Foundations of logic programming in hybrid logics with user-defined sharing

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Abstract

The present contribution advances an abstract notion of hybrid logic by supplementing the definition of institution with an additional structure to extract frames. The foundation of logic programming is set in the general framework proposed by defining the basic concepts such as Horn clause, query and solution, and proving fundamental results such as the existence of initial model of Horn clauses and Herbrand's theorem. The abstract results are then applied to hybrid logics with user-defined sharing, where the possible worlds share a common domain and the variables used for quantification are interpreted uniformly across the worlds.

Keywords: institution, logic programming, Herbrand's theorem, initiality

1. Introduction

Hybrid logics [6] are a type of standard modal logics that have symbols for naming individual states/worlds in models. Hybrid (propositinal) logic was introduced by Arthur Prior [40] in the fifties, and further developed in contributions such as [39, 2, 8]. Recently, hybrid logics have been studied in the abstract framework provided by the institutional model theory [35, 17, 21, 27, 33, 38]. Nowadays, hybrid logics have been found useful to provide logical support for developing reconfigurable systems, i.e. software systems with reconfigurable features governing the evolution of their configurations in response to external stimuli or internal performance measures. See [43] for an overview of the reconfiguration paradigm and [34] for a specification method based on hybrid logics. From a mathematical perspective, hybrid logics allow a more uniform proof-theoretical approach than the ordinary modal logics [8].

The concept of institution [22] formalizes the intuitive notion of logical system in a category-based setting including syntax, semantics and the satisfaction between them. This paper defines the notion of hybrid institution which supplements the definition of institution with an additional structure to extract *frames*

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from models. Hybrid institutions are stratified institutions [1, 16] with *nominal* and frame extraction. While the notion of stratified institution formalizes modal logics, the definition of hybrid institution describes hybrid logics. The hybridization method [35, 17, 21] is a construction technique of hybrid logical systems over an arbitrary institution. Its parameters are very general yielding to an abstract framework that can be instantiated to many hybrid logics. However, the hybridization process consists of a 'bottom-up' technique, whereas the present approach provides a 'top-down' methodology to the developing of logical and computing science results over classical and unconventional hybrid logics, in the spirit of universal logic [4, 5].

This paper investigates a series of model-theoretic properties of hybrid logics such as substitution [14, 28] (which is closely related to quantification), reachable model [31, 30, 28] and basic set of sentences [12]. This fundamental work is crucial for proving not only initiality and Herbrand's theorem but also completeness, downward Lowenheim-Skolem property, interpolation, etc.

Initial semantics [26] is closely related to good computational properties of logical systems. For example, in logic programing, the initial models are called the least Herbrand models [32], and the initiality supports the execution of programs by resolution. In algebraic specification, initiality supports the execution of specification languages by rewriting, thus increasing significantly the automation of the formal verification process. Our approach to initiality in hybrid logics is layered, and it is intimately linked to the structure of sentences, in the style of [29]. The existence of initial models of sets of atomic sentences is assumed in the abstract setting, but it is developed in concrete examples of hybrid logical systems; then the initiality property is shown to be closed under certain sentence building operators in the general framework provided by the definition of hybrid institution.

Another contribution of the paper is an abstract variant of Herbrand's theorem applicable to hybrid logics, which reduces the satisfiability of a query by a theory at a given state to the search of a suitable substitution. The logic programming paradigm [32], in its classical form, can be described as follows: Given a program (Σ, Γ) (that consists of a signature Σ and a set of Horn clauses Γ) and a query $(\exists Y)\rho$ (that consists of an existentially quantified conjunction of atoms) find a solution θ, i.e. values for the variables Y such that the corresponding instance $\theta(\rho)$ of ρ is satisfied by (Σ, Γ). The essence of this paradigm is however independent of any logical system of choice. The basic logic programming concepts, query, solutions, and the the fundamental results, such as Herbrand's theorem, are developed over an arbitrary institution (satisfying certain hypotheses) in [14] by employing institution-independent concepts of variables, substitution, quantifiers and atomic formulae. Our work sets the foundation of logic programming over a large variety of hybrid logics by developing the results in the framework of hybrid institutions.

One major class of applications consists of hybrid logics with user-defined sharing, where the possible worlds share a common domain and the variables used for quantification are interpreted uniformly across worlds. In the literature, this semantic approach is called rigid quantification [8]. We propose a new type of hybrid logics by upgrading the definition of some hybrid logics with user-defined sharing that can be found in the area of computer science. These new hybrid logics increase the syntactic expressivity of their classical counterparts and allow a more structured model-theoretic approach to proving the desired properties.

Some preliminary results were reported in [27]. That study is performed in the framework provided by the hybridization method and it is applicable to hybrid logical systems where the quantified variables may be interpreted differently across the worlds, which amounts to the world-line semantics of [42].

1.1. *Contributions and structure of the paper*

The paper is organized as follows:

- Section 2 is devoted to the brief presentation of general institution theoretic notions that are needed by the present work.
- Section 3 proposes a notion of hybrid logic by upgrading the definition of institution. This section lays the foundation in which the abstract results are proved.
- Section 4 introduces hybrid logics with user-defined sharing and annotation, and it shows that the newly defined hybrid logics are more expressive than their standard counterparts. The notion of substitution is generalized to hybrid logics to cover the concept of solution for queries.
- Section 5 recalls the necessary fundamental concepts of institutional model theory such as basic set of sentences and reachable model. One important contribution, not only for the present study, but also for future research, is the investigation of the concepts of basic set of sentences and reachable model in concrete hybrid logics.
- Section 6 is dedicated to the development of the general layered initiality result on top of the foundational work completed in the previous section.
- Section 7 presents a version of Herbrand's theorem in the context of hybrid institutions, and its applications to concrete hybrid logics.
- Section 8 concludes the paper and discusses the future work.

2. Institutions

In this section we describe the abstract framework which is the basis for developing the general results.

2.1. Definition

The concept of institution formalises the intuitive notion of logical system, and has been defined by Goguen and Burstall in the seminal paper [22].

Definition 1. *An* institution $\mathtt{I} = (\mathtt{Sig}^{\mathtt{I}}, \mathtt{Sen}^{\mathtt{I}}, \mathtt{Mod}^{\mathtt{I}}, \models^{\mathtt{I}})$ *consists of*

(1) a category $\mathtt{Sig}^{\mathtt{I}}$*, whose objects are called* signatures*,*

(2) a functor $\mathtt{Sen}^{\mathtt{I}} : \mathtt{Sig}^{\mathtt{I}} \to \mathbb{S}et$*, providing for each signature* Σ *a set whose elements are called* (Σ-)sentences*,*

(3) a functor $\mathtt{Mod}^{\mathtt{I}} : \mathtt{Sig}^{\mathtt{I}} \to \mathbb{CAT}^{op}$*, providing for each signature* Σ *a category whose objects are called* (Σ-)models *and whose arrows are called* (Σ-)homomorphisms*,*

(4) a relation $\models^{\mathtt{I}}_{\Sigma} \subseteq |\mathtt{Mod}^{\mathtt{I}}(\Sigma)| \times \mathtt{Sen}^{\mathtt{I}}(\Sigma)$ *for each signature* $\Sigma \in |\mathtt{Sig}^{\mathtt{I}}|$*, called* ($\Sigma$-)satisfaction*, such that for each morphism* $\varphi : \Sigma \to \Sigma'$ *in* $\mathtt{Sig}^{\mathtt{I}}$*, the following* satisfaction condition *holds:*

$$M' \models^{\mathtt{I}}_{\Sigma'} \mathtt{Sen}^{\mathtt{I}}(\varphi)(e) \text{ iff } \mathtt{Mod}^{\mathtt{I}}(\varphi)(M') \models^{\mathtt{I}}_{\Sigma} e$$

for all $M' \in |\mathtt{Mod}^{\mathtt{I}}(\Sigma')|$ *and* $e \in \mathtt{Sen}^{\mathtt{I}}(\Sigma)$*.*

We let $\mathtt{I}$ range over institutions of the form $(\mathtt{Sig}^{\mathtt{I}}, \mathtt{Sen}^{\mathtt{I}}, \mathtt{Mod}^{\mathtt{I}}, \models^{\mathtt{I}})$. When there is no danger of confusion, we omit the superscript from the notations of the institution components; for example $\mathtt{Sig}^{\mathtt{I}}$ may be simply denoted by $\mathtt{Sig}$. We denote the *reduct* functor $\mathtt{Mod}(\varphi)$ by $\_\upharpoonright_{\varphi}$ and the sentence translation $\mathtt{Sen}(\varphi)$ by $\varphi(\_)$. When $M = M' \upharpoonright_{\varphi}$ we say that M is the *φ-reduct* of M' and M' is a *φ-expansion* of M. A signature morphism $\varphi : \Sigma \to \Sigma'$ is *conservative* if all Σ-models have a φ-expansion. Given a signature Σ and two sets of Σ-sentences E_1 and E_2, we write (a) $E_1 \models E_2$ whenever E_2 is a *semantic consequence* of E_2, i.e. $M \models E_1$ implies $M \models E_2$ for all Σ-models M, and (b) $E_1 \models\!\mid E_2$ whenever E_1 and E_2 are *semantically equivalent*, i.e. $E_1 \models E_2$ and $E_2 \models E_1$.

The following institutional notions dealing with the semantics of Boolean operators and quantifiers were defined in [45].

Definition 2 (Internal logic)**.** *Given an institution* $\mathtt{I}$ *then*

(1) $M \models \rho_1 \wedge \rho_2$ *iff* $M \models \rho_1$ *and* $M \models \rho_2$*;*

(2) $M \models \neg\rho$ *iff* $M \not\models \rho$*;*

(3) $M \models (\forall\chi)\gamma$ *iff* $M' \models \gamma$ *for all χ-expansions* M' *of* M*.*

where $\Sigma \xrightarrow{\chi} \Sigma' \in \mathtt{Sig}^{\mathtt{I}}$*,* $M \in |\mathtt{Mod}^{\mathtt{I}}(\Sigma)|$*,* $\rho \in \mathtt{Sen}^{\mathtt{I}}(\Sigma)$*,* $\rho_i \in \mathtt{Sen}^{\mathtt{I}}(\Sigma)$ *and* $\gamma \in \mathtt{Sen}^{\mathtt{I}}(\Sigma')$*.*

Based on these constructors for sentences we can also define $\_\vee\_$, *false*, $(\exists\chi)\_$ using the classical definitions. For example, $false = \vee\emptyset$.

2.2. Examples

We give a few examples of institutions which are often in met in algebraic specification literature.

Example 3 (Propositional Logic (**PL**)). *The category of signatures* $\mathtt{Sig}^{\mathbf{PL}}$ *is* $\mathbb{S}\mathrm{et}$. *For any set* P*, the set of sentences* $\mathtt{Sen}^{\mathbf{PL}}(P)$ *is generated by the grammar* $S ::= P \mid S \wedge S \mid \neg S$. *The category of models* $\mathtt{Mod}^{\mathbf{PL}}(P)$ *is* $(2^P, \subseteq)$. *For any mapping* $\varphi : P \to P'$*, the function* $\mathtt{Sen}^{\mathbf{PL}}(\varphi)$ *replaces each element* $p \in P$ *that occur in a sentence of* $\mathtt{Sen}^{\mathbf{PL}}(P)$ *by* $\varphi(p)$. *For each model* $M' \in 2^{P'}$*, we have* $\mathtt{Mod}^{\mathbf{PL}}(\varphi)(M') = \varphi; M'$. *The satisfaction relation is defined by induction on the structure of the sentences:*

- $M \models e_0$ *iff* $e_0 \in M$,
- $M \models e_1 \wedge e_2$ *iff* $M \models e_1$ *and* $M \models e_2$,
- $M \models \neg e_3$ *iff* $M \not\models e_3$.

where $M \in 2^P$*,* $e_0 \in P$*, and* $e_i \in \mathtt{Sen}^{\mathbf{PL}}(P)$ *for all* $i \in \{1, 2, 3\}$.

Example 4 (First-Order Logic (**FOL**) [22]). *The signatures are triplets of the form* (S, F, P)*, where* S *is the set of sorts,* $F = \{F_{a\to s}\}_{(a,s)\in S^*\times S}$ *is the* $(S^* \times S$ *-indexed) set of operation symbols, and* $P = \{P_a\}_{a\in S^*}$ *is the* (S^**-indexed) set of relation symbols.* [1] *If* $a = \varepsilon$ *then an element of* $F_{a\to s}$ *is called a* constant symbol*, or a* constant. *A signature morphism between* (S, F, P) *and* (S', F', P') *is a triplet* $\varphi = (\varphi^{st}, \varphi^{op}, \varphi^{rl})$*, where* $\varphi^{st} : S \to S'$*,* $\varphi^{op} = \{\varphi^{op}_{a\to s} : F_{a\to s} \to F'_{\varphi^{st}(a)\to\varphi^{st}(s)} \mid a \in S^*, s \in S\}$*,* $\varphi^{rl} = \{\varphi^{rl}_a : P_a \to P'_{\varphi^{st}(a)} \mid a \in S^*\}$. *When there is no danger of confusion, we may let* φ *denote each of* φ^{st}*,* $\varphi^{op}_{a\to s}$*,* φ^{rl}_a.

Given a signature $\Sigma = (S, F, P)$*, a* Σ*-model is a triplet*

$$M = (\{M_s\}_{s\in S}, \{M_\sigma\}_{(a,s)\in S^*\times S, \sigma\in F_{a\to s}}, \{M_\pi\}_{a\in S^*, \pi\in P_a})$$

interpreting each sort s *as a set* M_s*, each operation symbol* $\sigma \in F_{a\to s}$ *as a function* $M_\sigma : M_a \to M_s$ *(where* M_a *stands for* $M_{s_1} \times \ldots \times M_{s_n}$ *if* $a = s_1 \ldots s_n$*), and each relation symbol* $\pi \in P_a$ *as a relation* $M_\pi \subseteq M_a$. *The model morphisms are the usual* Σ*-homomorphisms, i.e.,* S*-sorted functions that preserve the structure. The initial* Σ*-model of terms is denoted by* T_Σ.

The Σ*-sentences are obtained from (a) equations* $t =_s t'$*, where* $t \in (T_\Sigma)_s$*,* $t' \in (T_\Sigma)_s$*,* $s \in S$*, and (b) relations* $\pi(t)$*, where* $\pi \in P_a$*,* $t \in (T_\Sigma)_a$ *and* $a \in S^*$*, by applying for a finite number of times Boolean operators and quantification over finite sets of variables. When there is no danger of confusion we may omit the subscript* s *from* $t =_s t'$.

