaction of atomic sentences. Hence, atomic sentences are not basic in HPL (the same example works for any modal logic). Note however that local satisfaction (satisfiaction at a world) is preserved, and in hybrid logic the retrieve operator (@) lifts local satisfaction to global. This motivates the next definition which is from [28].

Definition 26 (Locally basic set of sentences). A set of sentences Γ over a signature Δ is *locally (epi-)basic* if $@\Gamma := \{@_k\, \gamma \mid k \in F^{\mathsf{n}} \text{ and } \gamma \in \Gamma\}$ is (epi-)basic.

Notice that $@\Gamma$ is semantically equivalent to $@@\Gamma$. We denote by $\mathsf{Sen}_0^{\mathsf{HDFOLR}}(\Delta)$ the set of all *extended atomic* sentences, which consists of:

1. nominals $k \in F^{\mathsf{n}}$,
2. nominal relations $\langle\lambda\rangle k$, where $\lambda \in P^{\mathsf{n}}$ is a binary modality and $k \in F^{\mathsf{n}}$,
3. hybrid equations $t_1 = t_2$, where $t_1, t_2 \in T_{\overline{\Sigma}}$, and
4. hybrid relations $\pi(t)$, where $\pi \in \overline{P}_{\mathsf{ar}}$, $t \in (T_{\overline{\Sigma}})_{\mathsf{ar}}$ and $\mathsf{ar} \in (\overline{S})^*$.

We denote by $\mathsf{Sen}_b^{\mathsf{HDFOLR}}(\Delta)$ the set of all sentences obtained from an extended atomic sentence by applying retrieve @ at most once.

Proposition 27 (Locally basic set of sentences)**.** *Given a signature Δ, every set of sentences $B \subseteq \mathsf{Sen}_b^{\mathsf{HDFOLR}}(\Delta)$ is locally basic. Moreover, if Δ is non-void, then B is locally epi-basic and its basic model (W^B, M^B) is reachable.*

See [28, Proposition 3.33] for a proof of the above proposition.

Definition 28 (Rigidification). For any signature $\Delta = (\Sigma^{\mathsf{n}}, \Sigma^{\mathsf{r}} \subseteq \Sigma)$, the *rigidification function* $\mathsf{at}_k\,\_\colon T_{\overline{\Sigma}} \to T_{@\Sigma}$, where $k \in F^{\mathsf{n}}$, is recursively defined by:

- $\mathsf{at}_k\, \sigma(t) := \begin{cases} (@_k\sigma)(\mathsf{at}_k\, t) & \text{if } (\sigma\colon \mathsf{ar} \to s) \in F^{\mathsf{f}}, \\ \sigma(\mathsf{at}_k\, t) & \text{if } (\sigma\colon \mathsf{ar} \to s) \in F^{\mathsf{r}} \cup @F^{\mathsf{f}}. \end{cases}$

Its extension $\mathsf{at}_k\,\_\colon \mathsf{Sen}^{\mathsf{HFOLR}}(\Delta) \to \mathsf{Sen}^{\mathsf{HFOLR}}(\Delta)$ is recursively defined by:

- $\mathsf{at}_k\, k' := @_k\, k'$
- $\mathsf{at}_k\, \langle\lambda\rangle(k') := @_k\, \langle\lambda\rangle(k')$
- $\mathsf{at}_k\, (t_1 = t_2) := (\mathsf{at}_k\, t_1 = \mathsf{at}_k\, t_2)$
- $\mathsf{at}_k\, \pi(t) := \begin{cases} (@_k\pi)(\mathsf{at}_k\, t) & \text{if } \pi \in P^{\mathsf{f}} \\ \pi(\mathsf{at}_k\, t) & \text{if } \pi \in P^{\mathsf{r}} \cup @P^{\mathsf{f}} \end{cases}$

- $\mathsf{at}_k \neg\gamma := \neg\mathsf{at}_k\,\gamma$
- $\mathsf{at}_k \vee \Gamma := \vee\mathsf{at}_k\,\Gamma$
- $\mathsf{at}_k\, @_{k'}\,\gamma := \mathsf{at}_{k'}\,\gamma$
- $\mathsf{at}_k\, \exists X \cdot \gamma := \exists X \cdot \mathsf{at}_k\,\gamma$

Any sentence semantically equivalent to a sentence in the image of at_k is called a *rigid sentence*.

Rigidification pushes the operator retrieve inside the structure of terms and it was developed in [28], where it plays a role in proving completeness of hybrid logics captured as stratified institutions.

Lemma 29. *Any sentence $@_k\,\gamma$ is semantically equivalent to $\mathsf{at}_k\,\gamma$. Hence, $@_k\,\gamma$ is rigid.*

The above lemma is due to [28].

5. Forcing

Forcing is a method of constructing models satisfying some properties forced by some conditions. In this section, we generalize the forcing relation for hybrid logics defined in [28] to hybrid dynamic first-order logic with rigid symbols. It is worth mentioning that the present developments can be cast in the framework of stratified institutions following the ideas presented in [28].

Framework 1. The results in this paper will be developed in a fragment $\mathcal{L}$ of HDFOLR that is semantically closed under negation and retrieve. [9] We make the following notational conventions:

- We let $\mathsf{Sen}_0^{\mathcal{L}}$ to denote the subfunctor of $\mathsf{Sen}^{\mathcal{L}}$ which maps each signature Δ to the set of extended atomic sentences of $\mathcal{L}$ over the signature Δ. This means that $\mathsf{Sen}_0^{\mathcal{L}}(\Delta) = \mathsf{Sen}^{\mathcal{L}}(\Delta) \cap \mathsf{Sen}_0^{\mathsf{HDFOLR}}(\Delta)$ for all signatures Δ.
- We let $\mathsf{Sen}_b^{\mathcal{L}}$ to denote the subfunctor of $\mathsf{Sen}^{\mathcal{L}}$ which maps each signature Δ to the set of basic sentences of $\mathcal{L}$ over the signature Δ. This means that $\mathsf{Sen}_b^{\mathcal{L}}(\Delta) = \mathsf{Sen}^{\mathcal{L}}(\Delta) \cap \mathsf{Sen}_b^{\mathsf{HDFOLR}}(\Delta)$ for all signatures Δ.

Since $\mathcal{L}$ is the logic in which we develop our results, we drop the superscript $\mathcal{L}$ from the notations $\mathsf{Sen}^{\mathcal{L}}$, $\mathsf{Sen}_0^{\mathcal{L}}$ and $\mathsf{Sen}_b^{\mathcal{L}}$ if there is no danger of confusion.

Examples of fragments can be found in Section 4.2. The following definition is due to [28].

Definition 30 (Forcing property). Given a signature Δ, a forcing property over Δ is a triple $\mathbb{P} = \langle P, \leq, f\rangle$ such that:

1. $(P, \leq)$ is a partially ordered set with a least element 0.

   The elements of p are traditionally called *conditions*.
2. $f\colon P \to \mathcal{P}(\mathsf{Sen}_b(\Delta))$ is a function,
3. if $p \leq q$ then $f(p) \subseteq f(q)$, and
4. if $f(p) \models @_k\,\gamma$ then $@_k\,\gamma \in f(q)$ for some $q \geq p$,

where $p \in P$, $q \in P$, $k \in F^{\mathsf{n}}$ and $\gamma \in \mathsf{Sen}_0(\Delta)$.

As for ordinary first-order logics, a forcing property generates a forcing relation on the set of all sentences.

[9] $\mathcal{L}$ is semantically closed under negation whenever for all $\mathcal{L}$-sentences γ there exists another $\mathcal{L}$-sentence φ such that we have: $(W, M) \models^w \varphi$ iff $(W, M) \not\models^w \gamma$ for all Kripke structures (W, M) and all possible worlds $w \in |W|$. When there is no danger of confusion, we denote φ by $\neg\gamma$. Similarly, one can define the semantic closer of $\mathcal{L}$ under any sentence building operator.

Definition 31 (Forcing relation). Let $\mathbb{P} = \langle P, \leq, f \rangle$ be a forcing property over Δ.

The family of relations $\Vdash = \{\Vdash^k\}_{k \in F^{\mathfrak{n}}}$, where $\Vdash^k \subseteq P \times \mathsf{Sen}(\Delta)$, is inductively defined as follows:

1. *For γ extended atomic:* $p \Vdash^k \gamma$ if $@_k \gamma \in f(p)$.
2. *For $\langle \mathfrak{a}_1 \mathbin{⨾} \mathfrak{a}_2 \rangle k''$:* $p \Vdash^k \langle \mathfrak{a}_1 \mathbin{⨾} \mathfrak{a}_2 \rangle k''$ if $p \Vdash^k \langle \mathfrak{a}_1 \rangle k'$ and $p \Vdash^{k'} \langle \mathfrak{a}_2 \rangle k''$ for some $k' \in F^{\mathfrak{n}}$.
3. *For $\langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''$:* $p \Vdash^k \langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''$ if $p \Vdash^k \langle \mathfrak{a}_1 \rangle k''$ or $p \Vdash^k \langle \mathfrak{a}_2 \rangle k''$.
4. *For $\langle \mathfrak{a}^* \rangle k''$:* $p \Vdash^k \langle \mathfrak{a}^* \rangle k''$ if $p \Vdash^k \langle \mathfrak{a}^n \rangle k''$ for some $n \in \mathbb{N}$.
5. *For $\langle \mathfrak{a} \rangle \gamma$ with $\gamma \notin F^{\mathfrak{n}}$:* $p \Vdash^k \langle \mathfrak{a} \rangle \gamma$ if $p \Vdash^k \langle \mathfrak{a} \rangle k'$ and $p \Vdash^{k'} \gamma$ for some nominal $k' \in F^{\mathfrak{n}}$.
6. *For $\neg\gamma$:* $p \Vdash^k \neg\gamma$ if there is no $q \geq p$ such that $q \Vdash^k \gamma$.
7. *For $\vee\Gamma$:* $p \Vdash^k \vee\Gamma$ if $p \Vdash^k \gamma$ for some $\gamma \in \Gamma$.
8. *For $@_{k'} \gamma$:* $p \Vdash^k @_{k'} \gamma$ if $p \Vdash^{k'} \gamma$.
9. *For $\downarrow z \cdot \gamma$:* $p \Vdash^k \downarrow z \cdot \gamma$ if $p \Vdash^k \gamma(z \leftarrow k)$.
10. *For $\exists X \cdot \gamma$:* $p \Vdash^k \exists X \cdot \gamma$ if $p \Vdash^k \theta(\gamma)$ for some substitution $\theta \colon X \to \emptyset$ over Δ.

The forcing relation defined in the present contribution consists of the forcing relation defined in [28] plus the items 2—4 of Definition 31. The notation $p \Vdash^k \gamma$ is read *p forces γ at k*.

Remark 32. Notice that Definition 31 does not rely on the fact that $\mathcal{L}$ is closed under disjunction or quantifiers. For example, the last item from Definition 31 should be interpreted as follows: if $\exists X \cdot \gamma$ is a sentence in $\mathcal{L}$ and $p \Vdash^k \theta(\gamma)$ for some substitution $\theta : X \to \emptyset$ over Δ then $p \Vdash^k \exists X \cdot \gamma$.

In regard to the satisfaction relation, one may consider a global forcing relation: $p \Vdash \gamma$ iff $p \Vdash^k \gamma$ for all nominals k. This remark establishes a connection between the results in the present contribution and the results in [34] and [24], where there exists only a global forcing relation.

Lemma 33. *Let $\mathbb{P} = \langle P, \leq, f \rangle$ be a forcing property as in Definition 30. We have:*

1. *$p \Vdash^k \neg\neg\gamma$ iff for each $q \geq p$ there is $r \geq q$ such that $r \Vdash^k \gamma$.*
2. *If $q \geq p$ and $p \Vdash^k \gamma$ then $q \Vdash^k \gamma$.*
3. *If $p \Vdash^k \gamma$ then $p \Vdash^k \neg\neg\gamma$.*
4. *We cannot have both $p \Vdash^k \gamma$ and $p \Vdash^k \neg\gamma$.*

Proof. Notice that the statements 1 and 3 are well-defined as $\mathcal{L}$ is semantically closed under negation.

1. $p \Vdash^k \neg\neg\gamma$ iff for each $q \geq p$ we have $q \nVdash^k \neg\gamma$ iff
   for each $q \geq p$ there is $r \geq q$ such that $r \Vdash^k \gamma$.
2. By induction on the structure of sentences:

   [*For γ extended atomic*] The conclusion follows easily from $f(p) \subseteq f(q)$.

   [*For $\langle \mathfrak{a}_1 \mathbin{⨾} \mathfrak{a}_2 \rangle k''$*] $p \Vdash^k \langle \mathfrak{a}_1 \mathbin{⨾} \mathfrak{a}_2 \rangle k''$ iff $p \Vdash^k \langle \mathfrak{a}_1 \rangle k'$ and $p \Vdash^{k'} \langle \mathfrak{a}_2 \rangle k''$ for some $k' \in F^{\mathfrak{n}}$. By the induction hypothesis, $q \Vdash^k \langle \mathfrak{a}_1 \rangle k'$ and $q \Vdash^{k'} \langle \mathfrak{a}_2 \rangle k''$. Hence, $q \Vdash^k \langle \mathfrak{a}_1 \mathbin{⨾} \mathfrak{a}_2 \rangle k''$.

   [*For $\langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''$*] $p \Vdash^k \langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''$ iff $p \Vdash^k \langle \mathfrak{a}_1 \rangle k''$ or $p \Vdash^k \langle \mathfrak{a}_2 \rangle k''$. By the induction hypothesis, $q \Vdash^k \langle \mathfrak{a}_1 \rangle k''$ or $q \Vdash^k \langle \mathfrak{a}_2 \rangle k''$. Hence, $q \Vdash^k \langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''$.

[*For* $\langle \mathfrak{a}^* \rangle k''$] $p \Vdash^k \langle \mathfrak{a}^* \rangle k''$ iff there exists $n \in \mathbb{N}$ such that $p \Vdash^k \mathfrak{a}^n \langle k'' \rangle$. By the induction hypothesis, $q \Vdash^k \langle \mathfrak{a}^n \rangle k''$. Hence, $q \Vdash^k \langle \mathfrak{a}^* \rangle k''$.

[*For* $\langle \mathfrak{a} \rangle \gamma$ *with* $\gamma \notin F^{\mathrm{n}}$] $p \Vdash^k \langle \mathfrak{a} \rangle \gamma$ iff $p \Vdash^k \langle \mathfrak{a} \rangle k'$ and $p \Vdash^{k'} \gamma$. By the induction hypothesis, $q \Vdash^k \langle \mathfrak{a} \rangle k'$ and $q \Vdash^{k'} \gamma$. Hence, $q \Vdash^k \langle \mathfrak{a} \rangle \gamma$.

[*For* $@_{k'} \gamma$] We have $p \Vdash^k @_{k'} \gamma$ iff $p \Vdash^{k'} \gamma$. By induction hypothesis, $q \Vdash^{k'} \gamma$. Hence, $q \Vdash^k @_{k'} \gamma$.

[*For* $\neg\gamma$] We have $p \Vdash^k \neg\gamma$. This means $r \nVdash^k \gamma$ for all $r \geq p$. In particular, $r \nVdash^k \gamma$ for all $r \geq q$. Hence, $q \Vdash^k \neg\gamma$.

[*For* $\vee\Gamma$] $p \Vdash^k \gamma$ for some $\gamma \in \Gamma$. By induction hypothesis, $q \Vdash^k \gamma$ which implies $q \Vdash^k \vee\Gamma$.

[*For* $\downarrow z \cdot \gamma$] We have $p \Vdash^k \downarrow z \cdot \gamma$ iff $p \Vdash^k \gamma(z \leftarrow k)$. By the induction hypothesis, $q \Vdash^k \gamma(z \leftarrow k)$, which implies $q \Vdash^k \downarrow z \cdot \gamma$.

[*For* $\exists X \cdot \gamma$] Since $p \Vdash^k \exists X \cdot \gamma$ then $p \Vdash^k \theta(\gamma)$ for some substitution $\theta \colon X \to \emptyset$ over Δ. By the induction hypothesis, $q \Vdash^k \theta(\gamma)$. Hence, $q \Vdash^k \exists X \cdot \gamma$.

3. It follows from 1 and 2.

4. By the reflexivity of $(P, \leq)$. □

Lemma 33 is a generalization of [28, Lemma 4.4] from hybrid logics to hybrid dynamic logics. Only the proof of the second statement requires an update, since we need to consider the case of possibility over structured actions for the induction.

Definition 34 (Generic set [28]). Let $\mathbb{P} = \langle P, \leq, f \rangle$ be a forcing property over a signature Δ.
A subset $G \subseteq P$ is generic if it has the following properties:

1. $r \in G$ if $r \leq p$ and $p \in G$;
2. there exists $r \in G$ such that $r \geq p$ and $r \geq q$, for all $p, q \in G$;
3. there exists $r \in G$ such that $r \Vdash^k \gamma$ or $r \Vdash^k \neg\gamma$, for all Δ-sentences $@_k \gamma$.

We write $G \Vdash^k \gamma$ whenever $p \Vdash^k \gamma$ for some $p \in G$.

In Definition 34, G is well-defined, since $\mathcal{L}$ is semantically closed under negation. The following lemma is due to [28] and it shows that generic sets exist provided that the underlying signature consists of a countable number of symbols.

Lemma 35 (Existence of generic sets). *Let* $\mathbb{P} = \langle P, \leq, f \rangle$ *be a forcing property over a signature* Δ. *If* $\mathsf{Sen}(\Delta)$ *is countable then every* p *belongs to a generic set.*

For the semantic forcing property defined in the next section it is possible to construct generic sets even if the underlying signature consists of an uncountable number of symbols. Notice that the definition of forcing relation and the definition of generic set are based on syntactic compounds. The following definition was proposed in [28] and it gives a semantics/meaning to the syntactic concepts defined above.

Definition 36 (Generic model). Let $\mathbb{P} = \langle P, \leq, f \rangle$ be a forcing property over a signature Δ.

- (W, M) is a model for a generic set $G \subseteq P$ when $(W, M) \models @_k \gamma$ iff $G \Vdash^k \gamma$, for all Δ-sentences $@_k \gamma$.
- (W, M) is a model for $p \in P$ if there is a generic set $G \subseteq P$ such that $p \in G$ and (W, M) is a model for G.

The models (W, M) from Definition 36 are called, traditionally, *generic models*. The following result ensures the existence of generic models.

Theorem 37 (Generic Model Theorem)**.** *Let $\mathbb{P} = \langle P, \leq, f \rangle$ be a forcing property over Δ. Then each generic set G of $\mathbb{P}$ has a generic Kripke structure (W, M). If in addition Δ is non-void, (W, M) is reachable.*

Proof. Let G be a generic set. We define $T = \{ @_k\, \gamma \in \mathsf{Sen}(\Delta) \mid G \Vdash^k \gamma \}$ and $B = T \cap \mathsf{Sen}_b(\Delta)$. By Proposition 27, B is basic, and there exists a basic model (W^B, M^B) for B that is reachable. We show that $(W^B, M^B) \models @_k\, \gamma$ iff $G \Vdash^k \gamma$, for all Δ-sentences $@_k\, \gamma$.

[*For γ extended atomic* **]** Assume that $(W^B, M^B) \models @_k\, \gamma$.

1	B and $\{@_k\, \gamma\}$ are basic	by Proposition 27
2	there exists an arrow $(W^{@_k \gamma}, M^{\{@_k \gamma\}}) \to (W^B, M^B)$	since $\{@_k\, \gamma\}$ is basic and $(W^B, M^B) \models @_k\, \gamma$
3	$B \models @_k\, \gamma$	since both B and $\{@_k\, \gamma\}$ are basic
4	there exists $B_f \subseteq B$ finite such that $B_f \models @_k\, \gamma$	since HDFOLR_b is compact
5	$B_f = \{@_{k_1} \gamma_1, \ldots, @_{k_n} \gamma_n\}$ for some $\gamma_i \in \mathsf{Sen}_0(\Delta)$ and some $k_i \in F^{\mathfrak{n}}$	by the definition of B
6	for all $i \in \{1, \ldots, n\}$, there exists $p_i \in G$ such that $p_i \Vdash^{k_i} \gamma_i$	by the definition of B
7	there exists $p \in G$ such that $p \geq p_i$ for all $i \in \{1, \ldots, n\}$	since G is generic
8	$B_f \subseteq f(p)$	since $B_f \subseteq \mathsf{Sen}_b(\Delta)$
9	$q \Vdash^k \gamma$ or $q \Vdash^k \neg\gamma$ for some $q \in G$	since G is generic
10	suppose towards a contradiction that $q \Vdash^k \neg\gamma$	
10.1	$r \geq p$ and $r \geq q$ for some $r \in G$	since G is generic
10.2	$r \Vdash^k \neg\gamma$	by Lemma 33 (2), since $r \geq q$ and $q \Vdash^k \neg\gamma$
10.3	$B_f \subseteq f(r)$	since $B_f \subseteq f(p)$ and $r \geq p$
10.4	there exists $s \geq r$ such that $@_k\, \gamma \in f(s)$	since $B_f \models @_k\, \gamma$, we have $f(r) \models @_k\, \gamma$
10.5	$s \Vdash^k \gamma$	by Definition 31
10.6	$s \Vdash^k \neg\gamma$	by Lemma 33 (2)
10.7	contradiction	by Lemma 33 (4)
11	$q \Vdash^k \gamma$	by 9 and 10
12	$G \Vdash^k \gamma$	since $q \in G$

If $G \Vdash^k \gamma$ then by the definition of B, we have $@_k\, \gamma \in B$, which implies $B \models @_k\, \gamma$; hence, $(W^B, M^B) \models @_k\, \gamma$.

[*For $\langle \mathfrak{a}_1 \fatsemi \mathfrak{a}_2 \rangle k''$* **]** Assume that $(W^B, M^B) \models @_k \langle \mathfrak{a}_1 \fatsemi \mathfrak{a}_2 \rangle k''$.

1	$(W^B_k, W^B_{k''}) \in W^B_{(\mathfrak{a}_1 \fatsemi \mathfrak{a}_2)}$	by definition
2	$(W^B_k, w) \in W^B_{\mathfrak{a}_1}$ and $(w, W^B_{k''}) \in W^B_{\mathfrak{a}_2}$ for some $w \in \lvert W^B \rvert$	since $\mathfrak{a}_1 \fatsemi \mathfrak{a}_2$ is the composition of the relations $\mathfrak{a}_1$ and $\mathfrak{a}_2$
3	$w = W^B_{k'}$ for some nominal $k' \in F^{\mathfrak{n}}$	since (W^B, M^B) is reachable
4	$(W^B_k, W^B_{k'}) \in W^B_{\mathfrak{a}_1}$ and $(W^B_{k'}, W^B_{k''}) \in W^B_{\mathfrak{a}_2}$	by 2 and 3
5	$G \Vdash^k \langle \mathfrak{a}_1 \rangle k'$ and $G \Vdash^{k'} \langle \mathfrak{a}_2 \rangle k''$	by the induction hypothesis
6	$p \Vdash^k \langle \mathfrak{a}_1 \rangle k'$ for some $p \in G$ and $q \Vdash^{k'} \langle \mathfrak{a}_2 \rangle k''$ for some $q \in G$	by Definition 34
7	$r \geq p$ and $r \geq q$ for some $r \in G$	since G is generic
8	$r \Vdash^k \langle \mathfrak{a}_1 \rangle k'$ and $r \Vdash^{k'} \langle \mathfrak{a}_2 \rangle k''$	by Lemma 33 (2) applied to 6 and 7
9	$r \Vdash^k \langle \mathfrak{a}_1 \fatsemi \mathfrak{a}_2 \rangle k''$	by Definition 31
10	$G \Vdash^k \langle \mathfrak{a}_1 \fatsemi \mathfrak{a}_2 \rangle k''$	by Definition 34

Assume that $G \Vdash^k \langle \mathfrak{a}_1 \fatsemi \mathfrak{a}_2 \rangle k''$.

1	$p \Vdash^k \langle \mathfrak{a}_1 \fatsemi \mathfrak{a}_2 \rangle k''$ for some $p \in G$	
2	$p \Vdash^k \langle \mathfrak{a}_1 \rangle k'$ and $p \Vdash^{k'} \langle \mathfrak{a}_2 \rangle k''$ for some $k' \in F^{\mathfrak{n}}$	by Definition 31
3	$(W^B, M^B) \models @_k \langle \mathfrak{a}_1 \rangle k'$ and $(W^B, M^B) \models @_{k'} \langle \mathfrak{a}_2 \rangle k''$	by the induction hypothesis

4 $(W^B, M^B) \models @_k \langle \mathfrak{a}_1 \fatsemi \mathfrak{a}_2 \rangle k''$ by the semantics of $\mathfrak{a}_1 \fatsemi \mathfrak{a}_2$

[*For* $\langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''$] The following are equivalent:

1 $G \Vdash^k \langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''$
2 $p \Vdash^k \langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''$ for some $p \in G$ by Definition 31
3 $p \Vdash^k \langle \mathfrak{a}_1 \rangle k''$ or $p \Vdash^k \langle \mathfrak{a}_2 \rangle k''$ by Definition 31
4 $G \Vdash^k \langle \mathfrak{a}_1 \rangle k''$ or $G \Vdash^k \langle \mathfrak{a}_2 \rangle k''$ by Definition 31
5 $(W^B, M^B) \models @_k \langle \mathfrak{a}_1 \rangle k''$ or $(W^B, M^B) \models @_k \langle \mathfrak{a}_2 \rangle k''$ by the induction hypothesis
6 $(W^B, M^B) \models @_k \langle \mathfrak{a}_1 \cup \mathfrak{a}_2 \rangle k''$ by the semantics of $\mathfrak{a}_1 \cup \mathfrak{a}_2$

[*For* $\langle \mathfrak{a}^* \rangle k''$] The following are equivalent:

1 $(W^B, M^B) \models @_k \langle \mathfrak{a}^* \rangle k''$
2 $(W^B, M^B) \models @_k \langle \mathfrak{a}^n \rangle k''$ for some $n \in \mathbb{N}$ by the semantics of $\mathfrak{a}^*$
3 $G \Vdash^k \langle \mathfrak{a}^n \rangle k''$ for some $n \in \mathbb{N}$ by the induction hypothesis
4 $G \Vdash^k \langle \mathfrak{a}^* \rangle k''$ by Definition 31

[*For* $\langle \mathfrak{a} \rangle \gamma$ *with* $\gamma \notin F^{\mathsf{n}}$] The following are equivalent:

1 $(W^B, M^B) \models @_k \langle \mathfrak{a} \rangle \gamma$
2 $(W^B, M^B) \models^{w_1} \gamma$ for some $w_1 \in |W^B|$ such that $(W^B_k, w_1) \in W^B_{\mathfrak{a}}$ by the definition of $\models$
3 $(W^B, M^B) \models @_k \langle \mathfrak{a} \rangle k_1$ and $(W^B, M^B) \models @_{k_1} \gamma$ for some $k_1 \in \mathsf{F}^{\mathsf{n}}$ such that $W^B_{k_1} = w_1$ by Proposition 23, since (W^B, M^B) is reachable
4 $G \Vdash^k \langle \mathfrak{a} \rangle k_1$ and $G \Vdash^{k_1} \gamma$ for some $k_1 \in F^{\mathsf{n}}$ by the induction hypothesis
5 $G \Vdash^k \langle \mathfrak{a} \rangle \gamma$ since G is generic

[*For* $\neg\gamma$] The following are equivalent:

1 $(W^B, M^B) \models @_k \neg\gamma$
2 $(W^B, M^B) \not\models @_k \gamma$ by the semantics of negation
3 $G \not\Vdash^k \gamma$ by the induction hypothesis
4 $p \not\Vdash^k \gamma$ for all $p \in G$ by the definition of $\Vdash$
5 $p \Vdash^k \neg\gamma$ for some $p \in G$ since G is generic
6 $G \Vdash^k \neg\gamma$

[*For* $\vee\Gamma$] The following are equivalent:

1 $(W^B, M^B) \models @_k \vee \Gamma$
2 $(W^B, M^B) \models @_k \gamma$ for some $\gamma \in \Gamma$ by the semantics of disjunction
3 $G \Vdash^k \gamma$ for some $\gamma \in \Gamma$ by the induction hypothesis
4 $G \Vdash^k \vee\Gamma$ by the definition of $\Vdash$

[*For* $\exists X \cdot \gamma$] Let $w = W^B_k$. The following are equivalent:

1 $(W^B, M^B) \models @_k \exists X \cdot \gamma$
2 $(W', M') \models^w \gamma$ for some expansion (W', M') of (W^B, M^B) to $\Delta[X]$ by the definition of $\models$
3 $(W^B, M^B) \models^w \theta(\gamma)$ for some substitution $\theta \colon X \to \emptyset$ over Δ such that $(W^B, M^B)\upharpoonright_\theta = (W', M')$ since (W^B, M^B) is reachable
4 $G \Vdash^k \theta(\gamma)$ for some substitution $\theta \colon X \to \emptyset$ over Δ by the induction hypothesis
5 $G \Vdash^k \exists X \cdot \gamma$ by the definition of $\Vdash$

[*For* $\downarrow z \cdot \gamma$] This case is straightforward since $@_k \downarrow z \cdot \gamma$ is semantically equivalent to $@_k \gamma(z \leftarrow k)$.

[*For* $@_{k'}\,\gamma$] This case is straightforward since $@_k\,@_{k'}\,\gamma$ is semantically equivalent to $@_{k'}\,\gamma$. $\square$

Theorem 37 is a generalization of Generic Model Theorem for hybrid logics from [28]. The new cases from the present contribution correspond to structured actions, which are the second, the third and the fourth cases.

6. Semantic forcing property

We study a semantic forcing property, which will be used to prove the Omitting Types Theorem for a fragment $\mathcal{L}$ of HDFOLR semantically closed under negation and retrieve.

Framework 2. In this section, we arbitrarily fix

1. a signature $\Delta = (\Sigma^{\mathsf{n}}, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ of $\mathcal{L}$,
2. a class $\mathcal{K}$ of Kripke structures over the signature Δ, and
3. a sorted set $C = \{C_s\}_{s\in S^{\mathsf{e}}}$ of new rigid constants for Δ such that $\mathsf{card}(C_s) = \alpha$ for all sorts $s \in S^{\mathsf{e}}$, where (a) α is the power of Δ, (b) $S^{\mathsf{e}} = S^{\mathsf{r}} \cup \{\mathsf{any}\}$ is the extended set of rigid sorts, and (c) any is the sort of nominals.

If the set of sorts in Σ is empty then C consists only of nominals.

Definition 38. The semantic forcing property $\mathbb{P} = (P, \leq, f)$ over the signature $\Delta[C]$ relative to the class of Kripke structures $\mathcal{K}$ is defined as follows:

1. $P = \{p \subseteq \mathsf{Sen}(\Delta[C]) \mid \mathsf{card}(p) < \alpha$ and $(W, M) \models p$ for some $(W, M) \in |\mathsf{Mod}(\Delta[C])|$ s.t. $(W, M)\upharpoonright_{\Delta} \in \mathcal{K}\}$,
2. $\leq$ is the inclusion relation, and
3. $f(p) = p \cap \mathsf{Sen}_b(\Delta[C])$ for all $p \in P$.

The set of conditions P consists of all sets of sentences over $\Delta[C]$ of cardinality strictly less than α which are satisfied by at least one expansion of a Kripke structure in $\mathcal{K}$. Given a condition $p \in P$ as input, the function f returns the set of basic sentences in p.

Lemma 39. *$\mathbb{P} = \langle P, \leq, f\rangle$ described in Definition 38 is a forcing property.*

Proof. All conditions enumerated in Definition 30 obviously hold except the last one. Assume that $f(p) \models @_k\,\gamma$, where $p \in P$ and $@_k\,\gamma \in \mathsf{Sen}_b(\Delta)$. Since $f(p) \subseteq p$, we have $p \models @_k\,\gamma$. By Definition 38, $(W, M) \models p$ for some $(W, M) \in |\mathsf{Mod}(\Delta[C])|$ such that $(W, M)\upharpoonright_{\Delta} \in \mathcal{K}$. Since $(W, M) \models p$ and $p \models @_k\,\gamma$, $(W, M) \models p \cup \{@_k\,\gamma\}$. Hence, $q \coloneqq p \cup \{@_k\,\gamma\} \in P$ and $p \leq q$ $\square$

Proposition 40. *$\mathbb{P} = \langle P, \leq, f\rangle$ described in Definition 38 has the following properties:*

P1) If $p \in P$ and $@_k\,\langle \mathfrak{a}_1 \mathbin{;} \mathfrak{a}_2\rangle k'' \in p$ then $p \cup \{@_k\,\langle \mathfrak{a}_1\rangle k', @_{k'}\,\langle \mathfrak{a}_2\rangle k''\} \in P$ for some nominal $k' \in C_{\mathsf{any}}$.

P2) If $p \in P$ and $@_k\,\langle \mathfrak{a}\rangle\gamma \in p$ then $p \cup \{@_k\,\langle \mathfrak{a}\rangle k', @_{k'}\,\gamma\} \in P$ for some nominal $k' \in C_{\mathsf{any}}$.

P3) If $p \in P$ and $@_k\, \vee\Gamma \in p$ then $p \cup \{@_k\,\gamma\} \in P$ for some $\gamma \in \Gamma$.