Satisfaction is the usual first-order satisfaction and it is defined using the natural interpretations of ground terms t *as elements* M_t *in models* M. *The definitions of functors* $\mathtt{Sen}^{\mathbf{FOL}}$ *and* $\mathtt{Mod}^{\mathbf{FOL}}$ *on signature morphisms are the*

[1] If S is a set then S^* is the set of strings over symbols in S, including the empty string ε.

natural ones: for any arrow $\varphi : \Sigma \to \Sigma' \in \mathtt{Sig}^{\mathbf{FOL}}$*, the function* $\mathtt{Sen}^{\mathbf{FOL}}(\varphi) : \mathtt{Sen}^{\mathbf{FOL}}(\Sigma) \to \mathtt{Sen}^{\mathbf{FOL}}(\Sigma')$ *translates sentences symbol-wise, and* $\mathtt{Mod}^{\mathbf{FOL}}(\varphi) : \mathtt{Mod}^{\mathbf{FOL}}(\Sigma') \to \mathtt{Mod}^{\mathbf{FOL}}(\Sigma)$ *is the forgetful functor.*

Notice that **PL** *is a fragment of* **FOL** *determined by the signatures with empty sets of sort symbols.*

Example 5 (**REL**)**.** *The institution* **REL** *is the sub-institution of single-sorted first-order logic with signatures having only constants and relational symbols.*

Example 6 (First-Order Logic with Rigid symbols (**FOLR**))**.** *This institution is used in Example 24 to define a hybrid logic and it is not intended for any other application. The signatures* $(S^r, F^r, P^r) \subseteq (S, F, P)$ *consist of* **FOL** *signatures* (S, F, P) *enhanced with a sub-signature* (S^r, F^r, P^r) *of 'rigid' symbols. Signature morphisms* $\varphi : (S^r, F^r, P^r) \subseteq (S, F, P) \to (S'^r, F'^r, P'^r) \subseteq (S', F', P')$ *are* **FOL** *signature morphisms* $(S, F, P) \to (S', F', P')$ *that map rigid symbols to rigid symbols. The set* $\mathtt{Sen}^{\mathbf{FOLR}}((S^r, F^r, P^r) \subseteq (S, F, P))$ *consists of those sentences in* $\mathtt{Sen}^{\mathbf{FOL}}(S, F, P)$ *that contain only quantifiers over variables of rigid sorts. The category of models* $\mathtt{Mod}^{\mathbf{FOLR}}((S^r, F^r, P^r) \subseteq (S, F, P))$ *is* $\mathtt{Mod}^{\mathbf{FOL}}(S, F, P)$*. The satisfaction relation in* **FOLR** *is induced from the satisfaction relation in* **FOL***, i.e.* $\models^{\mathbf{FOLR}}_{(S^r, F^r, P^r) \subseteq (S, F, P)} = \models^{\mathbf{FOL}}_{(S, F, P)}$.

Example 7 (Preorder Algebra (**POA**) [19, 20])**.** *The* **POA** *signatures are just ordinary algebraic signatures, i.e.* **FOL** *signatures without relation symbols. The* **POA** *models are* preordered algebras *which are interpretations of the signatures into the category of preorders* $\mathbb{P}re$ *rather than the category of sets* $\mathbb{S}et$*. This means that each sort gets interpreted as a preorder, and each operation as a preorder functor, which means a preorder-preserving (i.e. monotonic) function. The preorder relation is denoted by* $\leq$*. A* preordered algebra morphism *is just a family of preorder functors (preorder-preserving functions) which is also an algebra morphism.*

The sentences have two kinds of atoms: equations and transitions*. A transition* $t \Rightarrow t'$ *is satisfied by a preorder algebra* M *when the interpretations of the terms are in the preorder relation of the carrier, i.e.* $M_t \leq M_{t'}$ *. Full sentences are constructed from equations and transitions by applying Boolean operators and first-order quantification.*

Example 8 (Partial algebra (**PA**) [41, 9])**.** *A* partial algebraic signature (S, F) *consists of a set* S *of sorts and a set* F *of* partial *operations. We assume that there is a distinguished constant on each sort* $\perp_s : s$*. Signature morphisms map the sorts and operations in a compatible way, preserving* $\perp_s$*.*

A partial algebra is just like an ordinary algebra but interpreting the operations of F *as partial rather than total functions;* $\perp_s$ *is always interpreted as undefined. A* partial algebra (S, F)-morphism $h : M \to N$ *is a family of (total) functions* $\{M_s \xrightarrow{h_s} N_s\}_{s \in S}$ *such that* $h_s(M_\sigma(m)) = N_\sigma(h_a(m))$ *for each operation* $\sigma : a \to s$ *and each string of arguments* $m \in M_a$ *for which* $M_\sigma(m)$ *is defined.*

We consider one kind of atomic sentences: existence equality $t \overset{e}{=} t'$*. The existence equality* $t \overset{e}{=} t'$ *holds when both terms are defined and equal. The definedness predicate and strong equality can be introduced as notations:* $def(t)$

stands for $t \stackrel{e}{=} t$ *and* $t \stackrel{s}{=} t'$ *stands for* $(t \stackrel{e}{=} t') \vee (\neg def(t) \bigwedge \neg def(t'))$. *We consider the atomic sentences in* $\mathtt{Sen}^{\mathbf{PA}}(S,F)$ *to be the atomic existentence equalities. The sentences are formed from existence equalities by applying Boolean operators and quantification over variables (interpreted as partial constants). The definition of* **PA** *given here is slightly different from the one in [37, 3, 36] since it does not consider total operation symbols.*

2.3. Substitutions

We recall the notion of substitution in institutions.

Definition 9. *[14] For any signature morphisms* $\chi_1 : \Sigma \to \Sigma_1$ *and* $\chi_2 : \Sigma \to \Sigma_2$ *of an institution, a* Σ-substitution $\theta : \chi_1 \to \chi_2$ *consists of a pair* $(\mathtt{Sen}(\theta), \mathtt{Mod}(\theta))$, *where* $\mathtt{Sen}(\theta) : \mathtt{Sen}(\Sigma_1) \to \mathtt{Sen}(\Sigma_2)$ *is a function and* $\mathtt{Mod}(\theta) : \mathtt{Mod}(\Sigma_2) \to \mathtt{Mod}(\Sigma_1)$ *is a functor, such that* Σ *is preserved, i.e. the following diagrams commute,*

$$\begin{tikzcd} \mathtt{Sen}(\Sigma_1) \arrow[r, "\mathtt{Sen}(\theta)"] & \mathtt{Sen}(\Sigma_2) \\ \mathtt{Sen}(\Sigma) \arrow[u, "\mathtt{Sen}(\chi_1)"] \arrow[ur, "\mathtt{Sen}(\chi_2)"'] & \end{tikzcd} \qquad \begin{tikzcd} \mathtt{Mod}(\Sigma_1) \arrow[dr, "\mathtt{Mod}(\chi_1)"'] & \mathtt{Mod}(\Sigma_2) \arrow[l, "\mathtt{Mod}(\theta)"'] \arrow[d, "\mathtt{Mod}(\chi_2)"] \\ & \mathtt{Mod}(\Sigma) \end{tikzcd}$$

and the following satisfaction condition *holds:*

$$\mathtt{Mod}(\theta)(M_2) \models \rho_1 \text{ iff } M_2 \models \mathtt{Sen}(\rho_1)$$

for each Σ_2*-model* M_2 *and each* Σ_1*-sentence* ρ_1.

Note that a substitution $\theta : \chi_1 \to \chi_2$ is uniquely identified by its domain χ_1, codomain χ_2 and the pair $(\mathtt{Sen}(\theta), \mathtt{Mod}(\theta))$. We sometimes let $\_\!\upharpoonright_\theta$ denote the functor $\mathtt{Mod}(\theta)$, and let θ denote the sentence translation $\mathtt{Sen}(\theta)$.

Example 10 (**FOL** substitutions [14])**.** *Consider two signature extensions with constants* $\chi_1 : \Sigma \hookrightarrow \Sigma[C_1]$ *and* $\chi_2 : \Sigma \hookrightarrow \Sigma[C_2]$, *where* $\Sigma = (S,F,P) \in |\mathtt{Sig}^{\mathbf{FOL}}|$, C_i *is a set of constant symbols different from the the constants in* F *and* $\Sigma[C_i] = (S, F \cup C_i, P)$. *A function* $\theta : C_1 \to T_\Sigma(C_2)$ *represents a substitution between* χ_1 *and* χ_2.

On the syntactic side, θ *can be canonically extended to a function* $\mathtt{Sen}(\theta) : \mathtt{Sen}(\Sigma[C_1]) \to \mathtt{Sen}(\Sigma[C_2])$ *that substitutes terms in* $T_\Sigma(C_2)$ *for constants in* C_1 *according to* θ.

On the semantics side, θ *determines a functor* $\mathtt{Mod}(\theta) : \mathtt{Mod}(\Sigma[C_2]) \to \mathtt{Mod}(\Sigma[C_1])$ *such that for all* $\Sigma[C_2]$*-models* M *we have*

- $\mathtt{Mod}(\theta)(M)_x = M_x$, *for each sort* $x \in S$, *or operation symbol* $x \in F$, *or relation symbol* $x \in P$, *and*
- $\mathtt{Mod}(\theta)(M)_{c_1} = M_{\theta(c_1)}$ *for each* $c_1 \in C_1$.

Example 11 (**PA** substitutions [30]). *Consider two signature extensions with partial constants* $\chi_1 : \Sigma \hookrightarrow \Sigma[C_1]$ *and* $\chi_2 : \Sigma \hookrightarrow \Sigma[C_2]$*, where* $\Sigma = (S, F) \in |\mathtt{Sig}^{\mathbf{PA}}|$*,* C_i *is a set of partial constant symbols different from the the constants in* F*. A function* $\theta : C_1 \to T_\Sigma(C_2)$ *represents a substitution between* χ_1 *and* χ_2*.*

Substitution functors.

Let $\mathtt{I}$ be an institution. For any signature $\Sigma \in |\mathtt{Sig}^{\mathtt{I}}|$, Σ-substitutions form a category $\mathtt{Sb}^{\mathtt{I}}(\Sigma)$, where *the objects* are signature morphisms $\Sigma \xrightarrow{\chi} \Sigma' \in \mathtt{Sig}^{\mathtt{I}}$, and *the arrows* are substitutions $\theta : \chi_1 \to \chi_2$ described in Definition 9. Given $\Sigma_0 \xrightarrow{\varphi} \Sigma \in \mathtt{Sig}^{\mathtt{I}}$ there exists a reduct functor $\mathtt{Sb}^{\mathtt{I}}(\varphi) : \mathtt{Sb}^{\mathtt{I}}(\Sigma) \to \mathtt{Sb}^{\mathtt{I}}(\Sigma_0)$ that maps each Σ-substitution $\theta : \chi_1 \to \chi_2$ to the Σ_0-substitution $\mathtt{Sb}^{\mathtt{I}}(\varphi)(\theta) : \varphi; \chi_1 \to \varphi; \chi_2$ such that $\mathtt{Sen}^{\mathtt{I}}(\mathtt{Sb}^{\mathtt{I}}(\varphi)(\theta)) = \mathtt{Sen}^{\mathtt{I}}(\theta)$ and $\mathtt{Mod}^{\mathtt{I}}(\mathtt{Sb}^{\mathtt{I}}(\varphi)(\theta)) = \mathtt{Mod}^{\mathtt{I}}(\theta)$.

Fact 12. *[28]* $\mathtt{Sb}^{\mathtt{I}} : \mathtt{Sig}^{\mathtt{I}} \to \mathbb{CAT}^{op}$ *is a functor.*

In applications not all substitutions are of interest, and it is often assumed a substitution sub-functor $\mathtt{St}^{\mathtt{I}} : \mathtt{D}^{\mathtt{I}} \to \mathbb{CAT}^{op}$ of $\mathtt{Sb}^{\mathtt{I}}$ to work with, where $\mathtt{D}^{\mathtt{I}} \subseteq \mathtt{Sig}^{\mathtt{I}}$ is a broad subcategory of signature morphisms. When there is no danger of confusion we may drop the superscript $\mathtt{I}$ from notations.

Example 13 (**FOL** substitution functor [28]). *Let* $\mathtt{D}^{\mathbf{FOL}} \subseteq \mathtt{Sig}^{\mathbf{FOL}}$ *be the broad subcategory of signature extensions with constants. The first-order substitutions are represented by functions* $\theta : C_1 \to T_\Sigma(C_2)$*, where* $\Sigma \in |\mathtt{Sig}^{\mathbf{FOL}}|$ *and* C_i *are finite sets of new constants for* Σ*. Let* $\mathtt{St}^{\mathbf{FOL}} : \mathtt{D}^{\mathbf{FOL}} \to \mathbb{CAT}^{op}$ *denote the substitution functor which maps each signature* Σ *to the subcategory of* Σ*-substitutions represented by functions* $\theta : C_1 \to T_\Sigma(C_2)$ *as in Example 10.*

Example 14 (**PA** substitution functor). *Let* $\mathtt{D}^{\mathbf{PA}} \subseteq \mathtt{Sig}^{\mathbf{PA}}$ *be the broad subcategory of signature extensions with partial constants. The partial substitutions are represented by functions* $\theta : C_1 \to T_\Sigma(C_2)$*, where* $\Sigma \in |\mathtt{Sig}^{\mathbf{PA}}|$ *and* C_i *are finite sets of new partial constants for* Σ*. Let* $\mathtt{St}^{\mathbf{PA}} : \mathtt{D}^{\mathbf{PA}} \to \mathbb{CAT}^{op}$ *denote the substitution functor which maps each signature* Σ *to the subcategory of* Σ*-substitutions represented by functions* $\theta : C_1 \to T_\Sigma(C_2)$ *as in Example 11.*

3. Hybrid institutions

Given a base logic one can construct freely its hybridised version by applying the hybridisation process defined in [35, 17, 21]. In this paper, we give a definition of hybrid institution which provides a higher level of generality and a top-down approach to *Kripke semantics* in the spirit of universal logic.

3.1. Definition

Informally, hybrid institutions are institutions whose signatures are equipped with nominals and modalities and whose models are Kripke structures. Hybrid institutions are refinements of stratified institutions that were introduced in [1] to enhance the concept of institution with 'states' for the models.