P4) If $p \in P$ and $@_k\,\exists X \cdot \gamma \in p$ then there exists an injective mapping $f\colon X \to C$ such that $p \cup \{@_k\,\chi(\gamma)\} \in P$, where $\chi\colon \Delta[C, X] \to \Delta[C]$ is the unique extension of f to a signature morphism which preserves $\Delta[C]$.

Proof. Let $p \in P$ be a condition. By the definition of $\mathbb{P}$, we have that $p \subseteq \mathsf{Sen}(\Delta[C'])$ for some $C' \subset C$ with $\mathsf{card}(C'_s) < \alpha$ for all $s \in S^{\mathsf{e}}$.

P1) Assume that $@_k\,\langle \mathfrak{a}_1 \mathbin{;} \mathfrak{a}_2\rangle k'' \in p$. Since $\mathsf{card}(C_{\mathsf{any}}) = \alpha$ and $\mathsf{card}(C'_{\mathsf{any}}) < \alpha$, there exists $k' \in C_{\mathsf{any}} \setminus C'_{\mathsf{any}}$. We show that $p \cup \{@_k\,\langle \mathfrak{a}_1\rangle k', @_{k'}\,\langle \mathfrak{a}_2\rangle k''\} \in P$:

1	$(W,M) \models p$ for some model (W,M) over $\Delta[C]$ with $(W,M)\upharpoonright_\Delta \in \mathcal{K}$	by the definition of $\mathbb{P}$
2	$(W',M') := (W,M)\upharpoonright_{\Delta[C']} \models p$	by the satisfaction condition
3	$(W'_k, w) \in W'_{\mathfrak{a}_1}$ and $(w, W'_{k''}) \in W'_{\mathfrak{a}_2}$ for some $w \in \lvert W' \rvert$	since $(W',M') \models @_k \langle \mathfrak{a}_1 \mathbin{⨾} \mathfrak{a}_2 \rangle k''$
4	$(W'',M') \models @_k \langle \mathfrak{a}_1 \rangle k'$ and $(W'',M') \models @_{k'} \langle \mathfrak{a}_2 \rangle k''$, where (W'',M') is the unique expansion of (W',M') to $\Delta[C',k']$ interpreting k' as w	
5	$(V,N) \models p \cup \{@_k \langle \mathfrak{a}_1 \rangle k', @_{k'} \langle \mathfrak{a}_2 \rangle k''\}$, where (V,N) is any expansion of (W'',M') to $\Delta[C]$	by the satisfaction condition, since $(W'',M') \models p \cup \{@_k \langle \mathfrak{a}_1 \rangle k', @_{k'} \langle \mathfrak{a}_2 \rangle k''\}$
6	$p \cup \{@_k \langle \mathfrak{a}_1 \rangle k', @_{k'} \langle \mathfrak{a}_2 \rangle k''\} \in P$	since $(V,N) \models p \cup \{@_k \langle \mathfrak{a}_1 \rangle k', @_{k'} \langle \mathfrak{a}_2 \rangle k''\}$ and $(V,N)\upharpoonright_\Delta = (W',M')\upharpoonright_\Delta \in \mathcal{K}$

P2) Assume that $@_k \langle \mathfrak{a} \rangle \gamma \in p$. Since $\mathsf{card}(C_{\mathsf{any}}) = \alpha$ and $\mathsf{card}(C'_{\mathsf{any}}) < \alpha$, there exists $k' \in C_{\mathsf{any}} \setminus C'_{\mathsf{any}}$. We show that $p \cup \{@_k \langle \mathfrak{a} \rangle k', @_{k'} \gamma\} \in P$:

1	$(W,M) \models p$ for some model (W,M) over $\Delta[C]$ with $(W,M)\upharpoonright_\Delta \in \mathcal{K}$	by the definition of $\mathbb{P}$
2	$(W',M') := (W,M)\upharpoonright_{\Delta[C']} \models p$	by the satisfaction condition
3	$(W'_k, w) \in W'_{\mathfrak{a}}$ and $(W',M') \models^w \gamma$ for some $w \in \lvert W' \rvert$	since $(W',M') \models @_k \langle \mathfrak{a} \rangle \gamma$
4	$(W'',M') \models @_k \langle \mathfrak{a} \rangle k'$ and $(W'',M') \models @_{k'} \gamma$, where (W'',M') is the unique expansion of (W',M') to $\Delta[C,k']$ interpreting k' as w	by semantics
5	$(V,N) \models p \cup \{@_k \langle \mathfrak{a} \rangle k', @_{k'} \gamma\}$, where (V,N) is any expansion of (W'',M') to $\Delta[C]$	by the satisfaction condition, since $(W'',M') \models p \cup \{@_k \langle \mathfrak{a} \rangle k', @_{k'} \gamma\}$
6	$p \cup \{@_k \langle \mathfrak{a} \rangle k', @_{k'} \gamma\} \in P$	since $(V,N)\upharpoonright_\Delta = (W',M')\upharpoonright_\Delta \in \mathcal{K}$

P3) Assume that $@_k \vee \Gamma \in p$. There exists a Kripke structure (W,M) over $\Delta[C]$ such that $(W,M)\upharpoonright_\Delta \in \mathcal{K}$ and $(W,M) \models p$. Since $(W,M) \models @_k \vee \Gamma$, we have $(W,M) \models @_k \gamma$ for some $\gamma \in \Gamma$. Since $(W,M) \models p$ and $(W,M) \models @_k \gamma$ and $(W,M)\upharpoonright_\Delta \in \mathcal{K}$, we get $p \cup \{@_k \gamma\} \in P$.

P4) Assume that $@_k \exists X \cdot \gamma \in p$. Since $\mathsf{card}(C'_s) < \alpha$ and $\mathsf{card}(C_s) = \alpha$ for all sorts $s \in S^{\mathsf{e}}$, by the finiteness of X, there exists an injective mapping $f \colon X \to C \setminus C'$. Let $C'' := C' \cup f(X)$. Let $\chi' \colon \Delta[C',X] \to \Delta[C'']$ be the unique extension of f to a signature morphism which preserves $\Delta[C']$. Let $\chi \colon \Delta[C,X] \to \Delta[C]$ be the unique extension of f to a signature morphism which preserves $\Delta[C]$. Let $\iota \colon \Delta[C''] \hookrightarrow \Delta[C]$ and $\iota' \colon \Delta[C',X] \hookrightarrow \Delta[C,X]$ be inclusions. Since χ and χ' agree on X and they preserve the rest of the symbols, we have $\chi' \mathbin{⨾} \iota = \iota' \mathbin{⨾} \chi$.

$$\begin{tikzcd}
& \Delta[C',X] \arrow[rr, dashed, "\iota'"] \arrow[d, dashed, "\chi'"] & & \Delta[C,X] \arrow[d, dashed, "\chi"] \\
\Delta[C'] \arrow[ur, hook, dashed] \arrow[r, hook] & \Delta[C''] \arrow[r, "\iota"'] & \Delta[C] \arrow[ur, hook] \arrow[r, "1_{\Delta[C]}"'] & \Delta[C]
\end{tikzcd}$$

We show that $p \cup \{@_k \chi(\gamma)\} \in P$:

1	$(W,M) \models p$ for some Kripke structure (W,M) over the signature $\Delta[C]$ such that $(W,M)\upharpoonright_\Delta \in \mathcal{K}$	by the definition of $\mathbb{P}$
2	$(W',M') := (W,M)\upharpoonright_{\Delta[C']} \models p$	by the satisfaction condition
3	$(V',N') \models^w \gamma$ for some expansion (V',N') of (W',M') to the signature $\Delta[C',X]$, where $w = W'_k = V'_k$	since $@_k \exists X \cdot \gamma \in p$ and $(W',M') \models p$
4	let (V'',N'') be the unique χ'-expansion of (V',N')	(V'',N'') exists, as χ' is a bijection
5	let (V,N) be any expansion of (V'',N'') to $\Delta[C]$	
6	$(V,N)\upharpoonright_\chi \upharpoonright_{\iota'} = (V,N)\upharpoonright_\iota \upharpoonright_{\chi'} = (V'',N'')\upharpoonright_{\chi'} = (V',N')$	from 4 and 5, since $\iota' \mathbin{⨾} \chi = \chi' \mathbin{⨾} \iota$
7	$(V,N)\upharpoonright_\chi \models^w \gamma$	by the local satisfaction condition, since $(V,N)\upharpoonright_\chi \upharpoonright_{\iota'} = (V',N') \models^w \gamma$
8	$(V,N) \models^w \chi(\gamma)$	by the local satisfaction condition
9	$(V,N) \models @_k \chi(\gamma)$	since $w = V'_k = (V\upharpoonright_\chi \upharpoonright_{\iota'})_k = V_k$
10	$(V,N) \models p$	by the satisfaction condition, since $(V,N)\upharpoonright_{\Delta[C']} = (W',M') \models p$

11	$(V,N)\restriction_\Delta \in \mathcal{K}$	since $(V,N)\restriction_{\Delta[C']} = (W',M')$ and $(W',M')\restriction_\Delta \in \mathcal{K}$
12	$p \cup \{@_k\, \chi(\gamma)\} \in P$	from 9—11

□

Proposition 40 sets the basis for the following important result concerning semantic forcing properties, which says that all sentences of a given condition are forced eventually by some condition greater or equal than the initial one.

Theorem 41 (Semantic Forcing Theorem). *Let $\mathbb{P} = \langle P, \leq, f\rangle$ be the semantic forcing property described in Definition 38. For all $\Delta[C]$-sentences $@_k\,\gamma$ and conditions $p \in P$ we have:*

$$q \Vdash^k \gamma \text{ for some } q \geq p \text{ iff } p \cup \{@_k\,\gamma\} \in P.$$

Proof. We proceed by induction on the structure of γ.

[*For γ extended atomic* **]** Assume that there is $q \geq p$ such that $q \Vdash^k \gamma$. We show that $p \cup \{@_k\,\gamma\} \in P$:

1	$@_k\,\gamma \in q$	by Definition 31
2	$p \cup \{@_k\,\gamma\} \leq q$	since $q \geq p$
3	$(W,M) \models q$ for some Kripke structure (W,M) over the signature $\Delta[C]$ such that $(W,M)\restriction_\Delta \in \mathcal{K}$	since $q \in P$
4	$p \cup \{@_k\,\gamma\} \in P$	since $(W,M) \models p \cup \{@_k\,\gamma\}$ and $(W,M)\restriction_\Delta \in \mathcal{K}$

Assume that $p \cup \{@_k\,\gamma\} \in P$. Let $q = p \cup \{@_k\,\gamma\}$. By Definition 31, $q \Vdash^k \gamma$.

[*For $\langle \mathfrak{a}_1 \cup \mathfrak{a}_2\rangle k''$* **]** The following are equivalent:

1	$q \Vdash^k \langle \mathfrak{a}_1 \cup \mathfrak{a}_2\rangle k''$ for some $q \geq p$	
2	$q \Vdash^k \langle \mathfrak{a}_1\rangle k''$ or $q \Vdash^k \langle \mathfrak{a}_2\rangle k''$	by Definition 31
3	$p \cup \{@_k\,\langle \mathfrak{a}_1\rangle k''\} \in P$ or $p \cup \{@_k\,\langle \mathfrak{a}_2\rangle k''\} \in P$	by the induction hypothesis
4	$(W,M) \models p \cup \{@_k\,\langle \mathfrak{a}_1\rangle k''\}$ or $(W,M) \models p \cup \{@_k\,\langle \mathfrak{a}_2\rangle k''\}$ for some Kripke structure (W,M) over $\Delta[C]$ such that $(W,M)\restriction_\Delta \in \mathcal{K}$	by Definition 38
5	$(W,M) \models p \cup \{@_k\,\langle \mathfrak{a}_1 \cup \mathfrak{a}_2\rangle k''\}$ for some Kripke structure (W,M) over $\Delta[C]$ such that $(W,M)\restriction_\Delta \in \mathcal{K}$	by the semantics of $\mathfrak{a}_1 \cup \mathfrak{a}_2$
6	$p \cup \{@_k\,\langle \mathfrak{a}_1 \cup \mathfrak{a}_2\rangle k''\} \in P$	since $(W,M) \models p \cup \{@_k\,\langle \mathfrak{a}_1 \cup \mathfrak{a}_2\rangle k''\}$ and $(W,M)\restriction_\Delta \in \mathcal{K}$

[*For $\langle \mathfrak{a}_1 \fatsemi \mathfrak{a}_2\rangle k''$* **]** Assume that $q \Vdash^k \langle \mathfrak{a}_1 \fatsemi \mathfrak{a}_2\rangle k''$ for some $q \geq p$. We show that $p \cup \{\langle \mathfrak{a}_1 \fatsemi \mathfrak{a}_2\rangle k''\} \in P$:

1	$q \Vdash^k \langle \mathfrak{a}_1\rangle k'$ and $q \Vdash^{k'} \langle \mathfrak{a}_2\rangle k''$ for some nominal k'	by Definition 31
2	$q \cup \{@_k\,\langle \mathfrak{a}_1\rangle k'\} \in P$	from $q \leq q$, by the induction hypothesis
3	$q \cup \{@_k\,\langle \mathfrak{a}_1\rangle k'\} \Vdash^{k'} \langle \mathfrak{a}_2\rangle k''$	from $q \Vdash^{k'} \langle \mathfrak{a}_2\rangle k''$ and $q \leq q \cup \{@_k\,\langle \mathfrak{a}_1\rangle k'\}$, by Lemma 33 (2)
4	$p \cup \{@_k\,\langle \mathfrak{a}_1\rangle k'\} \cup \{@_{k'}\,\langle \mathfrak{a}_2\rangle k''\} \in P$	from $p \cup \{\langle \mathfrak{a}_1\rangle k'\} \leq q \cup \{\langle \mathfrak{a}_1\rangle k'\}$, by the induction hypothesis
5	$p \cup \{@_k\,\langle \mathfrak{a}_1 \fatsemi \mathfrak{a}_2\rangle k''\} \in P$	since $\{@_k\,\langle \mathfrak{a}_1\rangle k', @_{k'}\,\langle \mathfrak{a}_2\rangle k''\} \models @_k\,\langle \mathfrak{a}_1 \fatsemi \mathfrak{a}_2\rangle k''$

Assume that $p \cup \{@_k\,\langle \mathfrak{a}_1 \fatsemi \mathfrak{a}_2\rangle k''\} \in P$. We show that $q \Vdash^k \langle \mathfrak{a}_1 \fatsemi \mathfrak{a}_2\rangle k''$ for some $q \geq p$:

1	$p \cup \{@_k\,\langle \mathfrak{a}_1 \fatsemi \mathfrak{a}_2\rangle k'', @_k\,\langle \mathfrak{a}_1\rangle k', @_{k'}\,\langle \mathfrak{a}_2\rangle k''\} \in P$ for some $k \in C_{\mathsf{any}}$	by Proposition 40 (P1)
2	let $r := p \cup \{@_k\,\langle \mathfrak{a}_1 \fatsemi \mathfrak{a}_2\rangle k'', @_k\,\langle \mathfrak{a}_1\rangle k', @_{k'}\,\langle \mathfrak{a}_2\rangle k''\}$	
3	$s \Vdash^k \langle \mathfrak{a}_1\rangle k'$ for some $s \geq r$	by the induction hypothesis, since $r \cup \{@_k\,\langle \mathfrak{a}_1\rangle k'\} = r \in P$
4	$q \Vdash^{k'} \langle \mathfrak{a}_2\rangle k''$ for some $q \geq s$	by the induction hypothesis, since $s \cup \{@_{k'}\,\langle \mathfrak{a}_2\rangle k''\} = s \in P$
5	$q \Vdash^k \langle \mathfrak{a}_1\rangle k'$	by Lemma 33 (2), since $s \Vdash^k \langle \mathfrak{a}_1\rangle k'$ and $q \geq s$
6	$q \Vdash^k \langle \mathfrak{a}_1 \fatsemi \mathfrak{a}_2\rangle k''$	from 4 and 5

[*For* $\langle\mathfrak{a}^*\rangle k''$] The following are equivalent:

1	$q \Vdash^k \langle\mathfrak{a}^*\rangle k''$ for some $q \geq p$	
2	$q \Vdash^k \langle\mathfrak{a}^n\rangle k''$ for some $q \geq p$ and $n \in \mathbb{N}$	by Definition 31
3	$p \cup \{@_k \langle\mathfrak{a}^n\rangle k''\} \in P$ for some $n \in \mathbb{N}$	by the induction hypothesis
4	$p \cup \{@_k \langle\mathfrak{a}^*\rangle k''\} \in P$	by the semantics of $\mathfrak{a}^*$ and the definition of $\mathbb{P}$

[*For* $\langle\mathfrak{a}\rangle\gamma$ *with* $\gamma \notin F^{\mathsf{n}} \cup C_{\mathsf{any}}$] Assume that $q \Vdash^k \langle\mathfrak{a}\rangle\gamma$ for some $q \geq p$. We show that $p \cup \{@_k \langle\mathfrak{a}\rangle\gamma\} \in P$:

1	$q \Vdash^k \langle\mathfrak{a}\rangle k'$ and $q \Vdash^{k'} \gamma$ for some nominal k'	from $q \Vdash^k \langle\mathfrak{a}\rangle\gamma$, by Definition 31
2	$q \cup \{@_k \langle\mathfrak{a}\rangle k'\} \in P$	from $q \leq q$ and $q \Vdash^k \langle\mathfrak{a}\rangle k'$, by the induction hypothesis
3	$q \cup \{@_k \langle\mathfrak{a}\rangle k'\} \Vdash^{k'} \gamma$	from 1 and 2, by Lemma 33 (2)
4	$p \cup \{@_k \langle\mathfrak{a}\rangle k'\} \cup \{@_{k'} \gamma\} \in P$	from $p \cup \{@_k \langle\mathfrak{a}\rangle k'\} \leq q \cup \{@_k \langle\mathfrak{a}\rangle k'\}$ and $q \cup \{@_k \langle\mathfrak{a}\rangle k'\} \Vdash^{k'} \gamma$, by the induction hypothesis
5	$p \cup \{@_k \langle\mathfrak{a}\rangle\gamma\} \in P$	since $\{@_k \langle\mathfrak{a}\rangle k', @_{k'} \gamma\} \models @_k \langle\mathfrak{a}\rangle\gamma$

Assume that $p \cup \{@_k \langle\mathfrak{a}\rangle\gamma\} \in P$. We show that $q \Vdash^k \langle\mathfrak{a}\rangle\gamma$ for some $q \geq p$:

1	$(p \cup \{@_k \langle\mathfrak{a}\rangle\gamma\}) \cup \{@_k \langle\mathfrak{a}\rangle k'\} \cup \{@_{k'} \gamma\} \in P$ for some $k' \in C_{\mathsf{any}}$	by Proposition 40 (P2)
2	let $p_1 := p \cup \{@_k \langle\mathfrak{a}\rangle\gamma\} \cup \{@_k \langle\mathfrak{a}\rangle k'\} \cup \{@_{k'} \gamma\}$	
3	$p_2 \Vdash^k \langle\mathfrak{a}\rangle k'$ for some $p_2 \geq p_1$	from $p_1 \cup \{@_k \langle\mathfrak{a}\rangle k'\} = p_1 \in P$, by the induction hypothesis
4	$q \Vdash^{k'} \gamma$ for some $q \geq p_2$	from $p_2 \cup \{@_{k'} \gamma\} = p_2 \in P$, by the induction hypothesis
5	$q \Vdash^k \langle\mathfrak{a}\rangle k'$	from $p_2 \Vdash^k \langle\mathfrak{a}\rangle k'$ and $q \geq p_2$, by Lemma 33 (2)
6	$q \Vdash^k \langle\mathfrak{a}\rangle\gamma$	from 4 and 5

[*For* $\neg\gamma$] By the induction hypothesis, for each $q \in P$ we have

(S1) $r \Vdash^k \gamma$ for some $r \geq q$ iff $q \cup \{@_k \gamma\} \in P$, which is equivalent to

(S2) $r \nVdash^k \gamma$ for all $r \geq q$ iff $q \cup \{@_k \gamma\} \notin P$, which is equivalent to

(S3) $q \Vdash^k \neg\gamma$ iff $q \cup \{@_k \gamma\} \notin P$.

Assume that $q \Vdash^k \neg\gamma$ for some $q \geq p$. We show that $p \cup \{@_k \neg\gamma\} \in P$:

1	$q \cup \{@_k \gamma\} \notin P$	by statement S3
2	$(W, M) \models q$ for some Kripke structure (W, M) over $\Delta[C]$ such that $(W, M)\upharpoonright_\Delta \in \mathcal{K}$	by Definition 38, since $q \in P$
3	$(W, M) \not\models @_k \gamma$	since $q \cup \{@_k \gamma\} \notin P$
4	$(W, M) \models @_k \neg\gamma$	by the semantics of $\neg$
5	$q \cup \{@_k \neg\gamma\} \in P$	since $(W, M) \models q \cup \{@_k \neg\gamma\}$ and $(W, M)\upharpoonright_\Delta \in \mathcal{K}$
6	$p \cup \{@_k \neg\gamma\} \in P$	since $p \cup \{@_k \neg\gamma\} \leq q \cup \{@_k \neg\gamma\}$

Assume that $p \cup \{@_k \neg\gamma\} \in P$. We show that $q \Vdash^k \neg\gamma$ for some $q \geq p$:

1	let $q = p \cup \{@_k \neg\gamma\}$	
2	$q \cup \{@_k \gamma\} \notin P$	since $@_k \neg\gamma \in q$
3	$q \Vdash^k \neg\gamma$	by statement S3

[*For* $\vee\Gamma$] Assume that there exists $q \geq p$ such that $q \Vdash^k \vee\Gamma$. We show that $p \cup \{@_k \vee\Gamma\} \in P$:

1	$q \Vdash^k \gamma$ for some $\gamma \in \Gamma$	by Definition 31
2	$p \cup \{@_k \gamma\} \in P$	by the induction hypothesis
3	$p \cup \{@_k \vee\Gamma\} \in P$	since $@_k \gamma \models @_k \vee\Gamma$

Assume that $p \cup \{@_k \vee\Gamma\} \in P$. We show that $q \Vdash^k \vee\Gamma$ for some $q \geq p$:

1	$(p \cup \{@_k \vee \Gamma\}) \cup \{@_k \gamma\} \in P$ for some $\gamma \in \Gamma$	by Proposition 40 (P3)
2	$q \Vdash^k \gamma$ for some $q \geq p \cup \{@_k \vee \Gamma\}$	by the induction hypothesis
3	$q \Vdash^k \vee\Gamma$ for some $q \geq p$	by Definition 31

[*For* $\exists X \cdot \gamma$ **]** Assume that $q \Vdash^k \exists X \cdot \gamma$ for some $q \geq p$. We show that $p \cup \{@_k \exists X \cdot \gamma\} \in P$:

1	$q \Vdash^k \theta(\gamma)$ for some substitution $\theta : X \to \emptyset$	by Definition 31
2	$p \cup \{@_k \theta(\gamma)\} \in P$	by the induction hypothesis
3	$p \cup \{@_k \exists X \cdot \gamma\} \in P$	since $@_k \theta(\gamma) \models @_k \exists X \cdot \gamma$

We assume that $p \cup \{@_k \exists X \cdot \gamma\} \in P$. We show that $q \Vdash^k \exists X \cdot \gamma$ for some $q \geq p$:

1	$(p \cup \{@_k \exists X \cdot \gamma\}) \cup \{@_k \chi(\gamma)\} \in P$ for some signature morphism $\chi \colon \Delta[C, X] \to \Delta[C]$ which preserves $\Delta[C]$	by Proposition 40 (P4)
2	$q \Vdash^k \chi(\gamma)$ for some $q \geq p \cup \{@_k \exists X \cdot \gamma\}$	by the induction hypothesis
3	$q \Vdash^k \exists X \cdot \gamma$ for some $q \geq p$	by Definition 31

[*For* $\downarrow z \cdot \gamma$ **]** This case is straightforward, as $@_k \downarrow z \cdot \gamma$ is semantically equivalent to $@_k \gamma(z \leftarrow k)$.

[*For* $@_{k'} \gamma$ **]** This case is straightforward, as $@_k @_{k'} \gamma$ is semantically equivalent to $@_{k'} \gamma$. □

The following result is a corollary of Theorem 41. It shows that each generic set of a given semantic forcing property has a reachable model that satisfies all its conditions.

Corollary 42. *Let $\mathbb{P} = \langle P, \leq, f\rangle$ be the semantic forcing property described in Definition 38. Then for each generic set G we have:*

C1) $G \Vdash^k \gamma$ for all conditions $p \in G$, sentences $\gamma \in p$ and nominals $k \in F^{\mathsf{n}} \cup C_{\mathsf{any}}$.

C2) There exists a generic structure (W^G, M^G) for G which is reachable and satisfies each condition $p \in G$.

Proof.

C1) Suppose towards a contradiction that $G \nVdash^k \gamma$ for some $p \in G$, $\gamma \in p$ and nominal $k \in F^{\mathsf{n}} \cup C_{\mathsf{any}}$. Then:

1	$q \Vdash^k \neg\gamma$ for some $q \in G$	from $G \nVdash^k \gamma$, since G is generic
2	$r \geq p$ and $r \geq q$ for some $r \in G$	since G is generic
3	$\gamma \in r$	since $\gamma \in p$ and $r \geq p$
4	$r \cup \{@_k \gamma\} \in P$	since $r \models @_k \gamma$
5	$s \Vdash^k \gamma$ for some $s \geq r$	by Theorem 41
6	$s \Vdash^k \neg\gamma$	from $q \Vdash^k \neg\gamma$ and $s \geq q$, by Lemma 33 (2)
7	contradiction	from 5 and 6, by Lemma 33 (4)

It follows that $G \Vdash^k \gamma$ for all $p \in G$, $\gamma \in p$ and nominals k.

C2) By Theorem 37, there exists a generic model (W^G, M^G) for G which is reachable. Let $p \in G$, $\gamma \in p$ and $w \in |W^G|$. Since (W^G, M^G) is reachable, w is the denotation of some nominal $k \in F^{\mathsf{n}} \cup C_{\mathsf{any}}$. By the first part of the proof, $G \Vdash^k \gamma$. Since (W^G, M^G) is a model for G, we get $(W^G, M^G) \models @_k \gamma$, which means $(W^G, M^G) \models^w \gamma$. As $w \in |W^G|$ was arbitrary chosen, $(W^G, M^G) \models \gamma$. □

7. Omitting Types Theorem

Let $\Delta = (\Sigma^{\mathsf{n}}, \Sigma^{\mathsf{r}} \subseteq \Sigma^{\mathsf{n}})$ be a countable signature. Let $X = \{X_s\}_{s \in S^{\mathsf{e}}}$ be a set of variables for Δ such that X_s is finite for all sorts $s \in S^{\mathsf{e}}$ which can be used for quantification. A Kripke structure (W, M) over Δ *realizes* a set Γ of sentences over $\Delta[X]$ iff there exists an expansion (V, N) of (W, M) to $\Delta[X]$ such that $(V, N) \models \Gamma$. (W, M) *omits* Γ if (W, M) does not realize Γ. A satisfiable set T of sentences over Δ *locally realizes* Γ if there exists a finite set p of sentences over $\Delta[X]$ such that $T \cup p$ is satisfiable, and $T \cup p \models \Gamma$. In the following we generalize these definitions to signatures of any power.

Definition 43 (Omitting Types semantically). Assume a signature $\Delta = (\Sigma^{\mathsf{n}}, \Sigma^{\mathsf{r}} \subseteq \Sigma^{\mathsf{n}})$, and let α be the power of Δ. Let $X = \{X_s\}_{s\in S^{\mathsf{e}}}$ be set of variables for Δ such that X_s is finite for all sorts $s \in S^{\mathsf{e}}$.

- A Kripke structure (W, M) over Δ realizes a type $\Gamma \subseteq \mathsf{Sen}(\Delta[X])$ if there exists an expansion (V, N) of (W, M) to $\Delta[X]$ such that $(V, N) \models \Gamma$.
- A Kripke structure (W, M) over Δ omits a set Γ of $\Delta[X]$-sentences if (W, M) does not realize Γ.

Classically, Γ from Definition 43 is called a type with free variables X.

Definition 44 (Omitting Types syntactically). Let Δ be a signature, and let α be the power of Δ. Let $X = \{X_s\}_{s\in S^{\mathsf{e}}}$ be a sorted set of variables for Δ such that X_s is finite for all sorts $s \in S^{\mathsf{e}}$. A theory $T \subseteq \mathsf{Sen}(\Delta)$ *α-realizes* a type $\Gamma \subseteq \mathsf{Sen}(\Delta[X])$ if there exist

- a sorted set $C = \{C_s\}_{s\in S^{\mathsf{e}}}$ of new constants for Δ with $\mathsf{card}(C_s) < \alpha$ for all $s \in S^{\mathsf{e}}$,
- a substitution $\theta : X \to C$, and
- a set of sentences p over $\Delta[C]$ with $\mathsf{card}(p) < \alpha$,

such that $T \cup p$ is satisfiable and $T \cup p \models \theta(\Gamma)$. We say that T α-omits Γ if T does not α-realize Γ.

Notice that the power of any signature is at least ω. If $\alpha = \omega$, we say that T locally omits Γ instead of T α-omits Γ. Definition 44 is similar to the definition of locally omitting types for first-order logic without equality from [41]. Our results are applicable to fragments $\mathcal{L}$ without equality. We say $\mathcal{L}$ *has equality* if for all signatures Δ of $\mathcal{L}$ we have:

(a) for any nominal k there exists an $\mathcal{L}$-sentence φ such that $(W, M) \models^{w} \varphi$ iff $w = W_k$ for all Kripke structures (W, M) and all possible worlds $w \in |W|$, and

(b) for any hybrid terms $t_1, t_2 \in T_{\overline{\Sigma}}$ there exists an $\mathcal{L}$-sentence φ such that $(W, M) \models^{w} \varphi$ iff $M_{w,t_1} = M_{w,t_2}$ for all Kripke structures (W, M) and all possible worlds $w \in |W|$.

We give a couple of equivalent descriptions of the omitting types property which can be found in the literature.

Lemma 45. *Assume that $\mathcal{L}$ has equality.*

L1) T α-realizes Γ as described in Definition 44 iff there exist (a) a sorted set $C = \{C_s\}_{s\in S^{\mathsf{e}}}$ of new constants for $\Delta[X]$ with $\mathsf{card}(C_s) < \alpha$ for all $s \in S^{\mathsf{e}}$, and (b) a set of sentences p over $\Delta[C, X]$ with $\mathsf{card}(p) < \alpha$, such that $T \cup p$ is satisfiable and $T \cup p \models \Gamma$.

L2) Assume that $\mathcal{L}$ is semantically closed under Boolean connectives and quantifiers. Then T locally realizes Γ iff there exists a $\Delta[X]$-sentence φ such that $T \cup \{\varphi\}$ is satisfiable and $T \cup \{\varphi\} \models \Gamma$.

L3) Assume that $\mathcal{L}$ is compact and semantically closed under Boolean connectives and quantifiers. Then T α-realizes Γ iff there exists a set of $\Delta[X]$-sentences p with $\mathsf{card}(p) < \alpha$ such that $T \cup p$ is satisfiable and $T \cup p \models \Gamma$.

Proof. The backward implication is straightforward for all cases. Therefore, we will focus on the forward implication.

$$\begin{tikzcd}
 & & \Delta[C,X] \arrow[rr, dash, dotted] & & p^{\theta} \\
\Gamma \arrow[r, dash, dotted] & \Delta[X] \arrow[ur, hook] \arrow[rr, dashed, "\theta"] & & \Delta[C] \arrow[ul, hook'] \arrow[r, dash, dotted] & p \\
 & & \Delta \arrow[ul, hook] \arrow[ur, hook'] \arrow[rr, dash, dotted] & & T
\end{tikzcd}$$

Consider (a) a signature Δ of power α, (b) a set of variables $X = \{X_s\}_{s\in S^{\mathsf{e}}}$ for Δ such that X_s is finite for all sorts $s \in S^{\mathsf{e}}$, (c) a set of new constants $C = \{C_s\}_{s\in S^{\mathsf{e}}}$ such that $\mathsf{card}(C_s) < \alpha$ for all sorts $s \in S^{\mathsf{e}}$, (d) a substitution $\theta\colon X \to C$, and (e) a set of sentences p over $\Delta[C]$ with $\mathsf{card}(p) < \alpha$ such that $T \cup p$ is satisfiable and $T \cup p \models \theta(\Gamma)$. Without loss of generality, we assume that $X \cap C = \emptyset$. Since $\mathcal{L}$ has equality, there exists a set of sentences p^θ over $\Delta[C, X]$ semantically equivalent with $\{x = \theta(x) \mid x \in X\}$. [10] Since $T \cup p$ is satisfiable, $T \cup p \cup p^\theta$ is satisfiable too.