Definition 15 (Hybrid institution)**.**

A hybrid institution $\mathtt{HI} = (\mathtt{Sig}^{\mathtt{HI}}, \mathtt{F}^{\mathtt{HI}}, \mathtt{Sen}^{\mathtt{HI}}, \mathtt{Mod}^{\mathtt{HI}}, \mathtt{K}^{\mathtt{HI}}, \models^{\mathtt{HI}})$ *is an institution* $(\mathtt{Sig}^{\mathtt{HI}}, \mathtt{Sen}^{\mathtt{HI}}, \mathtt{Mod}^{\mathtt{HI}}, \models^{\mathtt{HI}})$ *equipped with*

(1) a functor $\mathtt{F} : \mathtt{Sig}^{\mathtt{HI}} \to \mathtt{Sig}^{\mathbf{REL}}$*, which extracts from each signature* Δ *its relational part* $\mathtt{F}(\Delta) = (\mathtt{Nom}^{\Delta}, \Lambda^{\Delta})$*, where* $\mathtt{Nom}^{\Delta}$ *is a set of* nominals *and* $\Lambda^{\Delta} = \{\Lambda^{\Delta}_n\}_{n\in\mathbb{N}}$ *is a family of sets of* modalities*,*

(2) a natural transformation $\mathtt{K} : \mathtt{Mod}^{\mathtt{HI}} \Rightarrow (\mathtt{F}; \mathtt{Mod}^{\mathbf{REL}})$*, providing for each signature* Δ *a frame functor* $\mathtt{K}_{\Delta} : \mathtt{Mod}^{\mathtt{HI}}(\Delta) \to \mathtt{Mod}^{\mathbf{REL}}(\mathtt{Nom}^{\Delta}, \Lambda^{\Delta})$*, which extracts from each* Δ*-model* M *its* frame $\mathtt{K}_{\Delta}(M)$ *consisting of a set of* states/worlds $|\mathtt{K}_{\Delta}(M)|$ *together with their* accessibility relations $\mathtt{K}_{\Delta}(M)_{\lambda}$*, where* $\lambda \in \Lambda^{\Delta}_n$ *and* $n \in \mathbb{N}$*,*

(3) a local satisfaction relation $\{M \models^{-}_{\Delta} {}_{-}\}_{\Delta\in|\mathtt{Sig}^{\mathtt{HI}}|, M\in|\mathtt{Mod}^{\mathtt{HI}}(\Delta)|}$*,*
where $M \models^{-}_{\Delta} {}_{-} \subseteq |\mathtt{K}_{\Delta}(M)| \times \mathtt{Sen}^{\mathtt{HI}}(\Delta)$ *is a binary relation such that*

   (a) the (global) satisfaction holds iff the local satisfaction holds for all states, i.e. given a signature Δ*, for each* Δ*-model* M *and any* Δ*-sentence* ρ *we have:* $M \models^{\mathtt{HI}}_{\Delta} \rho$ *iff* $M \models^{w}_{\Delta} \rho$ *for all states* w *of* M*.*

   (b) the local satisfaction condition *holds, i.e. given a signature morphism* $\varphi : \Delta \to \Delta'$*, for every* Δ'*-model* M'*, each state* w' *of* M' *and any* Δ*-sentence* ρ *we have:* $M' \models^{w'}_{\Delta'} \varphi(\rho)$ *iff* $M'\restriction_{\varphi} \models^{w'}_{\Delta} \rho$*.*

The notation is overloaded such that $\models^{\mathtt{HI}}$ denote both families of relations $\{M \models^{-}_{\Delta} {}_{-}\}_{\Delta\in|\mathtt{Sig}^{\mathtt{HI}}|, M\in|\mathtt{Mod}^{\mathtt{HI}}|}$ and $\{{}_{-} \models^{\mathtt{HI}}_{\Delta} {}_{-}\}_{\Delta\in|\mathtt{Sig}^{\mathtt{HI}}|}$. Notice that the (global) satisfaction relation is determined by the local satisfaction relation. The consistency of the local satisfaction condition is ensured by the following result.

Lemma 16. *Given a signature morphism* $\varphi : \Delta \to \Delta'$ *of a hybrid institution, any* Δ'*-model* M' *has exactly the same set of states as its reduct* $M\restriction_{\varphi}$*.*

Proof. By the definition of $\mathtt{Mod}^{\mathbf{REL}}$, we have $|\mathtt{K}_{\Delta'}(M')| = |\mathtt{K}_{\Delta'}(M')\restriction_{F(\varphi)}|$, and since $\mathtt{K}$ is a natural transformation, we obtain $|\mathtt{K}_{\Delta'}(M')\restriction_{F(\varphi)}| = |\mathtt{K}_{\Delta}(M'\restriction_{\varphi})|$. It follows that $|\mathtt{K}_{\Delta'}(M')| = |\mathtt{K}_{\Delta}(M'\restriction_{\varphi})|$. □

In general, the semantics of hybrid institutions is constructed on top of the semantics of some base institution. This construction is presented in the following definition.

Definition 17 (Kripke structures)**.** *Let* $\mathtt{Mod}^{\mathtt{I}} : \mathtt{Sig}^{\mathtt{I}} \to \mathbb{CAT}^{op}$ *be a* base *model functor. The* Kripke *model functor* $\mathtt{Mod}^{\mathtt{I}}_{\kappa} : \mathtt{Sig}^{\mathbf{REL}} \times \mathtt{Sig}^{\mathtt{I}} \to \mathbb{CAT}^{op}$ *over* $\mathtt{Mod}^{\mathtt{I}}$ *is defined as follows:*

(1) for each signature $(\mathtt{Nom}, \Lambda, \Sigma) \in |\mathtt{Sig}^{\mathbf{REL}} \times \mathtt{Sig}^{\mathtt{I}}|$*, where* $(\mathtt{Nom}, \Lambda) \in |\mathtt{Sig}^{\mathbf{REL}}|$ *and* $\Sigma \in |\mathtt{Sig}^{\mathtt{I}}|$*,* $\mathtt{Mod}^{\mathtt{I}}_{\kappa}(\mathtt{Nom}, \Lambda, \Sigma)$ *is the category that consists of*

(a) *Kripke models of the form* (W, M)*, where* $W \in |\mathtt{Mod}^{\mathbf{REL}}(\mathtt{Nom}, \Lambda)|$ *and* $M : |W| \to |\mathtt{Mod}^{\mathtt{I}}(\Sigma)|$ *is a mapping from the set of states* $|W|$ *to the class of* Σ*-models* $|\mathtt{Mod}(\Sigma)|$*, and*

(b) *homomorphisms* $h : (W, M) \to (W', M')$ *of the form* $(h^{\mathbf{REL}}, h^{mod})$*, where* $\begin{cases} h^{\mathbf{REL}} : W \to W' \text{ is a homomorphism in } \mathbf{REL}, \\ h^{mod} : M \Rightarrow M' \circ h^{\mathbf{REL}} \text{ is a natural transformation.} \end{cases}$

(2) *for each signature morphism* $(\mathtt{Nom}, \Lambda, \Sigma) \xrightarrow{\varphi} (\mathtt{Nom}', \Lambda', \Sigma') \in \mathtt{Sig}^{\mathbf{REL}} \times \mathtt{Sig}^{\mathtt{I}}$*, where* $\varphi = (\varphi^{\mathbf{REL}}, \varphi^{\mathtt{I}})$*,* $(\mathtt{Nom}, \Lambda) \xrightarrow{\varphi^{\mathbf{REL}}} (\mathtt{Nom}', \Lambda') \in \mathtt{Sig}^{\mathbf{REL}}$ *and* $\Sigma \xrightarrow{\varphi^{\mathtt{I}}} \Sigma' \in \mathtt{Sig}^{\mathtt{I}}$*, the reduct functor* $\mathtt{Mod}^{\mathtt{I}}_{\kappa}(\varphi) : \mathtt{Mod}^{\mathtt{I}}_{\kappa}(\mathtt{Nom}', \Lambda', \Sigma') \to \mathtt{Mod}^{\mathtt{I}}_{\kappa}(\mathtt{Nom}, \Lambda, \Sigma)$ *is defined by*

(a) $\mathtt{Mod}^{\mathtt{I}}_{\kappa}(\varphi)(W', M') = (W, M)$ *such that*
- $W = W' \upharpoonright_{\varphi^{\mathbf{REL}}}$ *and* $M_w = M'_w \upharpoonright_{\varphi^{\mathtt{I}}}$ *for all states* $w \in |W|$*,*

where (W', M') *is a* $(\mathtt{Nom}', \Lambda', \Sigma')$*-model.*

(b) $\mathtt{Mod}^{\mathtt{I}}_{\kappa}(\varphi)(h') = h$ *such that*
- $h^{\mathbf{REL}} = h'^{\mathbf{REL}} \upharpoonright_{\varphi^{\mathbf{REL}}}$ *and* $h^{mod} = \{h'_w \upharpoonright_{\varphi^{\mathtt{I}}}\}_{w \in |W|}$*,*

where h' *is a* $(\mathtt{Nom}', \Lambda', \Sigma')$*-homomorphism.*

In our examples of hybrid institutions, the model functor is a sub-functor of some Kripke functor $\mathtt{Mod}^{\mathtt{I}}_{\kappa} : \mathtt{Sig}^{\mathbf{REL}} \times \mathtt{Sig}^{\mathtt{I}} \to \mathbb{CAT}^{op}$, the functor $\mathtt{F}^{\mathtt{HI}} : \mathtt{Sig}^{\mathbf{REL}} \times \mathtt{Sig}^{\mathtt{I}} \to \mathtt{Sig}^{\mathbf{REL}}$ is the first projection, and for all signatures Δ, the frame functor $\mathtt{K}_{\Delta}$ is the forgetful functor mapping each Kripke structure (W, M) to W. Definition 17 provides a pattern for describing the semantics of hybrid institutions but the results of this paper will be developed at a more abstract level provided by Definition 15, where the Kripke structures are implicitly assumed, not constructed. This approach corresponds to the universal logic ideas such that the results are developed at the most general level and hypothesis are introduced only by-need basis.

Assumption 18. *Throughout this paper we assume that* $\mathtt{HI}$ *range over hybrid institutions satisfying the following property: for each signature* Δ *and any nominal variable* j *there exists a designated signature morphism* $\chi_j : \Delta \to \Delta[j]$*, where* $\Delta[j]$ *is the notation of its target signature, such that*

(1) $\mathtt{F}(\chi_j) = \chi_j^{\mathbf{REL}}$*, where* $\chi_j^{\mathbf{REL}} : (\mathtt{Nom}^{\Delta}, \Lambda^{\Delta}) \hookrightarrow (\mathtt{Nom}^{\Delta} \cup \{j\}, \Lambda^{\Delta})$*, and*

(2) *for each* Δ*-model* M *and any* $\chi_j^{\mathbf{REL}}$*-expansion* W' *of* $\mathtt{K}_{\Delta}(M)$ *there exists a unique* χ_j*-expansion* M' *of* M *such that* $\mathtt{K}_{\Delta[j]}(M') = W'$*.*

The above assumption is easily justified by Definition 17: the signatures of hybrid institutions are of the form $\Delta = (\mathtt{Nom}, \Lambda, \Sigma)$, $\Delta[j] = (\mathtt{Nom} \cup \{j\}, \Lambda, \Sigma)$ and $\chi_j : (\mathtt{Nom}, \Lambda, \Sigma) \hookrightarrow (\mathtt{Nom} \cup \{j\}, \Lambda, \Sigma)$ is the inclusion. For any Δ-model (W, M) and any $\chi_j^{\mathbf{REL}}$-expansion W' of W, the model (W', M) is the unique χ_j-expansion of (W, M) such that $\mathtt{K}_{\Delta[j]}(W', M) = W'$.

Notation 19. *Assume a hybrid institution* $\mathtt{HI}$ *and a* Δ*-model* M*, where* Δ *is a signature of* $\mathtt{HI}$*. If* W *is the frame of* M*, in symbols* $K_\Delta(M) = W$*, then for every state* w *of* M *and each nominal variable* j*, we denote by*

(1) $W^{(j,w)}$ *the unique* $\chi_j^{\mathbf{REL}}$*-expansion of* W *such that the denotation of* j *in* $W^{(j,w)}$ *is* w*, in symbols* $W_j^{(j,w)} = w$*, and*

(2) $M^{(j,w)}$ *the unique* χ_j*-expansion of* M *such that* $W^{(j,w)}$ *is the frame of* $M^{(j,w)}$*, in symbols* $\mathtt{K}_{\Delta[j]}(M^{(j,w)}) = W^{(j,w)}$*.*

The semantics of each sentence operator is defined at the abstract level provided by Definition 15.

Definition 20 (Internal logic)**.** *Given a hybrid institution* $\mathtt{HI}$ *then*

(1) $M \models^w k_1 = k_2$ *iff* $W_{k_1} = W_{k_2}$*;*

(2) $M \models^w \lambda(k_1, \ldots, k_n)$ *iff* $(W_{k_1}, \ldots, W_{k_n}) \in W_\lambda$*;*

(3) $M \models^w @_k\rho$ *iff* $M \models^{W_k} \rho$*;*

(4) $M \models^w \rho_1 \wedge \rho_2$ *iff* $M \models^w \rho_1$ *and* $M \models^w \rho_2$*;*

(5) $M \models^w \neg\rho$ *iff* $M \not\models^w \rho$*;*

(6) $M \models^w [\lambda](\rho_1, \ldots, \rho_{n-1})$ *iff for all* $(w, w_1, \ldots, w_{n-1}) \in W_\lambda$ *we have* $M \models^{w_i} \rho_i$ *for some* $i \in \{1, \ldots, n-1\}$*;*

(7) $M \models^w (\forall\chi)\gamma$ *iff* $M' \models^w \gamma$ *for all* χ*-expansions* M' *of* M*;*

(8) $M \models^w_\Delta (\downarrow j)e$ *iff* $M^{(j,w)} \models^w_{\Delta[j]} e$*.*

where $\Delta \xrightarrow{\chi} \Delta'$ *is a signature morphism of* $\mathtt{HI}$*,* M *is a* Δ*-model,* W *is the frame of* M*,* w *is a state of* M*,* $k \in \mathtt{Nom}^\Delta$*,* $k_i \in \mathtt{Nom}^\Delta$*,* $n \in \mathbb{N}^*$*,* $\lambda \in \Lambda_n$*,* ρ *is a* Δ*-sentence,* ρ_i *is a* Δ*-sentence,* γ *is a* Δ'*-sentence,* j *is a nominal variable, and* e *is a* $\Delta[j]$*-sentence.*

We call $k_1 = k_2$ *nominal equations*, and $\lambda(k_1, \ldots, k_n)$ *nominal relations*. The operator @ is called *retrieve* because it changes the point of evaluation for a model. For any modality λ, the operator $[\lambda]$ is called traditionally *necessity*. The operator $\downarrow$ is called *store* because it allows us to give a name to the current state that can be referred later on in sentences.

Lemma 21. *Every (ordinary) institution* $\mathtt{I}$ *generates a hybrid institution* $\mathcal{H}(\mathtt{I}) = (\mathtt{Sig}^\mathtt{I}, \mathtt{F}^\mathtt{I}, \mathtt{Sen}^\mathtt{I}, \mathtt{Mod}^\mathtt{I}, \mathtt{K}^\mathtt{I}, \models^\mathtt{I})$ *where*

(1) $\mathtt{F}^\mathtt{I} : \mathtt{Sig}^\mathtt{I} \to \mathtt{Sig}^{\mathbf{REL}}$ *is defined by* $\mathtt{F}^\mathtt{I}(\Sigma) = (\emptyset, \emptyset)$ *for all* $\Sigma \in |\mathtt{Sig}^\mathtt{I}|$,

(2) for any $\Sigma \in |\mathtt{Sig}^\mathtt{I}|$*, the functor* $\mathtt{K}^\mathtt{I}_\Sigma : \mathtt{Mod}^\mathtt{I}(\Sigma) \to \mathtt{Mod}^{\mathbf{REL}}(\emptyset, \emptyset)$ *is defined by*

(a) $\mathtt{K}^{\mathtt{I}}_{\Sigma}(N) = (\mathtt{W}^N, \mathtt{M}^N)$ *for all* Σ*-models* N, *where* $|\mathtt{W}^N| = \{\star\}$ *and* $\mathtt{M}^N(\star) = N$.

(b) $\mathtt{K}^{\mathtt{I}}_{\Sigma}(h : N \to P) = (h^{\mathbf{REL}}, h^{mod}) : (\mathtt{W}^N, \mathtt{M}^N) \to (\mathtt{W}^P, \mathtt{M}^P)$ *for all* Σ*-homomorphisms* $h : N \to P$, *where* $h^{\mathbf{REL}}(\star) = \star$ *and* $h^{mod}_{\star} = h$.

(3) $(\mathtt{W}^N, \mathtt{M}^N) \models^{\star} \rho$ *iff* $N \models^{\mathtt{I}} \rho$ *for all* $\Sigma \in |\mathtt{Sig}^{\mathtt{I}}|$, $N \in |\mathtt{Mod}^{\mathtt{I}}(\Sigma)|$ *and* $\rho \in \mathtt{Sen}^{\mathtt{I}}(\Sigma)$.

Proof. It is easy to prove that $\mathtt{F}^{\mathtt{I}} : \mathtt{Sig}^{\mathtt{I}} \to \mathtt{Sig}^{\mathbf{REL}}$ is a functor and $\mathtt{K}^{\mathtt{I}} : \mathtt{Mod}^{\mathtt{I}} \Rightarrow \mathtt{F}^{\mathtt{I}}; \mathtt{Mod}^{\mathbf{REL}}$ defined by $\mathtt{K}^{\mathtt{I}} = \{\mathtt{K}^{\mathtt{I}}_{\Sigma} : \mathtt{Mod}^{\mathtt{I}}(\Sigma) \to \mathtt{Mod}^{\mathbf{REL}}(\emptyset, \emptyset)\}_{\Sigma \in |\mathtt{Sig}|}$ is a natural transformation. Given a signature Σ, for all Σ-models N and Σ-sentences ρ we have $N \models^{\mathcal{H}(\mathtt{I})} \rho$ iff $N \models^{\star} \rho$ iff $N \models^{\mathtt{I}} \rho$. It follows that the local satisfaction condition for $\mathcal{H}(\mathtt{I})$ is a consequence of the satisfaction condition for $\mathtt{I}$. $\square$

Lemma 21 shows that ordinary institutions can be regarded as particular cases of hybrid institutions such that the set of states for each model is a singleton. For this reason, the semantics of sentence operators is strictly more general for hybrid institutions than for ordinary ones. For example, given an institution $\mathtt{I}$, we have (a) $\neg\neg\rho \models^{\mathtt{I}} \rho$, and (b) $\Gamma \models^{\mathtt{I}} \bigwedge H \Rightarrow C$ iff $\Gamma \cup H \models^{\mathtt{I}} C$, for all $\Sigma \in |\mathtt{Sig}^{\mathtt{I}}|$, $\rho \in \mathtt{Sen}^{\mathtt{I}}(\Sigma)$, $\Gamma \subseteq \mathtt{Sen}^{\mathtt{I}}(\Sigma)$ and $H \cup \{C\} \subseteq \mathtt{Sen}^{\mathtt{I}}(\Sigma)$. The above properties do not hold in hybrid institutions. It follows that the abstract results developed at the level of institutions are not applicable to hybrid institutions, in general. On the other hand, Lemma 21 provides a solution to apply the results for hybrid institutions to ordinary institutions. See Section 7.

3.2. Examples

We give a few examples of hybrid institutions which can be found in other works as well.

Example 22 (Hybrid Propositional Logic (**HPL**) [2])**.** *The signatures of this institution* $\Delta = (\mathtt{Nom}, \Lambda, \mathtt{Prop})$ *consist of a set of nominals* $\mathtt{Nom}$*, a family of sets of modalities* $\Lambda = \{\Lambda_n\}_{n \in \mathbb{N}}$*, and a set of propositional symbols* $\mathtt{Prop}$*. The* Δ*-models are Kripke structures of the form* (W, M)*, where* W *is a* $(\mathtt{Nom}, \Lambda)$*-model in* **REL** *and* $M : |W| \to |\mathtt{Mod}^{\mathbf{PL}}(\mathtt{Prop})|$ *is a mapping. The atomic* Δ*-sentences consist of propositional symbols* $p \in \mathtt{Prop}$*. 'Full' sentences are constructed from atomic sentences, nominal equations and nominal relations by applying Boolean operators, retrieve, store, necessity and quantification over nominal variables. The satisfaction relation for atomic sentences is defined by* $(W, M) \models^{w} p$ *iff* $p \in M_w$ *for all* Δ*-models* (W, M)*, states* $w \in |W|$ *and propositional symbols* $p \in \mathtt{Prop}$.

Example 23 (Hybrid first-order logic (**HFOL**))**.** *This hybrid institution is a variation of* first-order hybrid logic *of [7] where models may have different carrier sets across the states. The functor* $\mathtt{Mod}^{\mathbf{HFOL}}$ *is the Kripke model functor* $\mathtt{Mod}^{\mathbf{FOL}}_{\kappa} : \mathtt{Sig}^{\mathbf{REL}} \times \mathtt{Sig}^{\mathbf{FOL}} \to \mathbb{CAT}^{op}$*. Given a* **HFOL** *signature* $\Delta = (\mathtt{Nom}, \Lambda, \Sigma)$*, the atomic sentences in* $\mathtt{Sen}^{\mathbf{HFOL}}(\Delta)$ *are the atomic sentences in* $\mathtt{Sen}^{\mathbf{FOL}}(\Sigma)$*. The sentences are constructed from atomic sentences, nominal* *equations and nominal relations by applying Boolean operators, retrieve, store, necessity and quantification over nominal variables. The satisfaction of atomic sentences is defined by* $(W, M) \models^w \rho$ *iff* $M_w \models^{\mathbf{FOL}} \rho$ *for all signatures* Δ*, atomic* Δ*-sentences* ρ*,* Δ*-models* (W, M) *and states* $w \in |W|$*.*

Example 24 (Hybrid First-Order Logic with user-defined Sharing (**HFOLS**) [35, 17])**.** *The functor* $\mathtt{Mod}^{\mathbf{HFOLS}} : \mathtt{Sig}^{\mathbf{REL}} \times \mathtt{Sig}^{\mathbf{FOLR}} \to \mathbb{CAT}^{op}$ *is a subfunctor of* $\mathtt{Mod}_\kappa^{\mathbf{FOLR}} : \mathtt{Sig}^{\mathbf{REL}} \times \mathtt{Sig}^{\mathbf{FOLR}} \to \mathbb{CAT}^{op}$ *which restricts the models and the homomorphisms of* $\mathtt{Mod}_\kappa^{\mathbf{FOLR}}$ *such that the rigid symbols are interpreted uniformly across the states, i.e. for all* $\Delta \in |\mathtt{Sig}^{\mathbf{HFOLS}}|$*, we have (a)* $(W, M) \in |\mathtt{Mod}^{\mathbf{HFOLS}}(\Delta)|$ *iff for all states* $w_1, w_2 \in |W|$ *and rigid symbols* x *in* Δ *we have* $(M_{w_1})_x = (M_{w_2})_x$*, and (b)* $h : (W, M) \to (W', M') \in \mathtt{Mod}^{\mathbf{HFOLS}}(\Delta)$ *iff for all states* $w_1, w_2 \in |W|$ *and rigid sorts* $\mathtt{sr}$ *in* Δ *we have* $(h^{mod}_{w_1})_{\mathtt{sr}} = (h^{mod}_{w_2})_{\mathtt{sr}}$*.*

Given a signature $\Delta = (\mathtt{Nom}, \Lambda, \Sigma)$*, the atomic sentences in* $\mathtt{Sen}^{\mathbf{HFOLS}}(\Delta)$ *are the atomic sentences in* $\mathtt{Sen}^{\mathbf{FOLR}}(\Sigma)$*. The sentences are constructed from atomic sentences, nominal equations and nominal relations by applying Boolean operators, retrieve, store, necessity and quantification over nominal variables and rigid variables. The satisfaction of atomic sentences is defined as follows:* $(W, M) \models^w \rho$ *iff* $M_w \models^{\mathbf{FOLR}} \rho$*, for all signatures* Δ*, atomic* Δ*-sentences* ρ*,* Δ*-models* (W, M) *and states* $w \in |W|$*.*

4. Hybrid institutions with annotated syntax

We propose a variation of hybrid institutions, where the atomic sentences are constructed from the terms of the initial model. This approach has the advantage of (a) increasing the expressivity of the logical syntax, and (b) allowing a more structured model theory, which leads to the proof of the desired results.

Given a hybrid institution, we let Δ and Δ' range over signatures of the form $(\mathtt{Nom}, \Lambda, \Sigma)$ and $(\mathtt{Nom}', \Lambda', \Sigma')$, respectively. If the hybrid institution is **HFOLS** then we let Σ and Σ' range over **FOLR** signatures of the form $(S^r, F^r, P^r) \subseteq (S, F, P)$ and $(S'^r, F'^r, P'^r) \subseteq (S', F', P')$, respectively.

4.1. Initial term model

We proceed by studying the existence of initial models in **HFOLS**.

Definition 25 (Hybrid term)**.** *Let* Δ *be a* **HFOLS** *signature. For all nominals* $k \in Nom$*, we define (simultaneously) the* S*-sorted sets* T_k^Δ *of hybrid* Δ*-terms:*

(1) $\dfrac{\tau \in (T_k^\Delta)_{\mathtt{ar}}}{\varsigma(\tau) \in (T_k^\Delta)_{\mathtt{sr}}}$ *for all rigid symbols* $\varsigma \in F^r$*,*

where $\mathtt{ar}$ *is the arity of* ς *and* $\mathtt{sr}$ *is the sort of* ς*;*

(2) $\dfrac{t \in (T_k^\Delta)_a}{\sigma_k(t) \in (T_k^\Delta)_s}$ *for all non-rigid symbols* $\sigma \in (F - F^r)$*,*

where a *is the arity of* σ *and* s *is the sort of* σ*;*

(3) $\dfrac{\tau \in (T^{\Delta}_{k_1})_{\mathtt{sr}}}{\tau \in (T^{\Delta}_{k})_{\mathtt{sr}}}$ *for all nominals* $k_1 \in \mathtt{Nom}$ *and rigid sorts* $\mathtt{sr} \in S^r$.

By the first statement of Definition 25, the rigid symbols do not receive annotation as they are interpreted uniformly across the states. By the second statement of Definition 25, the non-rigid symbols receive annotation as they are not shared between states. By the third statement of Definition 25, we have $(T^{\Delta}_{k_1})_{\mathtt{sr}} = (T^{\Delta}_{k_2})_{\mathtt{sr}}$ for all nominals $k_1, k_2 \in \mathtt{Nom}$ and rigid sorts $\mathtt{sr} \in S^r$.

Definition 26. *The interpretation of a hybrid term into Kripke structure is defined inductively on the structure of the hybrid terms. Given a* **HFOLS** *signature* Δ*, for any* Δ*-model* (W, M)*, we have:*

(1) $(W, M)_{\varsigma(\tau)} = (M_{W_k})_{\varsigma}((W, M)_{\tau})$,
where $\varsigma \in F^r_{\mathtt{ar}\to\mathtt{sr}}$*,* $\tau \in (T^{\Delta}_{k})_{\mathtt{ar}}$*,* $k \in \mathtt{Nom}$*,* $\mathtt{ar} \in (S^r)^*$ *and* $\mathtt{sr} \in S^r$*;*

(2) $(W, M)_{\sigma_k(t)} = (M_{W_k})_{\sigma}((W, M)_{t})$,
where $\sigma \in (F_{a\to s} - F^r_{a\to s})$*,* $k \in \mathtt{Nom}$*,* $t \in (T^{\Delta}_{k})_{a}$*,* $a \in S^*$ *and* $s \in S$*;*

(3) *Let* $\tau \in (T^{\Delta}_{k_1})_{\mathtt{sr}}$ *be a rigid term, where* $k_1 \in \mathtt{Nom}$ *and* $\mathtt{sr} \in S^r$.
For any nominal $k \in \mathtt{Nom}$*, assuming that* $(W, M)_{\tau} \in (M_{W_{k_1}})_{\mathtt{sr}}$ *is defined then the interpretation of* $\tau \in (T^{\Delta}_{k})_{\mathtt{sr}}$ *into* (W, M) *is* $(W, M)_{\tau} \in (M_{W_{k_1}})_{\mathtt{sr}}$.

The third statement of Definition 26 is consistent as $(M_{W_{k_1}})_{\mathtt{sr}} = (M_{W_k})_{\mathtt{sr}}$.