L1) Since $p \cup p^\theta$ is a set of sentences over $\Delta[C, X]$), it suffices to show that $T \cup p \cup p^\theta \models \Gamma$:

1	let $(W, M) \in \lvert\mathsf{Mod}(\Delta[C, X])\rvert$ such that $(W, M) \models T \cup p \cup p^\theta$	
2	$(W, M)\upharpoonright_{\Delta[C]}\upharpoonright_\theta = (W, M)\upharpoonright_{\Delta[X]}$	since $(W, M) \models p^\theta$
3	$(W, M)\upharpoonright_{\Delta[C]} \models T \cup p$	by the satisfaction condition, since $(W, M) \models T \cup p$
4	$(W, M)\upharpoonright_{\Delta[C]} \models \theta(\Gamma)$	since $T \cup p \models \theta(\Gamma)$ and $(W, M)\upharpoonright_{\Delta[C]} \models T \cup p$
5	$(W, M)\upharpoonright_{\Delta[X]} = (W, M)\upharpoonright_{\Delta[C]}\upharpoonright_\theta \models \Gamma$	by the satisfaction condition for substitutions
6	$(W, M) \models \Gamma$	by the satisfaction condition
7	$T \cup p \cup p^\theta \models \Gamma$	since (W, M) was arbitrarily chosen

L2) If $\alpha = \omega$, we show $T \cup \{\varphi\} \models \Gamma$ for a single sentence φ over $\Delta[X]$:

1	the sets C, p and p^θ are finite	since their cardinals are strictly less than ω
2	there exists a $\Delta[X]$-sentence φ semantically equivalent with $\exists C \cdot + \wedge(p \cup p^\theta)$	since $\mathcal{L}$ is semantically closed under conjunction, quantifiers and retrieve
3	$T \cup \{\varphi\}$ is satisfiable over $\Delta[X]$	since $T \cup p \cup p^\theta$ is satisfiable over $\Delta[C, X]$
4	$T \cup \{\varphi\} \models \Gamma$	since $T \cup p \cup p^\theta \models \Gamma$

L3) If $\mathcal{L}$ is compact, we show that $T \cup p' \models \Gamma$ for a set p' of sentences over $\Delta[X]$ with $\mathsf{card}(p') < \alpha$:

1	for each $\gamma \in \Gamma$ there exists $p^\gamma \subseteq p \cup p^\theta$ finite such that $T \cup p^\gamma \models \gamma$	by compactness, since $T \cup p \cup p^\theta \models \gamma$ for all $\gamma \in \Gamma$
2	let C^γ be all constants from C which occur in p^γ for all $\gamma \in \Gamma$	
3	there exists a set p' of $\Delta[X]$-sentences semantically equivalent with $\{\exists C^\gamma \cdot + \wedge p^\gamma \mid \gamma \in \Gamma\}$	since $\mathcal{L}$ is semantically closed under conjunction, quantifiers and retrieve
4	$T \cup p'$ is satisfiable over $\Delta[C]$	since $T \cup p \cup p^\theta$ is satisfiable over $\Delta[C, X]$
5	$T \cup p' \models \Gamma$	since $T \cup p^\gamma \models \gamma$ for all $\gamma \in \Gamma$
6	$\mathsf{card}(\mathcal{P}_\omega(p \cup p^\theta)) < \alpha$	since $\mathsf{card}(p) < \alpha$ and $\mathsf{card}(p^\theta) < \alpha$
7	$\mathsf{card}(\{p^\gamma \mid \gamma \in \Gamma\}) < \alpha$	since $\{p^\gamma \mid \gamma \in \Gamma\} \subseteq \mathcal{P}_\omega(p \cup p^\theta)$
8	$\mathsf{card}(p') < \alpha$	by its definition, p' is in one-to-one correspondence with $\{p^\gamma \mid \gamma \in \Gamma\}$

□

The following result is needed for proving the Omitting Types Theorem.

Lemma 46. *Assume that T α-omits Γ as described in Definition 44. Then for any substitution $\theta : X \to C$ over Δ such that $\mathsf{card}(C_s) < \alpha$ for all $s \in S^{\mathsf{e}}$, and any set of $\Delta[C]$-sentences p such that $\mathsf{card}(p) < \alpha$ and $T \cup p$ is satisfiable, there exists $\gamma \in \Gamma$ such that $T \cup p \cup \{@_z \neg\theta(\gamma)\}$ is satisfiable, where z is a nominal variable for $\Delta[C]$.*

Proof. Let $C = \{C_s\}_{s\in S^{\mathsf{e}}}$ be a set of new constants for Δ with $\mathsf{card}(C_s) < \alpha$ for all $s \in S^{\mathsf{e}}$. Let $\theta : X \to C$ be a substitution over Δ. Let p be a set of $\Delta[C]$-sentences such that $\mathsf{card}(p) < \alpha$ and $T \cup p$ satisfiable. Since T α-omits Γ, we have $T \cup p \not\models \theta(\Gamma)$. There exists a Kripke structure (W, M) over $\Delta[C]$ such that $(W, M) \models T \cup p$ and $(W, M) \not\models \theta(\Gamma)$. It follows that $(W, M) \models^w \neg\theta(\gamma)$ for some possible world $w \in |W|$ and some sentence $\gamma \in \Gamma$. Let z be a new nominal for $\Delta[C]$, and let $(W^{z\leftarrow w}, M)$ be the unique expansion of (W, M) to $\Delta[z, C]$ which interprets z as w. Since $(W, M) \models^w \neg\theta(\gamma)$, we get $(W^{z\leftarrow w}, M) \models @_z \neg\theta(\gamma)$. Hence, $T \cup p \cup \{@_z \neg\theta(\gamma)\}$ is satisfiable. □

Definition 47 (Omitting Types Property). We say that $\mathcal{L}$ has α-Omitting Types Property (α-OTP), where α is an infinite cardinal, whenever

[10] Here = is a shorthand from the metalanguage. In particular, for nominals $x = \theta(x)$ means that $@_x\, \theta(x)$ for all $x \in X_{\mathsf{any}}$.

- for all signatures Δ of power at most α,
- all satisfiable theories $T \subseteq \mathsf{Sen}(\Delta)$, and
- all families of types $\{\Gamma^i \subseteq \mathsf{Sen}(\Delta[X^i]) \mid i < \alpha\}$,
  where $X^i = \{X^i_s\}_{s \in S^{\mathsf{e}}}$ is a set of variables for Δ with X^i_s finite for all $s \in S^{\mathsf{e}}$,

such that T α-omits Γ^i for all $i < \alpha$, there exists a Kripke structure over Δ which satisfies T and omits Γ^i for all $i < \alpha$. If $\alpha = \omega$ then we say that $\mathcal{L}$ has OTP rather than $\mathcal{L}$ has ω-OTP.

All the ingredients for proving Omitting Types Theorem are in place.

Theorem 48 (Extended Omitting Types Theorem)**.** *Let α be an infinite cardinal. Assume that $\mathcal{L}$ is semantically closed under retrieve and negation, and if $\alpha > \omega$ assume that $\mathcal{L}$ is compact. Then $\mathcal{L}$ has α-OTP.*

Proof. Assume that T α-omits Γ^i as described in Definition 44. Let $C = \{C_s\}_{s \in S^{\mathsf{e}}}$ be a sorted set of new constants for Δ such that $\mathsf{card}(C_s) = \alpha$ for all $s \in S^{\mathsf{e}}$. Let $\mathbb{P} = (P, \leq, f)$ be the semantic forcing property described in Definition 38 with $\mathcal{K} = |\mathsf{Mod}(\Delta, T)|$. The proof is performed in four steps.

$$\begin{tikzcd}
 & \Delta[X^i] \arrow[rr, hook] \arrow[d, dashed, "\theta'"] & & \Delta[C, X^i] \arrow[d, dashed, "\theta"] \\
\Delta \arrow[ur, hook] \arrow[r, hook] & \Delta[C'] \arrow[r, hook] & \Delta[C] \arrow[ur, hook] \arrow[r, "1_{\Delta[C]}"'] & \Delta[C]
\end{tikzcd}$$

S1) We show that for any condition $p \in P$, any index $i < \alpha$, and any substitution $\theta : X^i \to \emptyset$ over $\Delta[C]$, there exist a sentence $\gamma \in \Gamma^i$ and a nominal $c \in C$ such that $p \cup \{@_c \neg\theta(\gamma)\} \in P$:

1	let C^p be the set of all constants from C which occur in p	
2	there exists $c \in C_{\mathsf{any}} \setminus (\theta(X^i_{\mathsf{any}}) \cup C^p_{\mathsf{any}})$	since $\mathsf{card}(\theta(X^i_{\mathsf{any}})) < \omega$, $\mathsf{card}(C^p_{\mathsf{any}}) < \alpha$ and $\mathsf{card}(C_{\mathsf{any}}) = \alpha$
3	let $C' \coloneqq \theta(X^i) \cup C^p \cup \{c\}$	
4	let $\theta' : X^i \to C'$ be the substitution over Δ defined by $\theta(x) = \theta'(x)$ for all $x \in X^i$	
5	$T \cup p \cup \{@_c \neg\theta'(\gamma)\}$ is satisfiable for some $\gamma \in \Gamma^i$	by Lemma 46, since T α-omits Γ^i
6	$T \cup p \cup \{@_c \neg\theta(\gamma)\}$ is satisfiable for some $\gamma \in \Gamma^i$	since $@_c \neg\theta(\gamma) = @_c \neg\theta'(\gamma)$
7	$p \cup \{@_c \neg\theta(\gamma)\} \in P$	since $(W, M) \models p \cup \{@_c \neg\theta(\gamma)\}$ for some $(W, M) \in |\mathsf{Mod}(\Delta[C])|$ such that $(W, M) \upharpoonright_\Delta \in |\mathsf{Mod}(\Delta, T)|$

S2) The cardinality of the set $\mathcal{S}^i$ of all substitutions $\theta : X^i \to \emptyset$ over $\Delta[C]$ is equal or less than α. It follows that the cardinality of $\mathcal{S} \coloneqq \bigcup_{i<\alpha} \mathcal{S}^i$ is equal or less than α. Let $\{\theta^j : X^{i_j} \to \emptyset \in \mathcal{S} \mid j < \alpha\}$ be an enumeration of $\mathcal{S}$. Let $\{@_{k_j} \varphi_j \in \mathsf{Sen}(\Delta[C]) \mid j < \alpha\}$ be an enumeration of the $\Delta[C]$-sentences with retrieve as the top operator. We define an increasing chain of conditions $p_0 \leq p_1 \leq \dots$ by induction on ordinals:

[$j = 0$] $p_0 \coloneqq \emptyset$.

[$j \Rightarrow j + 1$] If $p_j \Vdash^{k_j} \neg\varphi_j$ then let $q \coloneqq p_j$ else let $q \geq p_j$ be a condition such that $q \Vdash^{k_j} \varphi_j$. By the first part of the proof, there exist $\gamma \in \Gamma^{i_j}$ and $c \in C$ such that $q \cup \{@_c \neg\theta^j(\gamma)\} \in P$. Let $p_{j+1} \coloneqq q \cup \{@_c \neg\theta^j(\gamma)\}$.

[*$\beta < \alpha$ limit ordinal*] $p_\beta \coloneqq \bigcup_{j<\beta} p_j$. Since $\mathsf{card}(p_j) < \alpha$ for all $j < \beta$ and $\beta < \alpha$, we have $\mathsf{card}(p_\beta) < \alpha$. Since $p_j \in P$ for all $j < \beta$, the set $T \cup p_j$ is satisfiable for all $j < \beta$. By compactness[11], $(\bigcup_{j<\beta} p_j) \cup T$ is satisfiable too. Hence, $p_\beta \in P$.

[11] If there exists a limit ordinal $\beta < \alpha$ then α is not countable, so we assume $\mathcal{L}$ is compact.

The set $G = \{q \in P \mid q \le p_{j+1} \text{ for some } j < \alpha\}$ is generic. Let $k \in F^{\mathsf{n}} \cup C_{\mathsf{any}}$ and $\psi \in T$. Suppose towards a contradiction that $q \Vdash^k \neg\psi$ for some $q \in G$ then:

1	$q \cup \{@_k\, \psi\} \in P$	since $\psi \in T$ and $q \cup T$ is satisfiable
2	$r \Vdash^k \psi$ for some $r \ge q$	by Theorem 41
3	$r \Vdash^k \neg\psi$	since $q \Vdash^k \neg\psi$ and $r \ge q$
4	contradiction	by Lemma 33 (4) from 2 and 3

Since G is generic, $q \Vdash^k \psi$ for some $q \in G$.

S3) By Theorem 37, there exists a generic Kripke structure (W, M) for G that is reachable. Let $(V, N) \coloneqq (W, M)\upharpoonright_{\Delta}$. We show that $(V, N) \models T$:

1	let $w \in \lvert V\rvert$ and $\psi \in T$	
2	$W_k = w$ for some $k \in F^{\mathsf{n}} \cup C_{\mathsf{any}}$	since $\lvert W\rvert = \lvert V\rvert$ and (W, M) is reachable
3	$G \Vdash^k \psi$	by the second part of the proof
4	$(W, M) \models @_k\, \psi$	since (W, M) is generic for G
5	$(W, M) \models^w \psi$	by the semantics of @, since $W_k = w$
6	$(W, M) \models T$	since $w \in \lvert W\rvert$ and $\psi \in T$ were arbitrarily chosen
7	$(V, N) \models T$	by the satisfaction condition, since $(W, M)\upharpoonright_{\Delta} = (V, N)$

S4) We show that (V, N) omits Γ^i for all $i < \alpha$:

1	let (V', N') be an arbitrary expansion of (V, N) to $\Delta[X^i]$	
2	there exists an expansion (W', M') of (W, M) to $\Delta[C, X^i]$ such that $(W', M')\upharpoonright_{\Delta[X^i]} = (V', N')$	by interpreting X^i as (V', N') interprets X^i
3	there exists $\theta^j : X^i \to \emptyset \in \mathcal{S}$ such that $(W, M)\upharpoonright_{\theta^j} = (W', M')$	since (W, M) is reachable
4	there exist $c \in C$ and $\gamma \in \Gamma^i$ such that $@_c\, \neg\theta^j(\gamma) \in p_{j+1}$	by the construction of the chain $p_0 \le p_1 \le \ldots$
5	$(W, M) \models p_{j+1}$	by Corollary 42, since $p_{j+1} \in G$ and (W, M) is generic for G
6	$(W, M) \models^w \neg\theta^j(\gamma)$, where $w = W_c$	since $@_c\, \neg\theta^j(\gamma) \in p_{j+1}$
7	$(W', M') \models^w \neg\gamma$	by the local satisfaction condition for θ^j
8	$(V', N') \models^w \neg\gamma$	by the local satisfaction condition, since $(W', M')\upharpoonright_{\Delta[X^i]} = (V', N')$
9	$(V', N') \not\models \Gamma^i$	since $\gamma \in \Gamma^i$ and $(V', N') \not\models^w \gamma$
10	(V, N) omits Γ^i	since (V', N') is an arbitrary expansion of (V, N)

We conclude that (V, N) is a Kripke structure over Δ which satisfies T and omits Γ^i for all $i < \alpha$. $\square$

Any fragment $\mathcal{L}$ of HDFOLR free of the Kleene operator is compact. If, in addition, $\mathcal{L}$ is semantically closed under negation and retrieve, $\mathcal{L}$ is an instance of Theorem 48. In particular, any fragment presented in Examples 17 — 20 can be an instance of $\mathcal{L}$ from Theorem 48. Omitting Types Theorem is obtained from Theorem 48 by restricting the signatures Δ to countable ones. By Lemma 45 (L2), Omitting Types Theorem is a corollary of Extended Omitting Types Theorem. Any fragment $\mathcal{L}$ of HDFOLR with equality and closed under Boolean connectives, quantifiers and retrieve, is an instance of Omitting Types Theorem.

Notice that the forcing technique developed in the present contribution is not applicable to HFOLS as this logic lacks support for the substitutions described in Section 4.1. However, by Lemma 21, OTP can be borrowed from HFOLR to HFOLS.

Theorem 49. HFOLS *has α-OTP for all infinite cardinals α.*

Proof. Recall that for all HFOLS signatures Δ, we have:

- $\mathsf{Sen}^{\mathsf{HFOLS}}(\Delta) \subseteq \mathsf{Sen}^{\mathsf{HFOLR}}(\Delta)$, and
- by Lemma 21, for every sentence $\gamma \in \mathsf{Sen}^{\mathsf{HFOLR}}(\Delta)$ there exists a sentence $\gamma' \in \mathsf{Sen}^{\mathsf{HFOLS}}(\Delta)$ which is satisfied by the same class of Kripke structures as γ.

Assume that T α-omits Γ^i as described in Definition 44. By the remarks above, T α-omits Γ^i in HFOLR for all $i < \alpha$. By Theorem 48, there exists a Kripke structure (W, M) over Δ, which satisfies T and omits Γ^i for all $i < \alpha$. □

Theorem 49 is important in computer science, since variations of HFOLS has been used to implement theorem provers such as the one presented in [11] or [44]. It is worth noting that in general the Omitting Types Property cannot be borrowed from a given logic to its restrictions. If T omits Γ^i in a restriction then T might not omit Γ^i in the full underlying logic. This is the reason for developing Theorem 48 in an arbitrary fragment $\mathcal{L}$ of HDFOLR.