Definition 27. *Let* Δ *be a* **HFOLS** *signature and* $k \in \mathtt{Nom}$ *a nominal.*

(1) *We define the first-order* Σ*-model* M^{Δ}_{k} *as follows:*

  (a) *the carrier set of* M^{Δ}_{k} *is* T^{Δ}_{k}*;*

  (b) *the function* $(T^{\Delta}_{k})_{\varsigma} : (T^{\Delta}_{k})_{\mathtt{ar}} \to (T^{\Delta}_{k})_{\mathtt{sr}}$ *is defined by* $(M^{\Delta}_{k})_{\varsigma}(\tau) = \varsigma(\tau)$ *for all rigid function symbols* $\varsigma \in F^r$ *and hybrid terms* $\tau \in (T^{\Delta}_{k})_{\mathtt{ar}}$*, where* $\mathtt{ar}$ *is the arity of* ς *and* $\mathtt{sr}$ *is the sort of* ς*;*

  (c) *the function* $(T^{\Delta}_{k})_{\sigma} : (T^{\Delta}_{k})_{a} \to (T^{\Delta}_{k})_{s}$ *is defined by* $(M^{\Delta}_{k})_{\sigma}(t) = \sigma_k(t)$ *for all non-rigid function symbols* $\sigma \in (F - F^r)$ *and hybrid terms* $t \in (T^{\Delta}_{k})_{a}$*, where* a *is the arity of* σ *and* s *is the sort of* σ*;*

  (d) $(M^{\Delta}_{k})_{\pi} = \emptyset$ *for all* $\pi \in P$.

(2) *We define the* Δ*-model* (W^{Δ}, M^{Δ}) *as follows:*

  (a) W^{Δ} *is* $(\mathtt{Nom}, \Lambda)$*-model such that* $|W^{\Delta}| = \mathtt{Nom}$ *and* $W^{\Delta}_{\lambda} = \emptyset$ *for all* $\lambda \in \Lambda$*, and*

  (b) M^{Δ} *maps each nominal* $k \in \mathtt{Nom}$ *to* M^{Δ}_{k}.

Since the rigid function and relation symbols do not receive annotations and the terms of rigid sorts are shared between states, the definition of term model above is consistent.

Lemma 28. *The term model (W^Δ, M^Δ) defined in Definition 27 is initial.*

Proof. Let $(W, M) \in |\mathtt{Mod}^{\mathbf{HFOLS}}(\Delta)|$. One can straightforwardly prove that there exists a homomorphism $h : (W^\Delta, M^\Delta) \to (W, M)$ defined by $h^{mod}_k(t) = (W, M)_t$ for all $k \in \mathtt{Nom}$ and $t \in T^\Delta_k$.

For the uniqueness part, assume another homomorphism $g : (W^\Delta, M^\Delta) \to (W, M)$ and prove $g^{mod}_k = h^{mod}_k$ for all nominals $k \in \mathtt{Nom}$ by induction on the structure of the Δ-terms:

(1) Let $\varsigma \in F^r$, $k \in \mathtt{Nom}$, $\tau \in (T^\Delta_k)_{\mathtt{ar}}$, where $\mathtt{ar}$ is the arity of ς, and assume $g^{mod}_k(\tau) = h^{mod}_k(\tau)$.
We have $h^{mod}_k(\varsigma(\tau)) = (M_k)_\varsigma(h^{mod}_k(\tau)) = (M_k)_\varsigma(g^{mod}_k(\tau)) = g^{mod}_k(\varsigma(\tau))$.

(2) Let $\sigma \in (F - F^r)$, $k \in \mathtt{Nom}$, $t \in (T^\Delta_k)_a$, where a is the arity of σ, and assume $g^{mod}_k(t) = h^{mod}_k(t)$.
We have $h^{mod}_k(\sigma_k(t)) = (M_k)_\sigma(h^{mod}_k(t)) = (M_k)_\sigma(g^{mod}_k(t)) = g^{mod}_k(\sigma_k(t))$.

(3) Let $\mathtt{sr} \in S^r$, $k_1 \in \mathtt{Nom}$, $\tau \in (T^\Delta_{k_1})_{\mathtt{sr}}$, and assume $h^{mod}_{k_1}(\tau) = g^{mod}_{k_1}(\tau)$. We show that $h^{mod}_k(\tau) = g^{mod}_k(\tau)$ for a given nominal $k \in \mathtt{Nom}$. By the definition of g, $g^{mod}_k(\tau) = g^{mod}_{k_1}(\tau)$. By induction hypothesis, $g^{mod}_{k_1}(\tau) = h^{mod}_{k_1}(\tau)$. By the definition of h, $h^{mod}_{k_1}(\tau) = h^{mod}_k(\tau)$. Hence, $g^{mod}_k(\tau) = h^{mod}_k(\tau)$.

$\square$

One can replicate Definitions 25, 26 and 27 for hybrid preorder algebras with user-defined sharing (**HPOAS**) and define the hybrid terms, their interpretations into the models, and the hybrid term models. Note that in **HPOAS** the preorder relation on the hybrid term models is empty. It follows that the initiality result of Lemma 28 can be straightforwardly replicated to **HPOAS**.

4.2. Examples

We present two examples of hybridised logics where the atomic sentences are constructed with terms of the initial model.

Example 29 (Hybrid First-Order Logic with user-defined Sharing and Annotation (**HFOLSA**) [27])**.** *This institution has the same model functor as* **HFOLS**, *i.e.* $\mathtt{Mod}^{\mathbf{HFOLSA}} = \mathtt{Mod}^{\mathbf{HFOLS}}$.

The set of atomic Δ-sentences consist of

(a) hybrid equations $t =^k_s t'$, where $t \in (T^\Delta_k)_s$, $t' \in (T^\Delta_k)_s$, $k \in \mathtt{Nom}$ and $s \in S$;

(b) rigid relations $\varpi(\tau)$, where $\varpi \in P^r_{\mathtt{ar}}$, $\tau \in (T^\Delta_k)_{\mathtt{ar}}$, $\mathtt{ar} \in (S^r)^*$ and $k \in \mathtt{Nom}$;

(c) non-rigid relations $\pi_k(t)$, where $\pi \in (P_a - P^r_a)$, $t \in (T^\Delta_k)_a$, $a \in S^*$ and $k \in \mathtt{Nom}$.

When there is no danger of confusion we omit the subscript s and/or the superscript k from the notation $t =_s^k t'$. Full sentences are constructed from atomic sentences, nominal equations and nominal relations by applying Boolean operators, retrieve, store, necessity and quantification over nominal variables and rigid variables. Given a signature morphism $\Delta \xrightarrow{\varphi} \Delta'$, the atomic sentences get translated symbol-wise.

The satisfaction relation for atomic sentences is defined as follows:

(a) $(W, M) \models t = t'$ *iff* $(W, M)_t = (W, M)_{t'}$*;*

(b) $(W, M) \models \varpi(\tau)$ *iff* $(W, M)_\tau \in (M_k)_\varpi$*;*

(c) $(W, M) \models \pi_k(t)$ *iff* $(W, M)_t \in (M_k)_\pi$*.*

Lemma 30. *We have* $(W', M')_{\varphi(t)} = ((W', M') \upharpoonright_\varphi)_t$*, for all* $\Delta \xrightarrow{\varphi} \Delta' \in \mathtt{Sig}^{\mathbf{HFOLSA}}$*,* $(W', M') \in |\mathtt{Mod}^{\mathbf{HFOLSA}}(\Delta')|$*,* $k \in \mathtt{Nom}$ *and* $t \in T_k^\Delta$*.*

The local satisfaction condition is a corollary of Lemma 30.

Corollary 31. HFOLSA *is an institution.*

Similarly one can replicate the construction of **HFOLSA** in the context of preorder algebra.

Example 32 (Hybrid Preorder Algebra with user-defined Sharing and Annotation (**HPOASA**))**.** *The Kripke structures of* **HPOASA** *upgrade the Kripke structures of* **HFOLSA** *with preorder relations for the carrier sets of each sort. Given a signature* $\Delta = (\mathtt{Nom}, \Lambda, \Sigma)$*, where* $\Sigma = (S^r, F^r) \subseteq (S, F)$*, the atomic* Δ*-sentences consist of (a) equations* $t =_s^k t'$ *and (b) transitions* $t \Rightarrow_s^k t'$*, where* $t \in (T_k^\Delta)_s$*,* $t' \in (T_k^\Delta)_s$*,* $k \in \mathtt{Nom}$ *and* $s \in S$*. The sentences are constructed from atomic sentences, nominal equations and nominal relations by applying Boolean operators, retrieve, store, necessity and quantification over nominal variables and rigid variables. The satisfaction relation for atomic sentences is defined by (a)* $(W, M) \models (t = t')$ *iff* $(W, M)_t = (W, M)_{t'}$ *and (b)* $(W, M) \models (t \Rightarrow t')$ *iff* $(W, M)_t \leq (W, M)_{t'}$*.*

4.3. Standard approach vs. annotation

We investigate the connection between **HFOLS** and **HFOLSA** pointing out that **HFOLSA** is superior in terms of expressivity. We denote by $\mathbf{FOLR}_0$ and $\mathbf{HFOLSA}_0$ the restrictions of **FOLR** and **HFOLSA**, respectively, to atomic sentences.

Remark 33. *We have* $(W, M) \models^{w_1} \rho$ *iff* $(W, M) \models^{w_2} \rho$ *for all* **HFOLSA** *signatures* Δ*, models* $(W, M) \in |\mathtt{Mod}^{\mathbf{HFOLSA}}(\Delta)|$*, states* $w_1, w_2 \in |W|$*, and atomic sentences* $\rho \in \mathtt{Sen}^{\mathbf{HFOLSA}_0}(\Delta)$*.*

The above remark says that the satisfaction of **HFOLSA** atoms is invariant across the states. Consider a **HFOLS** signature $\Delta = (\mathtt{Nom}, \Lambda, \Sigma)$ and recall that in **HFOLS** we let Σ range over signatures of the form $(S^r, F^r, P^r) \subseteq (S, F, P)$. Let j be a nominal variable and define the functions:

(1) $\mathtt{at}_j : T_{(S,F,P)} \to T_j^{\Delta[j]}$ by induction on the structure of terms:

$$\mathtt{at}_j(\sigma(t)) = \begin{cases} \sigma(\mathtt{at}_j(t)) & \text{if } \sigma \in F^r \\ \sigma_j(\mathtt{at}_j(t)) & \text{if } \sigma \in F - F^r \end{cases}$$

where $t \in (T_{(S,F,P)})_a$ and a is the arity of σ.

(2) $\alpha_\Delta : \mathtt{Sen}^{\mathbf{HFOLS}_0}(\Delta) \to \mathtt{Sen}^{\mathbf{HFOLSA}}(\Delta)$ by extending $\mathtt{at}_j$ to atomic sentences:

(a) $\alpha_\Delta(t_1 = t_2) = (\downarrow j)(\mathtt{at}_j(t_1) = \mathtt{at}_j(t_2))$, where $t_1, t_2 \in T_{(S,F,P)}$;

(b) $\alpha_\Delta(\pi(t)) = \begin{cases} (\downarrow j)\pi(\mathtt{at}_j(t)) & \text{if } \pi \in P^r \\ (\downarrow j)\pi_j(\mathtt{at}_j(t)) & \text{if } \pi \in P - P^r \end{cases}$

where $t \in (T_{(S,F,P)})_a$, and a is the arity of π.

We overload the notation and let α_Δ denote the canonical extension of the function $\alpha_\Delta : \mathtt{Sen}^{\mathbf{HFOLS}_0}(\Delta) \to \mathtt{Sen}^{\mathbf{HFOLSA}}(\Delta)$ to all **HFOLS** sentences.

Proposition 34. *We have $(W, M) \models^w \rho$ iff $(W, M) \models^w \alpha_\Delta(\rho)$ for all* **HFOLS** *signatures Δ, models $(W, M) \in |\mathtt{Mod}^{\mathbf{HFOLS}}(\Delta)|$, states $w \in |W|$ and sentences $\rho \in \mathtt{Sen}^{\mathbf{HFOLS}}(\Delta)$.*

Proof. Firstly, we show that $(M_w)_t = (W^{(j,w)}, M)_{\mathtt{at}_j(t)}$ for all signatures Δ, models $(W, M) \in |\mathtt{Mod}^{\mathbf{HFOLS}}(\Delta)|$, states $w \in |W|$ and terms $t \in T_{(S,F,P)}$. We proceed by induction on the structure of terms. Let $\sigma(t) \in T_{(S,F,P)}$ such that

(a) $\sigma \in F^r$: $(M_w)_{\sigma(t)} = (M_w)_\sigma((M_w)_t) = (M_w)_\sigma((W^{(j,w)}, M)_{\mathtt{at}_j(t)}) = (W^{(j,w)}, M)_{\sigma(\mathtt{at}_j(t))} = (W^{(j,w)}, M)_{\mathtt{at}_j(\sigma(t))}$.

(b) $\sigma \in (F - F^r)$: $(M_w)_{\sigma(t)} = (M_w)_\sigma((M_w)_t) = (M_w)_\sigma((W^{(j,w)}, M)_{\mathtt{at}_j(t)}) = (W^{(j,w)}, M)_{\sigma_j(\mathtt{at}_j(t))} = (W^{(j,w)}, M)_{\mathtt{at}_j(\sigma(t))}$.

Secondly, we prove the statement of the proposition for the atoms of **HFOLS**.

(1) For any $t_1, t_2 \in T_{(S,F,P)}$ we have $(W, M) \models^w t_1 = t_2$ iff $(M_w)_{t_1} = (M_w)_{t_2}$ iff $(W^{(j,w)}, M)_{\mathtt{at}_j(t_1)} = (W^{(j,w)}, M)_{\mathtt{at}_j(t_2)}$ iff $(W^{(j,w)}, M) \models^w \mathtt{at}_j(t_1) = \mathtt{at}_j(t_2)$ iff $(W, M) \models^w (\downarrow j)\mathtt{at}_j(t_1) = \mathtt{at}_j(t_2)$ iff $(W, M) \models^w \alpha_\Delta(t_1 = t_2)$.

(2) There are two sub-cases for hybrid relations.

(a) For $\varpi \in P^r$ and $\tau \in (T_{(S,F,P)})_{\mathtt{ar}}$ where $\mathtt{ar}$ is the arity of ϖ: $(W, M) \models^w \varpi(\tau)$ iff $(M_w)_\tau \in (M_w)_\varpi$ iff $(W^{(j,w)}, M)_{\mathtt{at}_j(\tau)} \in (M_w)_\varpi$ iff $(W^{(j,w)}, M) \models^w \varpi(\mathtt{at}_j(\tau))$ iff $(W, M) \models^w (\downarrow j)\varpi(\mathtt{at}_j(\tau))$ iff $(W, M) \models^w \alpha_\Delta(\varpi(\tau))$.

(b) For any $\pi \in (P - P^r)$ and $t \in (T_{(S,F,P)})_a$, where a is the arity of π: $(W, M) \models^w \pi(t)$ iff $(M_w)_t \in (M_w)_\pi$ iff $(W^{(j,w)}, M)_{\mathtt{at}_j(t)} \in (M_w)_\pi$ iff $(W^{(j,w)}, M) \models^w \pi_j(\mathtt{at}_j(t))$ iff $(W, M) \models^w (\downarrow j)\pi_j(\mathtt{at}_j(t))$ iff $(W, M) \models^w \alpha_\Delta(\pi_j(t))$.

Thirdly, the proof of the statement for all **HFOLS** sentences is a straightforward extension of the case for atomic sentences. □

By Proposition 34, for any sentence in **HFOLS** there exists a semantically equivalent formula in **HFOLSA**. The sentence operator store is somehow integrated in the atomic sentences of **HFOLS** while the construction of **HFOLSA** avoids this situation, e.g. each **HFOLS** equation $t_1 = t_2$ is satisfied by the same models as the **HFOLSA** sentence $(\downarrow j)\mathtt{at}_j(t_1) = \mathtt{at}_j(t_2)$ and the top sentence operator of the latter sentence is store. It follows that **HFOLSA** allows a more structured approach to model-theoretic properties.

Notice that $\alpha : \mathtt{Sen}^{\mathbf{HFOLS}} \Rightarrow \mathtt{Sen}^{\mathbf{HFOLSA}}$ defined by $\alpha = \{\alpha_\Delta\}_{\Delta \in |\mathtt{Sig}^{\mathbf{HFOLS}}|}$ is a natural transformation. Using Proposition 34 one can easily define a *comorphism* [25] from **HFOLS** to **HFOLSA** which means that **HFOLSA** is at least as expressive as **HFOLS**. Annotated atomic sentences of the form $\sigma_{k_1} = \sigma_{k_2}$, which say that the non-rigid constant symbol σ is interpreted uniformly in the worlds denoted by the nominals k_1 and k_2, cannot be expressed in **HFOLS**. Therefore, we conclude that **HFOLSA** is more expressive than **HFOLS**.

4.4. Hybrid substitutions

In this subsection we define an abstract notion of hybrid substitution, which upgrades the notion of substitution defined in Section 2.3 to hybrid institutions.

Definition 35. *Let* $\mathtt{HI}$ *be a hybrid institution. A* hybrid substitution *between the signature morphisms* $\Delta \xrightarrow{\chi_1} \Delta' \in \mathtt{Sig}^{\mathtt{HI}}$ *and* $\Delta \xrightarrow{\chi_2} \Delta'' \in \mathtt{Sig}^{\mathtt{HI}}$ *is a substitution* $\chi_1 \xrightarrow{\theta} \chi_2$ *such that the following* local satisfaction condition *holds:* $M'' \models^w \theta(\rho')$ *iff* $M'' \upharpoonright_\theta \models^w \rho$*, for all* $M'' \in |\mathtt{Mod}^{\mathtt{HI}}(\Delta'')|$*,* $\rho \in \mathtt{Sen}^{\mathtt{HI}}(\Delta')$ *and* $w \in |\mathtt{K}_{\Delta''}(M'')|$.

The following result shows that the local satisfaction condition for substitutions is consistent.

Lemma 36. M'' *and* $M''\upharpoonright_\theta$ *from Definition 35 share the same set of states.*

Proof. By Lemma 16, $|\mathtt{K}_{\Delta''}(M'' \upharpoonright_\theta)| = |\mathtt{K}_{\Delta''}(M'' \upharpoonright_\theta \upharpoonright_{\chi_1})|$. It follows that $|\mathtt{K}_{\Delta''}(M''\upharpoonright_\theta)| = |\mathtt{K}_{\Delta''}(M''\upharpoonright_{\chi_2})|$. By Lemma 16, $|\mathtt{K}_{\Delta''}(M''\upharpoonright_{\chi_2})| = |\mathtt{K}_{\Delta''}(M'')|$. Hence, $|\mathtt{K}_{\Delta''}(M''\upharpoonright_\theta)| = |\mathtt{K}_{\Delta''}(M'')|$.

□

Example 37 (**HFOLSA** substitutions)**.** *Consider two signature extensions with nominals and rigid constant symbols* $\chi_1 : \Delta \hookrightarrow \Delta[C_1, D_1]$ *and* $\chi_2 : \Delta \hookrightarrow \Delta[C_2, D_2]$*, where* $\Delta = (\mathtt{Nom}, \Lambda, \Sigma)$*,* $\Sigma = (S^r, F^r, P^r) \subseteq (S, F, P)$*,* C_i *is a set of nominals different from the elements of* $\mathtt{Nom}$*, and* D_i *is a set of rigid constants different from the constants in* F*. A pair of functions* $\theta = (\theta_a : C_1 \to \mathtt{Nom} \cup C_2, \theta_b : D_1 \to T_k^{\Delta[C_2,D_2]})$*, where* $k \in \mathtt{Nom}$*, represents a substitution between* χ_1 *and* χ_2*:*

(1) *On the syntactic side, θ determines a sentence translation* $\mathtt{Sen}^{\mathbf{HFOLSA}}(\theta) : \mathtt{Sen}^{\mathbf{HFOLSA}}(\Delta[C_1, D_1]) \to \mathtt{Sen}^{\mathbf{HFOLSA}}(\Delta[C_2, D_2])$*, which preserves Δ and substitutes*

   (i) nominals in $\mathtt{Nom} \cup C_2$ for nominals in C_1 according to θ_a, and

   (ii) $\Delta[C_2, D_2]$-terms for constants in D_1 according to θ_b.

(2) *On the semantics side, θ determines a model functor* $\mathtt{Mod}^{\mathbf{HFOLSA}}(\theta) : \mathtt{Mod}^{\mathbf{HFOLSA}}(\Delta[C_2, D_2]) \to \mathtt{Mod}^{\mathbf{HFOLSA}}(\Delta[C_1, D_1])$ *such that for all* $(W'', M'') \in |\mathtt{Mod}^{\mathbf{HFOLSA}}(\Delta[C_2, D_2])|$ *the model* $(W'', M'') \upharpoonright_\theta$ *interprets*

   (i) every symbol in Δ as (W'', M''),

   (ii) each nominal $c_1 \in C_1$ as $(W'', M'')_{\theta_a(c_1)}$, and

   (iii) any rigid symbol $d_1 \in D_1$ as $(W'', M'')_{\theta_b(d_1)}$.

Lemma 38. *We have $((W'', M'') \upharpoonright_\theta)_t = (W'', M'')_{\theta(t)}$ for all $\Delta[C_2, D_2]$-models (W'', M'') and terms $t \in T_k^{\Delta[C_1, D_1]}$.*

Proof. Straightforward, by induction on the structure of the terms in $T_k^{\Delta[C_1, D_1]}$. □

The local satisfaction condition for substitutions is a direct consequence of Lemma 38.

Corollary 39. *We have $(W'', M'') \upharpoonright_\theta \models^w \rho$ iff $(W'', M'') \models^w \theta(\rho)$ for all $\Delta[C_2, D_2]$-models (W'', M''), states $w \in |W''|$, and $\Delta[C_1, D_1]$-sentences ρ.*

Hybrid substitution functors.

Given a hybrid institution $\mathtt{HI}$, for any signature $\Delta \in |\mathtt{Sig}^{\mathtt{HI}}|$, the hybrid Δ-substitutions form a subcategory $\mathtt{HSb}^{\mathtt{HI}}(\Delta)$ of $\mathtt{Sb}^{\mathtt{HI}}(\Delta)$.

Fact 40. $\mathtt{HSb}^{\mathtt{HI}} : \mathtt{Sig}^{\mathtt{HI}} \to \mathbb{CAT}^{op}$ *is a sub-functor of* $\mathtt{Sb}^{\mathtt{HI}}$.

In applications we work with a substitution sub-functor $\mathtt{HSt}^{\mathtt{HI}} : \mathtt{D}^{\mathtt{HI}} \to \mathbb{CAT}^{op}$ of $\mathtt{HSb}^{\mathtt{HI}}$, where $\mathtt{D}^{\mathtt{HI}} \subseteq \mathtt{Sig}^{\mathtt{HI}}$ is a broad subcategory of signature morphisms. When there is no danger of confusion we may drop the superscript $\mathtt{HI}$ from notations.

Example 41 (**HPL** substitution functor)**.** *Let $\mathtt{D}^{\mathbf{HPL}}$ be the broad subcategory of signature extensions with nominals. Let $\mathtt{HSt}^{\mathbf{HPL}} : \mathtt{D}^{\mathbf{HPL}} \to \mathbb{CAT}^{op}$ denote the sub-functor of $\mathtt{HSb}^{\mathbf{HPL}} : \mathtt{Sig}^{\mathbf{HPL}} \to \mathbb{CAT}^{op}$ which maps each signature $\Delta = (\mathtt{Nom}, \Lambda, \mathtt{Prop})$ to the subcategory of hybrid Δ-substitutions represented by functions $\theta : C_1 \to \mathtt{Nom} \cup C_2$, where C_i is a set of nominals different from the elements of $\mathtt{Nom}$. Notice that the substitutions given by $\mathtt{HSt}^{\mathbf{HPL}}$ determine signature morphisms. It follows that the local satisfaction condition is satisfied.*

Example 42 (**HFOLSA** substitution functor)**.** *Given a* **HFOLSA** *signature* $\Delta = (\mathtt{Nom}, \Lambda, (S^r, F^r, P^r) \subseteq (S, F, P))$*, only substitutions represented by pairs of functions* $(\theta_a : C_1 \to Nom \cup C_2, \theta_b : D_1 \to T_k^{\Delta[C_2,D_2]})$ *as described in Example 37 are relevant for the present study. Assume that* $\mathtt{D}^{\mathbf{HFOLSA}} \subseteq \mathtt{Sig}^{\mathbf{HFOLSA}}$ *is the broad subcategory of signature extensions with nominals and rigid constants. Let* $\mathtt{HSt}^{\mathbf{HFOLSA}} : \mathtt{D}^{\mathbf{HFOLSA}} \to \mathbb{CAT}^{op}$ *denote the substitution functor which maps each signature* Δ *to the subcategory of hybrid* Δ*-substitutions represented by pairs of functions* $(\theta_a : C_1 \to Nom \cup C_2, \theta_b : D_1 \to T_k^{\Delta[C_2,D_2]})$ *as described in Example 37.*

Proposition 43 (Hybridised substitutions)**.** *Given an institution* $\mathtt{I}$*, any substitution functor* $\mathtt{St}^{\mathtt{I}} : \mathtt{D}^{\mathtt{I}} \to \mathbb{CAT}^{op}$ *for* $\mathtt{I}$ *generates a hybrid substitution functor* $\mathtt{HSt}^{\mathcal{H}(\mathtt{I})} : \mathtt{D}^{\mathtt{I}} \to \mathbb{CAT}^{op}$ *for* $\mathcal{H}(\mathtt{I})$*:*

Proof. For any substitution $\theta : (\Sigma \xrightarrow{\chi_1} \Sigma_1) \to (\Sigma \xrightarrow{\chi_2} \Sigma_2) \in \mathtt{St}^{\mathtt{I}}(\Sigma)$ the following are equivalent $M \models^{\mathcal{H}(I)} \theta(\rho)$ iff $M \models^{\star} \theta(\rho)$ iff $M \models^{\mathtt{I}} \theta(\rho)$ iff $M \upharpoonright_{\theta} \models^{\mathtt{I}} \rho$ iff $M \upharpoonright_{\theta} \models^{\star} \rho$ iff $M \upharpoonright_{\theta} \models^{\mathcal{H}(\mathtt{I})} \rho$, where M is a Σ_2-model and ρ is a Σ_1-sentence. Hence, any substitution for $\mathtt{I}$ is also a substitution for $\mathcal{H}(I)$. $\square$

5. Institution-independent concepts

In this section, we investigate some of the institution-independent notions which are necessary to prove our abstract results.

5.1. Reachable models

We give an institution-independent characterisation of the models which consist of interpretations of syntactic elements.

Definition 44. *[30] Let* $\mathtt{I}$ *be an institution and* $\mathtt{St}^{\mathtt{I}} : \mathtt{D}^{\mathtt{I}} \to \mathbb{CAT}^{op}$ *a substitution functor for* $\mathtt{I}$*. Given a signature* $\Sigma \in |\mathtt{Sig}|$*, a* Σ*-model* M *is* $\mathtt{St}^{\mathtt{I}}$-reachable *if for each signature morphism* $\Sigma \xrightarrow{\chi} \Sigma' \in \mathtt{D}^{\mathtt{I}}$ *and any* χ*-expansion* M' *of* M *there exists a substitution* $\theta : \chi \to 1_{\Sigma} \in \mathtt{St}^{\mathtt{I}}(\Sigma)$ *such that* $M \upharpoonright_{\theta} = M'$*.*

If the substitution functor $\mathtt{St}^{\mathtt{I}}$ is fixed, $\mathtt{St}^{\mathtt{I}}$-reachable models may be called simply reachable.

Fact 45. *Given a substitution functor* $\mathtt{St}^{\mathtt{I}} : \mathtt{D}^{\mathtt{I}} \to \mathbb{CAT}^{op}$ *for an institution* $\mathtt{I}$*, a model is* $\mathtt{St}^{\mathtt{I}}$*-reachable iff it is* $\mathtt{HSt}^{\mathcal{H}(\mathtt{I})}$*-reachable.*

In what follows, we study the notion of reachability in concrete logical systems.

Proposition 46. *[30, 28] In* **FOL***, a model is* $\mathtt{St}^{\mathbf{FOL}}$*-reachable iff its elements consist of denotations of terms.*

Proposition 47. *[30] In* **PA***, a model is* $\mathtt{St}^{\mathbf{PA}}$*-reachable iff its elements consist of denotations of terms.*

In **HPL**, the reachability notion is connected to the states of models.

Proposition 48. *In* **HPL***, a model is* $\mathtt{HSt}^{\mathbf{HPL}}$*-reachable iff its states consist of denotations of nominals.*

Proof. For the direct implication, assume a reachable Δ-model (W, M), where $\Delta = (\mathtt{Nom}, \Lambda, \mathtt{Prop})$ is a **HPL** signature. Let $w \in |W|$ be an arbitrary state. We prove that there exists a nominal $k \in \mathtt{Nom}$ such that $W_k = w$. Let $\chi : \Delta \to \Delta[j]$ be a signature extension with the nominal variable j. Since (W, M) is reachable, there exists a substitution $\theta : \{j\} \to \mathtt{Nom}$ such that $(W, M)\upharpoonright_\theta = (W^{(j,w)}, M)$. It follows that $W_{\theta(j)} = W_j^{(j,w)} = w$. We define $k = \theta(j)$.