8. Constructor-based completeness

Constructor-based completeness is a modern approach to the well-known ω-completeness, which has applications in formal methods. We make the result independent of the arithmetic signature by working over an arbitrary vocabulary where we distinguish a set of constructors which determines a class of Kripke structures reachable by constructors. Throughout this section, we assume that the fragment $\mathcal{L}$ (a) has only countable signatures, (b) has equality, and (c) it is semantically closed under Boolean connectives, quantifiers and retrieve. An example of such fragment $\mathcal{L}$ is the restriction of HDFOLR or HDPL to countable signatures. In this case, the characterization of omitting types given by Lemma 45 (L2) is applicable.

8.1. Semantic restrictions

Given a theory T over a vocabulary Δ, not all Kripke structures are of interest. In many cases, formal methods practitioners are interested in the properties of a class Kripke structures that are reachable by a set of constructor operators. Let $\Delta = (\Sigma^{\mathrm{n}}, \Sigma^{\mathrm{r}} \subseteq \Sigma)$ be a signature and $\Sigma^{\mathrm{c}} \subseteq \Sigma^{\mathrm{r}}$ a subset of constructor operators. The constructors create a partition of the set of rigid sorts S^{r}. A *constrained* sort is a rigid sort $s \in S^{\mathrm{r}}$ that has a constructor, that is, there exists a constructor $\sigma\colon w \to s$ in Σ^{c}. A rigid sort that is not constrained it is called *loose*. We denote by S^{c} the set of all constrained sorts, and by S^{1} the set of all loose sorts. Let $Y = \{Y_s\}_{s \in S^1}$ be a set of loose variables such that Y_s is countably infinite for all $s \in S^1$. A constructor-based Kripke structure is a Kripke structure (W, M) such that

- for all possible worlds $w \in |W|$ there exists a nominal $k \in F^{\mathrm{n}}$ such that $w = W_k$, and
- for all rigid sorts $s \in S^{\mathrm{r}}$, all possible worlds $w \in |W|$, and all elements $m \in M_{w,s}$ there exist an expansion (W, N) of (W, M) to $\Delta[Y]$, and a term $t \in T_{\Sigma^{\mathrm{c}}}(Y)$ such that $m = N_{w,t}$.

Example 50. Let Δ^{c} be a constructor-based signature obtained from the signature Δ of Example 14 by defining $\Sigma^{\mathrm{c}} := \Sigma^{\mathrm{r}}$. It follows that the sort *List* is constrained while the sort *Elt* is loose. We define a theory T over Δ^{c}, which deletes n elements from a list in each possible world n:

- $\{@_n \langle\lambda\rangle n + 1 \mid n \geq 0\} \cup \{\neg @_n m \mid n \neq m\}$,
- $\{\forall N : \mathsf{any} \cdot (@_N\, \mathit{delete})(\mathit{empty}) = \mathit{empty}, \forall L : \mathit{List} \cdot (@_0\, \mathit{delete})(L) = L\}$, and
- $\{\forall E : \mathit{Elt}, L : \mathit{List} \cdot (@_{n+1}\, \mathit{delete})\mathit{cons}(E, L) = (@_n\, \mathit{delete})(L) \mid n > 0\}$.

Notice that the model (W, M) defined in Example 14 is a constructor-based Kripke structure which satisfies T.

By enhancing the syntax with a subset of rigid constructor operators and by restricting the semantics to constructor-based Kripke structures, we obtain a new logic $\mathcal{L}^{\mathrm{c}}$ from $\mathcal{L}$. Note that restricting the semantics also changes the relation $\models$, applied to theories: $T \models \varphi$ now means that all restricted models of T are models of φ, so there may be non-restricted models of T which are not models of φ.

8.2. Entailment systems

Given a system of proof rules for $\mathcal{L}$ which is sound and complete, the goal is to add some new proof rules such that the resulting proof system is sound and complete for $\mathcal{L}^{\mathrm{c}}$.

Definition 51 (Entailment relation). An *entailment relation* for $\mathcal{L}$ is a family of binary relations between sets of sentences indexed by signatures $\vdash = \{\vdash_\Delta\}_{\Delta \in |\mathsf{Sig}^{\mathcal{L}}|}$ with the following properties:

$$(Monotonicity)\frac{\Phi_1 \subseteq \Phi_2}{\Phi_2 \vdash \Phi_1} \qquad (Transitivity)\frac{\Phi_1 \vdash \Phi_2 \quad \Phi_2 \vdash \Phi_3}{\Phi_1 \vdash \Phi_3}$$

$$(Union)\ \frac{\Phi_1 \vdash \varphi_2 \text{ for all } \varphi_2 \in \Phi_2}{\Phi_1 \vdash \Phi_2} \qquad (Translation)\frac{\Phi_1 \vdash_\Sigma \Phi_2}{\chi(\Phi_1) \vdash \chi(\Phi_2)} \text{ where } \chi\colon \Delta \to \Delta'$$

The entailment relation is sound (complete) if $\vdash \subseteq \models$ ($\models \subseteq \vdash$). Examples of sound and complete entailment relations for HFOLR and HPL can be found in [28].

Definition 52 (Constructor-based entailment relation). Let $\vdash$ be an entailment relation for $\mathcal{L}$. The entailment relation $\vdash^{\mathsf{c}}$ for $\mathcal{L}^{\mathsf{c}}$ is the least entailment relation closed under the following proof rules:

$$\text{(R0)}\ \frac{\Phi \vdash \varphi}{\Phi \vdash^{\mathsf{c}} \varphi} \quad \text{(R1)}\ \frac{\Phi \vdash^{\mathsf{c}} @_{k_1}\, \varphi(k_2) \text{ for all } k_1, k_2 \in F^{\mathsf{n}}}{\Phi \vdash^{\mathsf{c}} \forall x \cdot \varphi(x)} \quad \text{(R2)}\ \frac{\Phi \vdash^{\mathsf{c}} @_k\, \forall Y_t \cdot \psi(t) \text{ for all } k \in F^{\mathsf{n}} \text{ and } t \in T_{\Sigma^{\mathsf{c}}}(Y)}{\Phi \vdash^{\mathsf{c}} \forall y \cdot \psi(y)}$$

where Y_t is the set of variables occurring in t

According to [16], the entailment relation $\vdash^{\mathsf{c}}$ exists. We say that a theory T in $\mathcal{L}$ is *semantically closed under (R1)* if $T \models @_{k_1}\, \varphi(k_2)$ for all $k_1, k_2 \in F^{\mathsf{n}}$ implies $T \models \forall x \cdot \varphi(x)$. Similarly, we define the closure under (R2), that is, $T \models @_k\, \forall Y_t \cdot \psi(t)$ for all $k \in F^{\mathsf{n}}$ and all $t \in T_{\Sigma^{\mathsf{c}}}(Y)$ implies $T \models \forall y \cdot \psi(y)$. It is not difficult to check that $\vdash^{\mathsf{c}}$ is sound for $\mathcal{L}^{\mathsf{c}}$ provided that $\vdash$ is sound for $\mathcal{L}$. Completeness is much more difficult to establish in general, but it can be done with the help of the OTP.

Theorem 53 (Constructor-based completeness)**.** *The entailment relation $\vdash^{\mathsf{c}}$ is complete for $\mathcal{L}^{\mathsf{c}}$ if $\vdash$ is complete for $\mathcal{L}$ and $\mathcal{L}$ has OTP.*

Proof. Let $\Delta = (\Sigma^{\mathsf{n}}, \Sigma^{\mathsf{r}} \subseteq \Sigma)$ be a signature and T a theory over Δ in $\mathcal{L}$. Let $\Sigma^{\mathsf{c}} \subseteq \Sigma^{\mathsf{r}}$ be a set of constructors, and $Y = \{Y_s\}_{s \in S^1}$ a set of loose variables such that Y_s is countably infinite for all $s \in S^1$.

(S1) We show that if T is satisfiable in $\mathcal{L}$ and semantically closed under (R1) and (R2) then T is satisfiable in $\mathcal{L}^{\mathsf{c}}$. Let $\Gamma^{\mathsf{n}} := \{x \neq k \mid k \in F^{\mathsf{n}}\}$ be a type in one nominal variable x, and let $\Gamma^{\mathsf{r}} := \{\forall Y_t \cdot y \neq t \mid t \in T_{\Sigma^{\mathsf{c}}}(Y)\}$ be a type in one constrained variable y. Any Kripke structure over Δ which omits Γ^{n} and Γ^{r} is reachable by the constructors in Σ^{c}. Firstly, we show that T locally omits Γ^{n}:

1	let $\rho(x)$ be a $\Delta[x]$-sentence such that $T \cup \{\rho(x)\}$ is satisfiable	
2	$T \cup \{+\rho(x)\}$ is satisfiable	by Lemma 12, $\rho(x) \boxminus +\rho(x)$
3	$T \not\models \forall x \cdot \neg + \rho(x)$	since $(W, M) \models T \cup \{+\rho(x)\}$ for some $(W, M) \in \lvert\mathsf{Mod}(\Delta[x])\rvert$
4	$T \not\models @_{k_1} \neg + \rho(k_2)$ for some nominals $k_1, k_2 \in F^{\mathsf{n}}$	since T is semantically closed under (R1)
5	$T \cup \{@_{k_1}\ + \rho(k_2)\}$ is satisfiable	by the semantics of retrieve and negation
6	$T \cup \{+\rho(k_2)\}$ is satisfiable	by Lemma 12, $@_{k_1}\ + \rho(k_2) \boxminus \rho(k_2)$
7	$T \cup \{+\rho(x)\} \cup \{@_{k_2}\, x\}$ is satisfiable	by semantics
8	T locally omits Γ^{n}	by Lemma 45 (L2), since $\rho(x)$ was arbitrarily chosen

Secondly, we show that T locally omits Γ^{r}:

1	let $\rho(y)$ be a $\Delta[y]$-sentence such that $T \cup \{\rho(y)\}$ is satisfiable	
2	$T \cup \{+\rho(y)\}$ is satisfiable	by Lemma 12, $+\rho(y) \boxminus \rho(y)$
3	$T \not\models \forall y \cdot \neg + \rho(y)$	since $(W, M) \models T \cup \{+\rho(y)\}$ for some $(W, M) \in \lvert\mathsf{Mod}(\Delta[y])\rvert$
4	$T \not\models @_k\, \forall Y_t \cdot \neg + \rho(t)$ for some $k \in F^{\mathsf{n}}$ and $t \in T_{\Sigma^{\mathsf{c}}}(Y)$	since T is semantically closed under (R2)
5	$T \not\models @_k \neg + \rho(t)$ over $\Delta[Y_t]$	by the semantics of quantifiers
6	$T \cup \{@_k\ + \rho(t)\}$ is satisfiable over $\Delta[Y_t]$	by the semantics of retrieve and negation
7	$T \cup \{\rho(t)\}$ is satisfiable over $\Delta[Y_t]$	by Lemma 12, $@_k\ + \rho(t) \boxminus \rho(t)$
8	$T \cup \{\rho(y)\} \cup \{\exists Y_t \cdot t = y\}$ is satisfiable	since $(W, M) \models T \cup \{\rho(t)\}$ for some $(W, M) \in \lvert\mathsf{Mod}(\Delta[Y_t])\rvert$
9	T locally omits Γ^{r}	by Lemma 45 (L2), since $\rho(y)$ was arbitrarily chosen

By Theorem 48, there exists a Kripke structure (W, M) which satisfies T and omits Γ^{n} and Γ^{r}. By the definition of Γ^{n} and Γ^{r}, (W, M) is a constructor-based Kripke structure.

(S2) Next we assume T is consistent in $\mathcal{L}^{\mathsf{c}}$ and show that T is satisfiable in $\mathcal{L}^{\mathsf{c}}$. Let $T' := \{\varphi \in \mathsf{Sen}(\Delta) \mid T \vdash^{\mathsf{c}} \varphi\}$. We have that T is consistent in $\mathcal{L}^{\mathsf{c}}$ iff T' is consistent in $\mathcal{L}$:

For the forward implication, suppose towards a contradiction that T' is not consistent in $\mathcal{L}$, that is, $T' \vdash \perp$. By (R0), $T' \vdash^{\mathsf{c}} \perp$. By *(Union)* , $T \vdash^{\mathsf{c}} T'$. By *(Transitivity)*, $T \vdash^{\mathsf{c}} \perp$, which is a contradiction with the consistency of T in $\mathcal{L}^{\mathsf{c}}$.

For the backward implication, suppose towards a contradiction that $T \vdash^{\mathsf{c}} \perp$. By the definition of T', $\perp \in T'$. By *(Monotonicity)*, $T' \vdash \perp$, which is a contradiction with the consistency of T' in $\mathcal{L}$.

Assume that T is consistent in $\mathcal{L}^{\mathsf{c}}$. It follows that T' is consistent in $\mathcal{L}$. By the completeness of $\vdash$ in $\mathcal{L}$, T' is satisfiable in $\mathcal{L}$. By the completeness of $\vdash$ in $\mathcal{L}$, T' is semantically closed under (R1) and (R2). By the first part of the proof, T' is satisfiable in $\mathcal{L}^{\mathsf{c}}$. Since $T \subseteq T'$, T is satisfiable in $\mathcal{L}^{\mathsf{c}}$. □

9. Omitting types and Löwenheim-Skolem Theorems

Downwards and Upwards Löwenheim-Skolem Theorems are consequences of the Omitting Types Theorem. Throughout this section, we assume that the fragment $\mathcal{L}$ has equality and it is semantically closed under Boolean connectives, quantifiers, retrieve, and possibility. An example of such fragment $\mathcal{L}$ is HDFOLR or HDPL, in which case $\mathcal{L}$ has ω-OTP. For cardinals greater than ω, we need to drop the Kleene operator $*$ in order to have compactness and be able to apply our OTP (we will show in the next section that compactness is necessary at least for certain strong fragments of $\mathcal{L}$). Some of the arguments in this and the next section are modeled after the technique used by Lindström [42] for first-order logic without equality.

Theorem 54 (Downwards Löwenheim-Skolem Theorem)**.** *Assume that $\mathcal{L}$ has α-OTP. Let T be a satisfiable theory over a signature Δ of power at most α. Then T has a Kripke structure (W, M) such that $\mathsf{card}(|W|) \leq \alpha$ and $\mathsf{card}(M_{w,s}) \leq \alpha$ for all rigid sorts $s \in S^{\mathsf{r}}$.*

Proof. Let $C = \{C_s\}_{s \in S^{\mathsf{e}}}$ be a sorted set of new constants for Δ such that $\mathsf{card}(C_s) = \alpha$ for all sorts $s \in S^{\mathsf{e}}$. Let $\Gamma^s := \{c \neq x \mid c \in C_s\}$ be a type [12] in one variable x of sort $s \in S^{\mathsf{e}}$. We show that T α-omits Γ^s:

1	let p be a set of sentences over $\Delta[C, x]$ such that $\mathsf{card}(p) < \alpha$ and $T \cup p$ is satisfiable	
2	$p \subseteq \Delta[C', x]$ for some $C' \subseteq C$ such that $\mathsf{card}(C'_s) < \alpha$	since $\mathsf{card}(p) < \alpha$
3	there exists $c \in C_s \setminus C'_s$	since $\mathsf{card}(C_s) = \alpha$ and $\mathsf{card}(C'_s) < \alpha$
4	$T \cup p \cup \{x = c\}$ is satisfiable	since $T \cup p$ is satisfiable and c does not occur in $T \cup p$
5	T α-omits Γ^s	by Lemma 45 (L3), since p was arbitrarily chosen

Since $\mathcal{L}$ has α-OTP, there exists a Kripke structure (W, M) over $\Delta[C]$ which satisfies T and omits Γ^s for all $s \in S^{\mathsf{e}}$. Notice that $\mathsf{card}(|W|) \leq C_{\mathsf{any}} = \alpha$ and $\mathsf{card}(M_{w,s}) \leq C_s = \alpha$ for all rigid sorts $s \in S^{\mathsf{r}}$. □

Theorem 55 (Upwards Löwenheim-Skolem Theorem)**.** *Assume that $\mathcal{L}$ has α-OTP, where α is a regular cardinal. Let T be a satisfiable theory over a signature Δ of power at most α. For each model (W, M) of T there exists another model (V, N) of T such that $\mathsf{card}((V, N)_s) \geq \alpha$ for all sorts $s \in S^{\mathsf{e}}$ interpreted by (W, M) as infinite.*

In fact, if Δ' is obtained from Δ by adding a rigid binary relation $\leq$ on each sort $s \in S^{\mathsf{e}}$ interpreted by (W, M) as infinite then there exists an expansion (V', N') of (V, N) to Δ' such that $\langle (V', N')_s, (V', N')_{\leq} \rangle$ is a linear ordering of cofinality α for all sorts $s \in S^{\mathsf{e}}$ interpreted by (W, M) as infinite.

Proof. Let $\Omega \subseteq S^{\mathsf{e}}$ be the set of all sorts interpreted by (W, M) as infinite. Let $C = \{C_s\}_{s \in \Omega}$ be a set of new rigid constants such that $C_s = \{c_i \mid i < \alpha\}$ for all $s \in \Omega$. Let T' be the theory over $\Delta'[C]$ obtained from T by adding:

$$\{\leq \text{ is a linear order on } s \text{ without the greatest element}\} \cup \{c_i \leq c_j \mid i < j < \alpha\} \text{ for each sort } s \in \Omega$$

[12] Notice that for nominals, $c \neq x$ means $\neg @_c\, x$. Compare Lemma 45 for a similar use.

The definition of T' relies on the semantic closure of $\mathcal{L}$ under the relevant sentence building operators. For example, for nominals, $c_i \leq c_j$ means $@_{c_i}\langle \leq \rangle c_j$. There exists an expansion (W', M') of (W, M) to the signature $\Delta'[C]$ such that $(W', M') \models T'$. For each sort $s \in \Omega$ we define the following type in one variable x of sort s:

$$\Gamma^s \coloneqq \{c_i \leq x \mid i < \alpha\}$$

We show that T' α-omits Γ^s:

1	let $p \subseteq \mathsf{Sen}(\Delta'[C, x])$ with $\mathsf{card}(p) < \alpha$ such that $T' \cup p$ is satisfiable	
2	$(V, N) \models T' \cup p$ for some Kripke structure (V, N) over $\Delta'[C, x]$	since $T' \cup p$ is satisfiable
3	$p \subseteq \mathsf{Sen}(\Delta'[C^\beta, x])$ for some $\beta < \alpha$, where C^β is obtained from C by restricting the constants of sort s to $C_s^\beta \coloneqq \{c_i \in C_s \mid i < \beta\}$	since α is regular
4	$(V^\beta, N^\beta) \models T \cup p$, where $(V^\beta, N^\beta) \coloneqq (V, N) \upharpoonright_{\Delta'[C^\beta, x]}$	since $(V, N) \models T' \cup p$ and $T \subseteq T'$
5	there exists $a > max\{(V, N)_x, (V, N)_{c_\beta}\}$	since $\langle (V, N)_s, (V, N)_\leq \rangle$ is a linear order without the greatest element
6	$a > (V, N)_{c_i}$ for all $i < \beta$	since $a > (V, N)_{c_\beta}$ and $(V, N)_{c_\beta} \geq (V, N)_{c_i}$ for all $i < \beta$
7	$(V', N') \models T' \cup p$, where (V', N') is the unique expansion of (V^β, N^β) to the signature $\Delta'[C, x]$ such that $(V', N')_{c_i} = a$ for all $i \geq \beta$	since $(V^\beta, N^\beta) \models T \cup p$ and a is greater than the interpretation of c_β in (V, N)
8	$(V', N') \not\models c_i \leq x$ for all $i \geq \beta$	since $(V, N)_x < a = (V', N')_{c_i}$ for all $i \geq \beta$
9	T' α-omits Γ^s	from 7 and 8, since p was arbitrarily chosen

By Theorem 48, there exists a model (V', N') which satisfies T' and omits Γ^s for all $s \in \Omega$. It follows that $\langle (V', N')_s, (V', N')_\leq \rangle$ is a linear ordering of cofinality α for all sorts $s \in \Omega$. [13] Let $(V, N) \coloneqq (V', N') \upharpoonright_\Delta$, and notice that (V, N) satisfies T and its carrier sets corresponding to the sorts in Ω have cardinalities greater than or equal to α. $\square$

10. Omitting types and compactness

In this section, we show that at least at some occasions, compactness is a necessary condition for proving the Omitting Types Theorem for uncountable signatures. We work within a fragment $\mathcal{L}$ with the following properties:

P1) $\mathcal{L}$ has equality and it is semantically closed under (a) Boolean connectives, (b) quantifiers, (c) retrieve, and (d) possibility.

P2) Signatures have only rigid sorts. For the sake of simplicity, we assume that all function symbols, except variables, are flexible.

Notice that predicates can be rigid.

10.1. Global substitutions

We begin by defining a notion of substitution which we then use to derive compactness for infinite models from α-OTP using a technique originally developed by Lindström for first-order logic with only relational symbols [42]. Consider a signature $\Delta = (\Sigma^{\mathtt{n}}, \Sigma^{\mathtt{r}} \subseteq \Sigma)$ in $\mathcal{L}$ with only one rigid sort: $S^{\mathtt{n}} = \{s_1\}$, $S = S^{\mathtt{r}} = S = \{s_2\}$ and $F^{\mathtt{r}} = \emptyset$. We define another signature $\Delta_+ = (\Sigma_+^{\mathtt{n}}, \Sigma_+^{\mathtt{r}} \subseteq \Sigma_+)$ on top of Δ as follows:

1. $\Sigma_+^{\mathtt{n}}$ consists of only one sort, let us say, s_0.
2. Σ_+ 'imports' s_1 and s_2 as rigid sorts, that is, $S_+ = S_+^{\mathtt{r}} = \{s_1, s_2\}$.
3. F_+ includes $F^{\mathtt{n}}$ and 'imports' all function symbols from F by adding the sort s_1 to their arity, which means that $F_+ = F_+^{\mathtt{f}} = F^{\mathtt{n}} \cup \{\sigma_+ : s_1 \underbrace{s_2 \dots s_2}_{m-times} \to s_2 \mid \sigma : \underbrace{s_2 \dots s_2}_{m-times} \to s_2 \in F\}$ and $F_+^{\mathtt{r}} = \emptyset$.

[13] To be more precise, we can select a subsequence $\{c_{i_j} : i_j < \alpha\}$ such that $\{(V', N')_{c_{i_j}} : i_j < \alpha\}$ is strictly increasing and unbounded. This sequence is order-isomorphic with an ordinal κ, and since α is regular, we have $\kappa = \alpha$.

4. Similarly, P_+ includes $P^{\mathtt{n}}$ and 'imports' all symbols from P by adding s_1 to their arity, which means that $P_+ = P_+^{\mathtt{f}} = P^{\mathtt{n}} \cup \{\pi_+ : s_1\, \underbrace{s_2 \dots s_2}_{m-times} \mid \pi : \underbrace{s_2 \dots s_2}_{m-times} \in P^{\mathtt{f}}\}$ and $P_+^{\mathtt{r}} = \emptyset$.

The signature Δ_+ provides a local environment for encoding Kripke structures over Δ. The following set of sentences over Δ_+ ensures that the interpretation of the rigid relation symbols in Δ is 'locally rigid' in Δ_+.

$$\Gamma_+ \coloneqq \{\forall x_1, x_2, y_1, \dots, y_m \cdot \pi_+(x_1, y_1, \dots, y_m) \Leftrightarrow \pi_+(x_2, y_1, \dots, y_m) \mid \pi : \underbrace{s_2 \dots s_2}_{m-times} \in P^{\mathtt{r}}\}$$

Let $\mathtt{z}$ be a distinguished nominal variable for Δ_+. We define a substitution $(\_)^+ : \Delta \dashrightarrow (\Delta_+[\mathtt{z}], \Gamma_+)$, that is,

1. a sentence function $(\_)^+ : \mathsf{Sen}(\Delta) \to \mathsf{Sen}(\Delta_+[\mathtt{z}], \Gamma_+)$ and
2. a reduct functor $(\_)^- : \mathsf{Mod}(\Delta_+[\mathtt{z}], \Gamma_+) \to \mathsf{Mod}(\Delta)$,

such that the following global satisfaction condition holds:

$$(W^{\mathtt{z}\leftarrow w}, M) \models \gamma^+ \text{ iff } (W^{\mathtt{z}\leftarrow w}, M)^- \models \gamma$$

for all Kripke structures $(W, M) \in |\mathsf{Mod}(\Delta_+, \Gamma_+)|$, all possible worlds $w \in |W|$ and all sentences $\gamma \in \mathsf{Sen}(\Delta)$.

Mapping on models. Notice that a model in $|\mathsf{Mod}(\Delta_+, \Gamma_+)|$ can be regarded as a collection of Kripke structures over the signature Δ. Once $\mathtt{z}$ is assigned to a node, the functor $(\_)^-$ extracts the Kripke structure corresponding to the node denoted by $\mathtt{z}$. Concretely, the functor $(\_)^- : \mathsf{Mod}(\Delta_+[\mathtt{z}], \Gamma_+) \to \mathsf{Mod}(\Delta)$ maps each Kripke structure $(W^{\mathtt{z}\leftarrow w}, M) \in |\mathsf{Mod}(\Delta_+[\mathtt{z}], \Gamma_+)|$ to $(V, N) \in |\mathsf{Mod}(\Delta)|$, where

1. $V \coloneqq M_w \upharpoonright_{\Sigma^{\mathtt{n}}}$, which is well defined since $\Sigma^{\mathtt{n}} \subseteq \Sigma_+$ and $M_w \in |\mathsf{Mod}(\Sigma_+)|$,
2. the mapping $N : M_{w,s_1} \to |\mathsf{Mod}(\Sigma)|$ is defined as follows:
   - For all $v \in M_{w,s_1}$, the carrier set N_{v,s_2} is M_{w,s_2}.
   - For all $v \in M_{w,s_1}$ and all $\sigma : \underbrace{s_2 \dots s_2}_{m-times} \to s_2 \in F$, the function $N_{v,\sigma} : \underbrace{M_{w,s_2} \times \dots \times M_{w,s_2}}_{m-times} \to M_{w,s_2}$ is defined by $N_{v,\sigma}(a_1, \dots, a_m) \coloneqq M_{w,\sigma_+}(v, a_1, \dots, a_m)$ for all $a_1, \dots, a_m \in M_{w,s_2}$.
   - For all $v \in M_{w,s_1}$ and all $\pi : \underbrace{s_2 \dots s_2}_{m-times} \in P$, the relation $N_{v,\pi} \subseteq \underbrace{M_{w,s_2} \times \dots \times M_{w,s_2}}_{m-times}$ is defined by $N_{v,\pi} \coloneqq \{(a_1, \dots, a_m) \mid (v, a_1, \dots, a_m) \in M_{w,\pi_+}\}$.

Since $(W, M) \models \Gamma_+$, the Kripke structure (V, N) interprets all rigid relation symbols in $P^{\mathtt{r}}$ uniformly across the possible worlds, which means that it is well-defined.

Fact 56. The functor $(\_)^- : \mathsf{Mod}(\Delta_+[z], \Gamma_+) \to \mathsf{Mod}(\Delta)$ can be extended to $(\_)^- : \mathsf{Mod}(\Delta_+[z, X], \Gamma_+) \to \mathsf{Mod}(\Delta[X])$, where $X = \{X_{s_i}\}_{i \in \{1,2\}}$ is a set of variables for Δ, such that the interpretation of all variables in X is preserved, that is, $(W^{z\leftarrow w}, M)_x = (W^{z\leftarrow w}, M)^-_x$ for all $x \in X$.

Mapping on sentences. We define a mapping on sentences $(\_)^+ : \mathsf{Sen}(\Delta[X]) \to \mathsf{Sen}(\Delta_+[\mathtt{z}, X])$ in three steps, where $X = \{X_{s_i}\}_{i \in \{1,2\}}$ is any set of variables for Δ.

S1) We define a mapping from the rigid terms over $\Delta[X]$ to the rigid terms over $\Delta_+[\mathtt{z}, X]$ by structural induction:

   - $x^+ \coloneqq x$, where x is any variable from X , and
   - $(@_k\, \sigma)(t_1, \dots, t_m)^+ \coloneqq (@_{\mathtt{z}}\, \sigma_+)(@_{\mathtt{z}}\, k, t_1^+, \dots, t_m^+)$, where $k \in F^{\mathtt{n}} \cup X_{s_1}$, and t_i are rigid terms over $\Delta_+[\mathtt{z}, X]$.

   Notice that $(\_)^+$ is well-defined on rigid terms, as $F_+^{\mathtt{r}} = \emptyset$.

Lemma 57. *For all Kripke structures* $(W, M) \in |\mathsf{Mod}(\Delta_+[X], \Gamma)|$*, all possible worlds* $w \in |W|$*, and all rigid terms* t *over* $\Delta[X]$*,*

$$(W^{z\leftarrow w}, M)_{t^+} = (W^{z\leftarrow w}, M)^-_t. \tag{1}$$

Proof. Let $(V, N) := (W^{z\leftarrow w}, M)^-$. We proceed by structural induction on terms:

[$x \in X$] By Fact 56, $(W^{z\leftarrow w}, M)_{x^+} = (V, N)_x$.

[$(@_k\, \sigma)(t_1, \ldots, t_m)$] Let $v := (W^{z\leftarrow w}, M)_{@_z\, k} = M_{w,k}$.
We have that $(W^{z\leftarrow w}, M)_{(@_k\, \sigma)(t_1,\ldots,t_m)^+} = (W^{z\leftarrow w}, M)_{(@_z\, \sigma_+)(@_z\, k, t_1^+,\ldots,t_m^+)} = M_{w,\sigma_+}(v, M_{w,t_1^+}, \ldots, M_{w,t_m^+})$. By the induction hypothesis, $M_{w,\sigma_+}(v, M_{w,t_1^+}, \ldots, M_{w,t_m^+}) = N_{v,\sigma}(N_{v,t_1}, \ldots, N_{v,t_m}) = (V, N)_{(@_k\, \sigma)(t_1,\ldots,t_m)}$.

Since $F^{\mathrm{r}} = \emptyset$, the cases considered above cover all possibilities. □

S2) We define the mapping $(\_)^+$ on rigid sentences of the form $@_k\, \varphi \in \mathsf{Sen}(\Delta[X])$ such that every rigid sentence will be mapped to a rigid sentence $(@_k\, \varphi)^+ \in \mathsf{Sen}(\Delta_+[z, X])$, which means that

$$(W^{z\leftarrow w}, M) \models (@_k\, \varphi)^+ \text{ iff } (W^{z\leftarrow w}, M) \models^w (@_k\, \varphi)^+$$

for all Kripke structures $(W, M) \in |\mathsf{Mod}(\Delta_+[X], \Gamma_+)|$ and all states $w \in |W|$. We proceed by structural induction:

- $(@_k\, k')^+ := @_z\, (k = k')$
- $(@_k\, \langle\lambda\rangle k')^+ := @_z\, \lambda(k, k')$
- $(@_k\, (t_1 = t_2))^+ := (\mathsf{at}_k\, t_1)^+ = (\mathsf{at}_k\, t_2)^+$
- $(@_k\, \pi(t_1, \ldots, t_m))^+ := (@_z\, \pi_+)(@_z\, k, (\mathsf{at}_k\, t_1)^+, \ldots, (\mathsf{at}_k\, t_m)^+)$
- $(@_k\, \vee \Phi)^+ := \vee_{\varphi\in\Phi}(@_k\, \varphi)^+$
- $(@_k\, \neg\varphi)^+ := \neg(@_k\, \varphi)^+$
- $(@_k\, \exists X' \cdot \varphi)^+ := \exists X' \cdot (@_k\, \varphi)^+$
- $(@_k\, @_{k'}\, \varphi)^+ := (@_{k'}\, \varphi)^+$
- $(@_k\, {\downarrow} x \cdot \varphi)^+ := (@_k\, \varphi(x \leftarrow k))^+$

Lemma 58 (Rigid satisfaction condition)**.** *For all sentences* $\varphi \in \mathsf{Sen}(\Delta[X])$*, all nominals* $k \in F^{\mathrm{n}} \cup X_{s_1}$*, all Kripke structures* $(W, M) \in |\mathsf{Mod}(\Delta_+[X], \Gamma_+)|$ *and all possible worlds* $w \in |W|$*,*

$$(W^{z\leftarrow w}, M) \models (@_k\, \varphi)^+ \textit{ iff } (W^{z\leftarrow w}, M)^- \models @_k\, \varphi. \tag{2}$$

Proof. Let $v := M_{w,k}$ and $(V, N) := (W^{z\leftarrow w}, M)^-$. We proceed by structural induction on φ:

[$k' \in F^{\mathrm{n}} \cup X_{s_1}$] $(W^{z\leftarrow w}, M) \models (@_k\, k')^+$ iff $(W^{z\leftarrow w}, M) \models @_z\, (k = k')$ iff $(W^{z\leftarrow w}, M) \models^w k = k'$ iff $M_{w,k} = M_{w,k'}$ iff $V_k = V_{k'}$ iff $(V, N) \models @_k\, k'$.

[$\langle\lambda\rangle k'$] $(W^{z\leftarrow w}, M) \models (@_k\, \langle\lambda\rangle k')^+$ iff $(W^{z\leftarrow w}, M) \models @_z\, \lambda(k, k')$ iff $(W^{z\leftarrow w}, M) \models^w \lambda(k, k')$ iff $(M_{w,k}, M_{w,k'}) \in M_{w,\lambda}$ iff $(V_k, V_{k'}) \in V_\lambda$ iff $(V, N) \models @_k\, \langle\lambda\rangle k'$.