For the converse implication, assume a Δ-model (W, M) such that its states consists of interpretations of nominals, where $\Delta = (\mathtt{Nom}, \Lambda, \mathtt{Prop})$ is a **HPL** signature. Let $\chi : \Delta \hookrightarrow \Delta[C]$ be a signature extension with nominals from C, and (W', M) a χ-expansion of (W, M). We define the functions (a) $f : \mathtt{Nom} \to |W|$ by $f(k) = W_k$ for all $k \in \mathtt{Nom}$, and (b) $g : C \to |W'|$ by $g(c) = W'_c$ for all $c \in C$. Note that f is surjective and $|W| = |W'|$. It follows that there exists $\theta : C \to \mathtt{Nom}$ such that $\theta; f = g$.

$$\begin{tikzcd} \mathtt{Nom} \arrow[rr, "f"] & & {|W| = |W'|} \\ & C \arrow[ul, dotted, "\theta"] \arrow[ur, "g"'] & \end{tikzcd}$$

Notice that $(W\upharpoonright_\theta)_c = W_{\theta(c)} = f(\theta(c)) = g(c) = W'_c$ for all $c \in C$. Hence, $(W, M)\upharpoonright_\theta = (W', M)$. □

In **HFOLSA**, the elements of reachable models denote nominals and hybrid terms.

Proposition 49. *In* **HFOLSA***, a model is* $\mathtt{HSt}^{\mathbf{HFOLSA}}$*-reachable iff (a) its set of states consists of denotations of nominals and (b) its carrier sets for the rigid sorts consist of denotations of hybrid terms.*

Proof. For the direct implication, let (W, M) be a reachable Δ-model.

(a) Let $w \in |W|$: We prove there exists $k \in \mathtt{Nom}$ such that $W_k = w$. Consider a signature extension $\iota : \Delta \hookrightarrow \Delta[j]$ with the nominal variable j. Since (W, M) is reachable there exists a substitution $\theta : \{j\} \to \mathtt{Nom}$ such that $(W, M)\upharpoonright_\theta = (W^{(j,w)}, M)$. It follows that $(W, M)_{\theta(j)} = ((W, M)\upharpoonright_\theta)_j = (W^{(j,w)}, M)_j = W_j^{(j,w)} = w$. Hence, $k = \theta(j)$.

(b) Let $k \in \mathtt{Nom}$, $s \in S^r$ and $m \in (M_{W_k})_s$: We prove that there exists $t \in (T_k^\Delta)_s$ such that $m = (W, M)_t$. Consider the signature extension $\chi : \Delta \hookrightarrow \Delta[x]$ with the rigid constant x and let (W, M') be the χ-expansion of (W, M) such that $(W, M')_x = m$. Since (W, M) is reachable, there exists a substitution $\theta : \{x\} \to T_k^\Delta$ such that $(W, M)\upharpoonright_\theta = (W, M')$. It follows that $(W, M)_{\theta(x)} = ((W, M)\upharpoonright_\theta)_x = (W, M')_x = m$. Hence, $t = \theta(x)$.

For the converse implication, consider a signature extension $\chi : \Delta \hookrightarrow \Delta[C, D]$ with nominals from C and rigid constants from D, a Δ-model (W, M) such that the conditions (a) and (b) are satisfied, and a χ-expansion (W', M') of (W, M). We prove that there exists a substitution $\chi \xrightarrow{\theta} 1_\Delta \in \mathtt{HSt}^{\mathbf{HFOLSA}}(\Delta)$ such that $(W, M)\upharpoonright_\theta = (W', M')$. Let $k \in \mathtt{Nom}$ be an arbitrary fixed nominal. We define the following functions:

(1) $f_a : C \to |W|$ by $f_a(c) = W'_c$ for all $c \in C$ and

(2) $f_b : D \to M_{W_k}$ by $f_b(d) = (W', M')_d$ for all $d \in D$.

Let $h : (W^\Delta, M^\Delta) \to (W, M)$ be the homomorphism given by the initiality of (W^Δ, M^Δ). Since the set of states $|W|$ consists of interpretations of nominals, $h^{rel} : \mathtt{Nom} \to |W|$ is surjective. It follows that there exists $\theta_a : C \to \mathtt{Nom}$ such that $\theta_a; h^{rel} = f_a$.

$$\begin{tikzcd} \mathtt{Nom} \arrow[rr, "h^{rel}"] & & {|W|} \\ & C \arrow[ul, dotted, "\theta_a"] \arrow[ur, "f_a"'] & \end{tikzcd}$$

Since the carrier sets of (W, M) for the rigid sorts consist of interpretations of terms, $h_k^{mod} : T_k^\Delta \to M_{W_k}$ is surjective on S^r. It follows that there exists $\theta_b : D \to T_k^\Delta$ such that $\theta_b; h_k^{mod} = f_b$.

$$\begin{tikzcd} T_k^\Delta \arrow[rr, "h_k^{mod}"] & & M_{W_k} \\ & D \arrow[ul, dotted, "\theta_b"] \arrow[ur, "f_b"'] & \end{tikzcd}$$

Let $\theta = (\theta_a, \theta_b) : \chi \to 1_\Delta$, where θ_a and θ_b are defined as above. It follows that

(1) $((W, M)\upharpoonright_\theta)_c = (W, M)_{\theta_a(c)} = h^{rel}(\theta_a(c)) = f_a(c) = W'_c$ for all $c \in C$, and

(2) $((W, M)\upharpoonright_\theta)_d(W, M)_{\theta_b(d)} = h_k^{mod}(\theta_b(d)) = f_b(d) = (W', M')_d$ for all $d \in D$.

Hence, $(W, M)\upharpoonright_\theta = (W', M')$. □

The expansion of models that consist of interpretations of syntactic elements along signature morphisms used for quantification does not generate substitutions in **HFOLS**. The abstract theorems of the following sections are not applicable to hybrid institutions with user-defined sharing in their standard versions.

We make another mild assumption about the hybrid substitution functors, which is satisfied in all examples of hybrid institutions given in this paper and will allows us to prove that the set of states of reachable models consists of interpretations of nominals.

Assumption 50. *Let* $\mathtt{HI}$ *be a hybrid institution. We assume that* $\mathtt{HSt}^{\mathtt{HI}}$ *range over substitution functors satisfying the following property: given a signature* Δ *and a nominal variable* j*, we have*

- $\Delta \xrightarrow{\chi_j} \Delta[j] \in \mathtt{D}^{\mathtt{HI}}$*, and*
- *each substitution* $\chi_j \xrightarrow{\theta} 1_\Delta \in \mathtt{HSt}(\Delta)$ *is represented by a signature morphism* $\Delta[j] \xrightarrow{\theta} \Delta \in \mathtt{Sig}^{\mathtt{HI}}$ *such that* $\chi_j;\theta = 1_\Delta$.

In concrete examples of hybrid institutions, θ from Assumption 50 range over substitutions $\theta : \{j\} \to \mathtt{Nom}$ that map j to a nominal.

Proposition 51. *Let* $\mathtt{HI}$ *be a hybrid institution with a hybrid substitution functor* $\mathtt{HSt}^{\mathtt{HI}} : \mathtt{D}^{\mathtt{HI}} \to \mathbb{CAT}^{op}$*. The set of states of reachable models consists of denotations of nominals.*

Proof. Let Δ be a signature. Assume a reachable Δ-model M and a state w of M. We show that there exists $k \in \mathtt{Nom}^{\Delta}$ such that $W_k = w$, where $W = \mathtt{K}_\Delta(M)$. Let j be a nominal variable. Since M is reachable, there exists a substitution $\chi_j \xrightarrow{\theta} 1_\Delta \in \mathtt{HSt}(\Delta)$ such that $M\upharpoonright_\theta = M^{(j,w)}$. By Assumption 50, θ is represented by a signature morphism $\theta : \Delta[j] \to \Delta$ such that $\chi_j;\theta = 1_\Delta$. We have $W_{\theta^{\mathbf{REL}}(j)} = (W\upharpoonright_{\theta^{\mathbf{REL}}})_j = (W^{(j,w)})_j = w$. We define $k = \theta^{\mathbf{REL}}(j)$. □

5.2. Basic sets of sentences

In concrete examples of institutions, basic sentences are the simplest sentences, which are intimately linked to the internal structure of models, in the sense that their satisfaction is preserved by homomorphisms. Basic sentences [12] tend to be the starting building blocks from which the complex sentences are constructed by using Boolean operators, quantification, or other sentence operators.

Definition 52. *Given an institution* $\mathtt{I}$*, a set of sentences* $B \subseteq \mathtt{Sen}^{\mathtt{I}}(\Sigma)$ *is* basic *if there exists a* Σ*-model* M^B *such that*

$$M \models B \text{ iff there exists a homomorphism } M^B \to M$$

for all Σ*-models* M*. We say that* M^B *is a* basic model *of* B*. If in addition the homomorphism* $M^B \to M$ *is unique then the set* B *is called* epi-basic*.*

Note that any set of epi-basic set of sentences has an initial model which is the basic model. We show that the sets of atomic sentences of the institutions presented above are epi-basic.

Lemma 53. *Any set of sentences in* $\mathbf{FOL}_b$ *is epi-basic and the basic model is* $\mathtt{St}^{\mathbf{FOL}}$*-reachable, where* $\mathbf{FOL}_b$ *is the restriction of* $\mathbf{FOL}$ *to atomic sentences.*

Proof. Let B be a set of atomic (S, F, P)-sentences in **FOL**. The basic model M^B is constructed as follows: on the quotient algebra $T_{(S,F)}/_{\equiv_B}$ obtained by partitioning the elements of $T_{(S,F)}$ into equivalence classes given by the congruence generated by the equational atoms of B, we interpret each relation symbol $\pi \in P$ by $M^B_\pi = \{(t_1/_{\equiv_B}, \ldots, t_n/_{\equiv_B}) \mid \pi(t_1, \ldots, t_n) \in B\}$. Since M^B consists of interpretations of terms, by Proposition 46, M^B is reachable. $\square$

The proof of Lemma 53 is well known, and it can be found, for example, in [12] or [15], but since we want to make use of the construction of the basic model, we include it for the convenience of the reader. By defining an appropriate notion of congruence for **POA** models compatible with the preorder (see [18] or [10]) one may obtain the same result for **POA**.

Lemma 54. *[30] Let $\mathbf{PA}_b$ be the restriction of* **PA** *to existence equations. Any set of sentences in $\mathbf{PA}_b$ is epi-basic and the basic model is $\mathtt{St}^{\mathbf{PA}}$-reachable.*

Proof. For a set of existence equations E we define S^E as the set of sub-terms appearing in E. Note that S^E is a partial algebra. The basic model M^E will be the quotient of S^E by the partial congruence induced by E. $\square$

Theorem 55. *Any set of sentences in $\mathbf{HPL}_b$ is epi-basic and the basic models are $\mathtt{HSt}^{\mathbf{HPL}}$-reachable, where $\mathbf{HPL}_b$ is the restriction of* **HPL** *to (a) nominal equations, (b) nominal relations and (c) sentences of the form $@_k p$ such that k is a nominal and p is a propositional symbol.*

Proof. Let $B \subseteq \mathtt{Sen}^{\mathbf{HPL}_b}(\Delta)$, where $\Delta = (\mathtt{Nom}, \Lambda, \mathtt{Prop}) \in |\mathtt{Sig}^{\mathbf{HPL}}|$. We define the following equivalence relation on nominals $\equiv^B = \{(k_1, k_2) \mid B \models^{\mathbf{HPL}} k_1 = k_2\}$. We define the basic model (W^B, M^B) as follows:

(1) Let W^B be the **REL** model such that
   (a) $|W^B| = \widehat{\mathtt{Nom}}$, where $\widehat{\mathtt{Nom}}$ is the factorisation of $\mathtt{Nom}$ to $\equiv^B$, and
   (b) $W^B_\lambda = \{\widehat{k} \mid B \models^{\mathbf{HPL}} \lambda(k)\}$ for all $\lambda \in \Lambda$.

(2) Given a nominal $k \in \mathtt{Nom}$, let $M^B_{\widehat{k}}$ be the **PL** model which consists of $\{p \in \mathtt{Prop} \mid B \models^{\mathbf{HPL}} @_k p\}$. Let $M^B : \widehat{\mathtt{Nom}} \to |\mathtt{Mod}^{\mathbf{PL}}(\mathtt{Prop})|$ be the mapping defined by $M^B(\widehat{k}) = M^B_{\widehat{k}}$ for all $k \in \mathtt{Nom}$.

We prove that

$$(W, M) \models B \text{ iff there exists a unique arrow } (W^B, M^B) \to (W, M)$$

for all Δ-models (W, M).

"$\Rightarrow$" Let $f^{\mathbf{REL}} : W^\Delta \to W$ and $g^{\mathbf{REL}} : W^\Delta \to W^B$ be the unique homomorphisms such that W^Δ is the initial model of $\mathtt{Mod}^{\mathbf{REL}}(\mathtt{Nom}, \Lambda)$. Since

$(W, M) \models^{\mathbf{HPL}} B$, we have $\equiv^B \subseteq Ker(f^{\mathbf{REL}})$. It follows that there exists a unique arrow $h^{\mathbf{REL}} : W^B \to W$ such that $g^{\mathbf{REL}}; h^{\mathbf{REL}} = f^{\mathbf{REL}}$.

$$\begin{tikzcd}
W^B \arrow[rr, dotted, "h^{\mathbf{REL}}"] & & W \\
& W^{\Delta} \arrow[ul, "g^{\mathbf{REL}}"] \arrow[ur, "f^{\mathbf{REL}}"'] &
\end{tikzcd}$$

Let $k \in \texttt{Nom}$ be an arbitrary nominal. Since $(W, M) \models^{\mathbf{HPL}} B$, for all $p \in \texttt{Prop}$ we have $B \models^{\mathbf{HPL}} @_k p$ implies $M_{W_k} \models^{\mathbf{PL}} p$. It follows that there exists a unique **PL** homomorphism $h^{mod}_{\widehat{k}} : M^B_{\widehat{k}} \to M_{W_k}$. Hence $h = (h^{\mathbf{REL}}, h^{mod}) : (W^B, M^B) \to (W, M)$ is unique.

"$\Leftarrow$" Straightforward.