[$t_1 = t_2$] $(W^{z\leftarrow w}, M) \models (@_k\, (t_1 = t_2))^+$ iff $(W^{z\leftarrow w}, M) \models (\mathsf{at}_k\, t_1)^+ = (\mathsf{at}_k\, t_2)^+$ iff
$(W^{z\leftarrow w}, M)_{(\mathsf{at}_k\, t_1)^+} = (W^{z\leftarrow w}, M)_{(\mathsf{at}_k\, t_2)^+}$ iff $N_{v,(\mathsf{at}_k\, t_1)} = N_{v,(\mathsf{at}_k\, t_2)}$ iff $N_{v,(@_k\, t_1)} = N_{v,(@_k\, t_2)}$ iff $(V, N) \models @_k\, (t_1 = t_2)$.

[$\pi(t_1, \ldots, t_m)$] $(W^{z\leftarrow w}, M) \models (@_k\, \pi(t_1, \ldots, t_m))^+$ iff $(W^{z\leftarrow w}, M) \models @_z\, \pi_+(@_z\, k, (\mathsf{at}_k\, t_1)^+, \ldots, (\mathsf{at}_k\, t_m)^+)$ iff
$(v, (W^{z\leftarrow w}, M)_{(\mathsf{at}_k\, t_1)^+}, \ldots, (W^{z\leftarrow w}, M)_{(\mathsf{at}_k\, t_m)^+}) \in M_{w,\pi_+}$ iff $(N_{v,(\mathsf{at}_k\, t_1)}, \ldots, N_{v,(\mathsf{at}_k\, t_m)}) \in N_{v,\pi}$ iff
$(N_{v,(@_k\, t_1)}, \ldots, N_{v,(@_k\, t_m)}) \in N_{@_k\, \pi}$ iff $(V, N) \models @_k\, \pi(t_1, \ldots, t_m)$.

[$\neg\varphi$] $(W^{z\leftarrow w}, M) \models (@_k\, \neg\varphi)^+$ iff $(W^{z\leftarrow w}, M) \models^w (@_k\, \neg\varphi)^+$ iff $(W^{z\leftarrow w}, M) \models^w \neg(@_k\, \varphi)^+$ iff $(W^{z\leftarrow w}, M) \not\models^w (@_k\, \varphi)^+$ iff $(W^{z\leftarrow w}, M) \not\models (@_k\, \varphi)^+$ iff $(V, N) \not\models @_k\, \varphi$ iff $(V, N) \models @_k\, \neg\varphi$.

[$\vee\Phi$] $(W^{\mathtt{z}\leftarrow w}, M) \models (@_k \vee\Phi)^+$ iff $(W^{\mathtt{z}\leftarrow w}, M) \models^w (@_k \vee\Phi)^+$ iff $(W^{\mathtt{z}\leftarrow w}, M) \models^w \vee_{\varphi\in\Phi}(@_k\, \varphi)^+$ iff $(W^{\mathtt{z}\leftarrow w}, M) \models^w (@_k\, \varphi)^+$ for some $\varphi \in \Phi$ iff $(W^{\mathtt{z}\leftarrow w}, M) \models (@_k\, \varphi)^+$ for some $\varphi \in \Phi$ iff $(V, N) \models @_k\, \varphi$ for some $\varphi \in \Phi$ iff $(V, N) \models @_k \vee \Phi$.

[$\exists X' \cdot \varphi$] Since $(\_)^-$ preserves the interpretation of variables, we have:

(a) for any expansion (W', M') of (W, M) to $\Delta_+[X, X']$, $(W'^{\mathtt{z}\leftarrow w}, M')^-$ is an expansion of (V, N) to $\Delta[X, X']$,

(b) for any expansion (V', N') of (V, N) to $\Delta[X, X']$, there exists a unique expansion (W', M') of (W, M) to $\Delta_+[X, X']$ such that $(W'^{\mathtt{z}\leftarrow w}, M')^- = (V', N')$.

$$\begin{tikzcd}
(V',N') \arrow[r, dotted, dash] & \Delta[X,X'] \arrow[r, dashed, "(_)^+"] & \Delta_+[\mathtt{z},X,X'] \arrow[r, dotted, dash] & (W'^{\mathtt{z}\leftarrow w},M') \\
(V,N) \arrow[r, dotted, dash] & \Delta[X] \arrow[u, hook] \arrow[r, dashed, "(_)^+"] & \Delta_+[\mathtt{z},X] \arrow[u, hook] \arrow[r, dotted, dash] & (W^{\mathtt{z}\leftarrow w},M) \\
& \Delta \arrow[u, hook] \arrow[r, dashed, "(_)^+"] & \Delta_+[\mathtt{z}] \arrow[u, hook] &
\end{tikzcd}$$

Based on the remark above, the following are equivalent:

1	$(W^{\mathtt{z}\leftarrow w}, M) \models (@_k\, \exists X' \cdot \varphi)^+$	
2	$(W^{\mathtt{z}\leftarrow w}, M) \models \exists X' \cdot (@_k\, \varphi)^+$	by the definition of $(\_)^+$
3	$(W'^{\mathtt{z}\leftarrow w}, M') \models (@_k\, \varphi)^+$ for some expansion (W', M') of (W, M) to $\Delta_+[X, X']$	since $(@_k\, \varphi)^+$ is rigid
4	$(V', N') \models @_k\, \varphi$ for some expansion (V', N') of (V, N) to $\Delta[X, X']$	by the induction hypothesis
5	$(V, N) \models @_k\, \exists X' \cdot \varphi$	by semantics

[$@_{k'}\, \varphi$] This case is straightforward, since $@_k\, @_{k'}\, \varphi \models\!\!\mid @_{k'}\, \varphi$.

[$\downarrow x \cdot \varphi$] This case is straightforward, since $@_k \downarrow x \cdot \varphi \models\!\!\mid @_k\, \varphi[x \leftarrow k]$. □

S3) The function $(\_)^+ : \mathsf{Sen}(\Delta[X]) \to \mathsf{Sen}(\Delta_+[\mathtt{z}, X])$ is defined by $\varphi^+ = \forall \mathtt{x} \cdot (@_{\mathtt{x}}\, \varphi)^+$ for all $\varphi \in \mathsf{Sen}(\Delta[X])$, where $\mathtt{x}$ is a distinguished nominal variable for $\Delta[X]$.

Proposition 59 (Global satisfaction condition)**.** *For all sentences $\varphi \in \mathsf{Sen}(\Delta[X])$, all Kripke structures $(W, M) \in |\mathsf{Mod}(\Delta_+[X])|$, and all possible worlds $w \in |W|$,*

$$(W^{\mathtt{z}\leftarrow w}, M) \models \varphi^+ \textit{ iff } (W^{\mathtt{z}\leftarrow w}, M)^- \models \varphi. \tag{3}$$

Proof. Let $(V, N) := (W^{\mathtt{z}\leftarrow w}, M)^-$.

$$\begin{tikzcd}
(V^{\mathtt{x}\leftarrow v},N) \arrow[r, dotted, dash] & \Delta[\mathtt{x},X] \arrow[r, dashed, "(_)^+"] & \Delta_+[\mathtt{z},\mathtt{x},X] \arrow[r, dotted, dash] & (W^{\mathtt{z}\leftarrow w},M^{\mathtt{x}\leftarrow v}) \\
(V,N) \arrow[r, dotted, dash] & \Delta[X] \arrow[u, hook] \arrow[r, dashed, "(_)^+"] & \Delta_+[\mathtt{z},X] \arrow[u, hook] \arrow[r, dotted, dash] & (W^{\mathtt{z}\leftarrow w},M) \\
& \Delta \arrow[u, hook] \arrow[r, dashed, "(_)^+"] & \Delta_+[\mathtt{z}] \arrow[u, hook] &
\end{tikzcd}$$

The following are equivalent:

1	$(W^{\mathtt{z}\leftarrow w}, M) \models \varphi^+$	
2	$(W^{\mathtt{z}\leftarrow w}, M) \models \forall \mathtt{x} \cdot (@_{\mathtt{x}}\, \varphi)^+$	by the definition of $(\_)^+$
3	$(W^{\mathtt{z}\leftarrow w}, M^{\mathtt{x}\leftarrow v}) \models (@_{\mathtt{x}}\, \varphi)^+$ for any expansion $(W^{\mathtt{z}\leftarrow w}, M^{\mathtt{x}\leftarrow v})$ of $(W^{\mathtt{z}\leftarrow w}, M)$ to $\Delta_+[\mathtt{z}, \mathtt{x}, X]$	since $(W^{\mathtt{z}\leftarrow w}, M) \models \forall \mathtt{x} \cdot (@_{\mathtt{x}}\, \varphi)^+$
4	$(V^{\mathtt{x}\leftarrow v}, N) \models @_{\mathtt{x}}\, \varphi$ for any expansion $(V^{\mathtt{x}\leftarrow v}, N)$ of (V, N) to $\Delta[\mathtt{x}, X]$	by Lemma 58, since $(W^{\mathtt{z}\leftarrow w}, M^{\mathtt{x}\leftarrow v})^- = (V^{\mathtt{x}\leftarrow v}, N)$
5	$(V, N) \models \forall \mathtt{x} \cdot @_{\mathtt{x}}\, \varphi$	by semantics
6	$(V, N) \models \varphi$	by semantics

$\square$

10.2. Inf-compactness

We say that $\mathcal{L}$ is *inf-compact* if each set of sentences Φ has an infinite model whenever each finite subset $\Phi_f \subseteq \Phi$ has an infinite model. We say that $\mathcal{L}$ is *α-inf-compact*, where α is an infinite cardinal, if each set of sentences Φ of cardinality α has an infinite model whenever each finite subset $\Phi_f \subseteq \Phi$ has an infinite model. We show that inf-compactness is a consequence of omitting type property.

Theorem 60. *If $\mathcal{L}$ has α-OTP, where α is a regular cardinal then $\mathcal{L}$ is β-inf-compact for all cardinals $\beta < \alpha$.*

Proof. Consider a signature $\Delta = (\Sigma^{\mathtt{n}}, \Sigma^{\mathtt{r}} \subseteq \Sigma)$ in $\mathcal{L}$ of power at most α with only one rigid sort: $S^{\mathtt{n}} = \{s_1\}$, $S = S^{\mathtt{r}} = \{s_2\}$ and $F^{\mathtt{r}} = \emptyset$. By induction, it suffices to prove that each sequence $\Phi_\beta = \{\varphi_i \in \mathsf{Sen}(\Delta) \mid i < \beta\}$ has an infinite model whenever each subsequence $\Phi_j := \{\varphi_i \mid i < j\}$ has an infinite model for all $j < \beta$. Let $\{(W^i, M^i) \in |\mathsf{Mod}(\Delta)| \mid 0 < i < \beta\}$ be a sequence of Kripke structures over Δ such that

- the carrier sets of (W^i, M^i) are infinite for all indexes j with $0 < j < \beta$, and
- $(W^j, M^j) \models \Phi_j$ for all indexes j with $0 < j < \beta$.

By Löwenheim-Skolem properties, we can assume that all carrier sets of (W^i, M^i) are of cardinality α. By renaming the elements, we assume furthermore that $|W^i| = |W^j|$ and $M^i_{w,s_2} = M^j_{w,s_2}$ for all $i < j < \beta$ and all possible worlds $w \in |W^i|$. Let Δ_+ be the signature obtained from Δ as described in Section 10.1. We define the following Kripke structure (W^+, M^+) over Δ_+:

- $|W^+| = \{w_i \mid 0 < i < \beta\}$, where $\{w_i \mid 0 < i < \beta\}$ is a sequence of pairwise distinct and new possible worlds. The carrier sets of (W^+, M^+) for the sorts s_1 and s_2 are the carrier sets of (W^i, M^i) for the sorts s_1 and s_2, where $0 < i < \beta$.
- For all $k \in F^{\mathtt{n}}$ and all $0 < i < \beta$, we define $M^+_{w_i,k} := W^i_k$.
- For all $\sigma : \underbrace{s_2 \dots s_2}_{m-times} \to s_2 \in F$ and all $0 < i < \beta$, the function $M^+_{w_i,\sigma^+} : M^+_{w_i,s_1} \times \underbrace{M^+_{w_i,s_2} \times \cdots \times M^+_{w_i,s_2}}_{m-times} \to M^+_{w_i,s_2}$ is defined by $M^+_{w_i,\sigma_+}(a, b_1, \dots, b_m) = M^i_{a,\sigma}(b_1, \dots, b_n)$ for all $(a, b_1, \dots, b_m) \in M^+_{w_i,s_1} \times \underbrace{M^+_{w_i,s_2} \times \cdots \times M^+_{w_i,s_2}}_{m-times}$.
- For all $\pi : \underbrace{s_2 \dots s_2}_{m-times} \in P$, we define $M^+_{w_i,\pi} := \{(a, b_1, \dots, b_m) \mid (b_1, \dots, b_m) \in M^i_{a,\pi}\}$.

By the definition of (W^+, M^+), we have

$$((W^+)^{\mathtt{z} \leftarrow w_i}, M^+)^- = (W^i, M^i) \text{ for all } i < \beta. \tag{4}$$

Let $\Delta_\bullet$ be the signature obtained from Δ_+ by adding a set of new nominals $C = \{k_i \mid 0 < i < \beta\}$ and a new binary relation symbol $\leq$ for nominals. Let $(W^\bullet, M^\bullet)$ be the expansion of (W^+, M^+) to $\Delta_\bullet$ such that

(a) $W^\bullet_{k_i} = w_i$ for all ordinals i with $0 < i < \beta$, and

(b) $(w_i, w_j) \in W^\bullet_<$ iff $i < j$.

Let $T = \Gamma_+ \cup \{\forall \mathtt{z} \cdot @_{k_i} \langle < \rangle \mathtt{z} \Rightarrow \varphi_i^+(\mathtt{z}) \mid i < \beta\}$. We show that $(W^\bullet, M^\bullet) \models T$:

1	$(W^\bullet, M^\bullet) \models \Gamma_+$	since $(W^+, M^+) \models \Gamma_+$
2	let i be an ordinal such that $0 < i < \beta$	
3	let $w_j \in \lvert W^\bullet \rvert$ such that $(w_i, w_j) \in W^\bullet_<$, meaning that $((W^\bullet)^{\mathtt{z} \leftarrow w_j}, M^\bullet) \models @_{k_i} \langle < \rangle z$	
4	$i < j$	by the definition of $(W^\bullet, M^\bullet)$, since $(w_i, w_j) \in W^\bullet_<$
5	$((W^+)^{\mathtt{z} \leftarrow w_j}, M^+) \models \Phi_j^+$	by Proposition 59 and statement 4, since $(W^j, M^j) \models \Phi_j$
6	$((W^\bullet)^{\mathtt{z} \leftarrow w_j}, M^\bullet) \models \Phi_j^+$	by the satisfaction condition
7	$((W^\bullet)^{\mathtt{z} \leftarrow w_j}, M^\bullet) \models \varphi_i^+$	since $\varphi_i \in \Phi_j$
8	$(W^\bullet, M^\bullet) \models \forall z \cdot @_{k_i} \langle < \rangle \mathtt{z} \Rightarrow \varphi_i^+$	from 3 and 7
9	$(W^\bullet, M^\bullet) \models T$	from 1 and 8

By Theorem 55, there exists a model $(V^\bullet, N^\bullet)$ of T such that $(|V^\bullet|, V^\bullet_\leq)$ is of cofinality α. We define $v_i := V^\bullet_{k_i}$ for all $i < \beta$. By cofinality, there exists $v \in V^\bullet_{s_0}$ such that $(v_i, v) \in V^\bullet_<$ for all $i < \beta$. It follows that $((V^\bullet)^{z \leftarrow v}, N^\bullet) \models \varphi_i^+$ for all $i < \beta$. Let $(V^+, N^+) := (V^\bullet, N^\bullet) \upharpoonright_{\Delta_+}$. By the satisfaction condition, $((V^+)^{z \leftarrow v}, N^+) \models \varphi_i^+$ for all $i < \beta$. By Proposition 59, $((V^+)^{z \leftarrow v}, N^+)^- \models \varphi_i$ for all $i < \beta$. □

11. Conclusion

In this paper we established an omitting types theorem for first-order hybrid dynamic logic and sufficiently expressive fragments. For countable signatures, the result followed without needing compactness whereas for uncountable signatures we had to restrict our attention to compact fragments of the logic. It turns out that the latter restriction is actually necessary for some of these fragments, as compactness is a consequence of OTT for uncountable signatures. We also provided two applications of the OTT: (1) Löwenheim-Skolem theorems and (2) a completeness theorem for the constructor-based version of first-order hybrid dynamic logic. For an application of OTT to Robinson Joint Consistency Property see [29]. In future work we intend to explore other interesting consequences of OTT in this setting.

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