Since W^B is constructed from factorisation of W^{Δ}, by Proposition 48, (W^B, M^B) is reachable. $\square$

Theorem 56. *Any set of sentences in* $\mathbf{HFOLSA}_b$ *is epi-basic and the basic models are* $\texttt{St}^{\mathbf{HFOLSA}}$*-reachable, where* $\mathbf{HFOLSA}_b$ *is the restriction of* **HFOLSA** *to (a) atomic sentences, (b) nominal equations and (c) nominal relations.*

Proof. Firstly we prove that any set of atomic sentences and nominal relations is epi-basic. Let $\Gamma \subseteq \texttt{Sen}^{\mathbf{HFOLSA}_b}(\Delta)$ be such a set, where $\Delta = (\texttt{Nom}, \Lambda, \Sigma)$ is a **HFOLSA** signature and $\Sigma = (S^r, F^r, P^r) \subseteq (S, F, P)$ is a **FOLR** signature. We define the congruences $\equiv^k = \{(t_1, t_2) \mid t_1, t_2 \in T^{\Delta}_k,\ \Gamma \models^{\mathbf{HFOLSA}} t_1 = t_2\}$ on the Σ-models M^{Δ}_k for all $k \in \texttt{Nom}$ (see Definition 27 for the construction of M^{Δ}_k). We show that $\equiv^k$ is indeed a congruence. There are two cases to consider:

(1) Let $\varsigma \in F^r$ and $\tau_1, \tau_2 \in (T^{\Delta}_k)_{\texttt{ar}}$ such that $\tau_1 \equiv^k \tau_2$, where $\texttt{ar}$ is the arity of ς. By the definition of $\equiv^k$, we have $\Gamma \models^{\mathbf{HFOLSA}} \tau_1 = \tau_2$, which implies $\Gamma \models^{\mathbf{HFOLSA}} \varsigma(\tau_1) = \varsigma(\tau_2)$. By the definition of $\equiv^k$, $\varsigma(\tau_1) \equiv^k \varsigma(\tau_2)$.

(2) Let $\sigma \in (F - F^r)$ and $t_1, t_2 \in (T^{\Delta}_k)_a$ such that $t_1 \equiv^k t_2$, where a is the arity of σ. By the definition of $\equiv^k$, $\Gamma \models^{\mathbf{HFOLSA}} t_1 = t_2$, which implies $\Gamma \models^{\mathbf{HFOLSA}} \sigma_k(t_1) = \sigma_k(t_2)$. By the definition of $\equiv^k$, $\sigma_k(t_1) \equiv^k \sigma_k(t_2)$.

We denote by M^{Γ}_k the Σ-model obtained by

(a) factorising M^{Δ}_k by the congruence $\equiv^k$, and

(b) interpreting each relation $\pi \in P$ by $(M^{\Gamma}_k)_{\pi} = \{t/_{\equiv} \mid \pi(t) \in \Gamma \text{ or } \pi_k(t) \in \Gamma\}$.

Let W^{Γ} be the $(\texttt{Nom}, \Lambda)$-model such that $|W^{\Gamma}| = \texttt{Nom}$ and $W^{\Gamma}_{\lambda} = \{k \mid \lambda(k) \in \Gamma\}$ for all $\lambda \in \Lambda$. Let $M^{\Gamma} : \texttt{Nom} \to |\texttt{Mod}(\Sigma)|$ be the mapping defined by $M^{\Gamma}(k) = M^{\Gamma}_k$ for all $k \in \texttt{Nom}$. The definition of (W^{Γ}, M^{Γ}) is correct as for any rigid sort $\texttt{sr} \in S^r$ and nominals $k_1, k_2 \in \texttt{Nom}$ we have $\equiv^{k_1}_{\texttt{sr}} = \equiv^{k_2}_{\texttt{sr}}$. We prove that

$$(W, M) \models \Gamma \text{ iff there exists a unique arrow } (W^{\Gamma}, M^{\Gamma}) \to (W, M)$$

for all Δ-models (W, M).

“$\Rightarrow$” Since for all $\lambda(k) \in \Gamma$ we have $W_k \in W_\lambda$, there exists a unique $(\texttt{Nom}, \Lambda)$-homomorphism $h^{rel} : W^\Gamma \to W$ defined by $h^{rel}(k) = W_k$ for all $k \in \texttt{Nom}$.

Let $f : (W^\Delta, M^\Delta) \to (W, M)$ and $g : (W^\Delta, M^\Delta) \to (W^\Gamma, M^\Gamma)$ be the unique homomorphisms given by the initiality of (W^Δ, M^Δ). Since $(W, M) \models \Gamma$, if $t_1 \equiv^k t_2$ then $f_k^{mod}(t_1) = f_k^{mod}(t_2)$ for all $t_1, t_2 \in T_k^\Delta$. We obtain $Ker(f_k^{mod}) \subseteq \equiv_k^\Gamma$. It follos that there exists a unique homomorphism $h_k^{mod} : M_k^\Gamma \to M_{W_k}$ such that the following diagram is commutative.

$$\begin{tikzcd}
M_k^\Gamma \arrow[rr, dotted, "h_k^{mod}"] & & M_{W_k} \\
& M_k^\Delta \arrow[ul, "g_k^{mod}"] \arrow[ur, "f_k^{mod}"'] &
\end{tikzcd}$$

Thus, we have defined a unique homomorphism $h : (W^\Gamma, M^\Gamma) \to (W, M)$ such that $g; h = f$.

”$\Leftarrow$” Straightforward.

Secondly, we prove that any set of atomic sentences, nominal equations and nominal relations $B \subseteq \texttt{Sen}^{\mathbf{HFOLSA}_b}(\Delta)$ is epi-basic. We define

(a) the equivalence relation $\equiv = \{(k_1, k_2) \mid \Gamma \models^{\mathbf{HFOLSA}} k_1 = k_2\}$ on $\texttt{Nom}$, and

(b) the signature morphism $\varphi : \Delta \to \widehat{\Delta}$, where $\widehat{\Delta} = (\widehat{\texttt{Nom}}, \Lambda, \Sigma)$, which maps each nominal k to its equivalence class $\widehat{k}$ w.r.t. $\equiv$ and it is the identity on the rest of the symbols.

We show that

$$(W, M) \models B \text{ iff there exists a unique arrow } (W^{\varphi(B)}, M^{\varphi(B)})\upharpoonright_\varphi \to (W, M)$$

where $(W^{\varphi(B)}, M^{\varphi(B)})$ is the basic model of $\varphi(B)$ defined above.

“$\Rightarrow$” If $(W, M) \models B$ then there exists a unique φ-expansion (W', M) of (W, M) such that $|W'| = |W|$, W' interprets each $\widehat{k}$ as W_k, and $W'_\lambda = W_\lambda$ for all $\lambda \in \Lambda$. By the satisfaction condition, $(W', M) \models \varphi(B)$. Since $\varphi(B)$ is epi-basic, there exists a unique arrow $h' : (W^{\varphi(B)}, M^{\varphi(B)}) \to (W', M)$. By the uniqueness of the φ-expansion (W', M) and the homomorphism h', it follows that $h'\upharpoonright_\varphi : (W^{\varphi(B)}, M^{\varphi(B)})\upharpoonright_\varphi \to (W, M)$ is also unique.

“$\Leftarrow$” Assume a homomorphism $h : (W^{\varphi(B)}, M^{\varphi(B)}) \upharpoonright_\varphi \to (W, M)$. Since $W^{\varphi(B)} = \widehat{\texttt{Nom}}$, the model (W, M) satisfies all nominal equations in B. There exists a unique φ-expansion $h' : (W^{\varphi(B)}, M^{\varphi(B)}) \to (W', M)$ of h. Since $\varphi(B)$ is epi-basic, $(W', M) \models^{\mathbf{HFOLSA}} \varphi(B)$, and by the satisfaction condition, $(W, M) \models^{\mathbf{HFOLSA}} B$.

Finally, notice that $(W^{\varphi(B)}, M^{\varphi(B)})$ is constructed by factorisations of some sets of signature symbols. By Proposition 49, $(\mathtt{M}^{\varphi(B)}, R^{\varphi(B)})$ is reachable, and since φ is surjective, the reduct of $(W^{\varphi(B)}, M^{\varphi(B)})$ along φ is also reachable. $\square$

The following result shows that the semantic consequences of a basic set of sentences can be reduced to the satisfaction by a base model.

Lemma 57. *Let* $\mathtt{I}$ *be an institution. Consider a signature* $\Sigma \in |\mathtt{Sig}^{\mathtt{I}}|$*, a basic set* B *of* Σ*-sentences and a basic model* M^B *for* B*. Then for all basic sentences* $\gamma \in \mathtt{Sen}(\Sigma)$ *we have* $B \models_\Sigma \gamma$ *iff* $M^B \models_\Sigma \gamma$*.*

Proof. The direct implication is trivial. For the converse implication assume that $M^B \models \gamma$ and let $M \in |\mathtt{Mod}(\Sigma)|$ such that $M \models_\Sigma B$. We show that $M \models_\Sigma \gamma$. Since $M^B \models \gamma$, there exists an arrow $M^\gamma \to M^B$, where M^γ is a basic model for γ. Since $M \models_\Sigma B$, there exists an arrow $M^B \to M$. It follows that there exists an arrow $M^\gamma \to M$. Hence, $M \models_\Sigma \gamma$. $\square$

6. Initiality

We have proved in concrete logical settings that certain sets of sentences are epi-basic. It follows that those sets of sentences have initial models. In this section, we prove in the general framework of hybrid institutions that the initiality property is closed to implication, store, quantification and necessity.

The framework in which the results of this section will be proved is given by a hybrid institution $\mathtt{HI}$ (such that Assumption 18 holds) equipped with a substitution functor $\mathtt{HSt}^{\mathtt{HI}} : \mathtt{D}^{\mathtt{HI}} \to \mathbb{CAT}^{op}$ (such that Assumption 50 holds). An example of such institution is **HPL**, where the substitution functor $\mathtt{HSt}^{\mathtt{HI}}$ is $\mathtt{HSt}^{\mathbf{HPL}}$ defined in Example 41. Another example of institution $\mathtt{HI}$ is **HFOLSA**, where the substitution functor $\mathtt{HSt}^{\mathtt{HI}}$ is $\mathtt{HSt}^{\mathbf{HFOLSA}}$ defined in Example 42.

6.1. Implication

Consider two sub-functors $\mathtt{Sen}^{\mathtt{HI}_i} \subseteq \mathtt{Sen}^{\mathtt{HI}}$ and $\mathtt{Sen}^{\mathtt{HI}_b} \subseteq \mathtt{Sen}^{\mathtt{HI}_i}$ such that

(1) all sentences of $\mathtt{HI}_i = (\mathtt{Sig}^{\mathtt{HI}}, \mathtt{F}^{\mathtt{HI}}, \mathtt{Sen}^{\mathtt{HI}_i}, \mathtt{Mod}^{\mathtt{HI}}, \mathtt{K}^{\mathtt{HI}}, \models^{\mathtt{HI}})$ are constructed from the sentences of $\mathtt{HI}_b = (\mathtt{Sig}^{\mathtt{HI}}, \mathtt{F}^{\mathtt{HI}}, \mathtt{Sen}^{\mathtt{HI}_b}, \mathtt{Mod}^{\mathtt{HI}}, \mathtt{K}^{\mathtt{HI}}, \models^{\mathtt{HI}})$ by applying logical implication, i.e. all sentences of $\mathtt{HI}_i$ (a) belong to $\mathtt{HI}_b$ or (b) are of the form $\bigwedge H \Rightarrow C$, where $H \cup \{C\}$ belongs to $\mathtt{HI}_b$, and

(2) $\mathtt{HI}_b$ is semantically closed to retrieve, i.e. if ρ is a sentence of $\mathtt{HI}_b$ then for any nominal k there exists a sentence in $\mathtt{HI}_b$ that is semantically equivalent to $@_k\rho$.

Theorem 58. *Every set of sentences in* $\mathtt{HI}$ *has a initial reachable model if every set of sentences in* $\mathtt{HI}_b$ *is basic and the basic models are reachable.*

Proof. Let Γ be a set of Δ-sentences in $\mathtt{HI}_i$, where Δ is a signature. We define $\Gamma_b = \{\rho \in \mathtt{Sen}^{\mathtt{HI}_b} \mid \Gamma \models^{\mathtt{HI}} \rho\}$. Let M^{Γ_b} be the basic model of Γ_b. Notice that for all Δ-models M such that $M \models^{\mathtt{HI}} \Gamma$ there exists a unique arrow $M^{\Gamma_b} \to M$ since $M \models^{\mathtt{HI}} \Gamma_b$ and M^{Γ_b} is the initial model of Γ_b. If we prove that $M^{\Gamma_b} \models^{\mathtt{HI}} \Gamma$ then M^{Γ_b} is the initial model of Γ.

Let $\bigwedge H \Rightarrow C \in \Gamma$ and $w \in |\mathtt{K}_\Delta(M^{\Gamma_b})|$, and assume that $M^{\Gamma_b} \models^{w} H$. By Proposition 51, there exists $k \in \mathtt{Nom}$ such that $w = \mathtt{K}_\Delta(M^{\Gamma_b})_k$. We have $M^{\Gamma_b} \models^{\mathtt{HI}} \{@_k h \mid h \in H\}$. Since $\mathtt{HI}_b$ is semantically closed to @, the set $\{@_k h \mid h \in H\}$ is basic. By Lemma 57, $\Gamma_b \models^{\mathtt{HI}} \{@_k h \mid h \in H\}$, which implies $\Gamma \models^{\mathtt{HI}} \{@_k h \mid h \in H\}$. Notice that $\Gamma \models^{\mathtt{HI}} \bigwedge_{h\in H} @_k h \Rightarrow @_k C$, and we get $\Gamma \models^{\mathtt{HI}} @_k C$. Because $\mathtt{HI}_b$ is semantically closed to @, there exists a sentence ρ in $\mathtt{HI}_b$ that is semantically equivalent to $@_k C$. We have $\rho \in \Gamma_b$, which implies $\Gamma_b \models^{\mathtt{HI}} \rho$. By Lemma 57, $M^{\Gamma_b} \models^{\mathtt{HI}} \rho$, which is equivalent to $M^{\Gamma_b} \models^{\mathtt{HI}} @_k C$. Hence, $M^{\Gamma_b} \models^{w} C$. $\square$

The following two results are corollaries of Theorem 58.

Corollary 59. *Any set of sentences in* $\mathbf{HPL}_i$ *has a initial reachable model, where* $\mathbf{HPL}_i$ *is the restriction of* **HPL** *to sentences constructed from* $\mathbf{HPL}_b$ *formulae by applying logical implication.*

Proof. By Theorem 55, any set of sentences in $\mathbf{HPL}_b$ is epi-basic and the basic models are reachable. $\mathbf{HPL}_b$ is semantically closed to @ as for all nominals k and sentences ρ of $\mathbf{HPL}_b$, we have $\rho \models\!\mid @_k\rho$. By Theorem 58, any set of sentences in $\mathbf{HPL}_i$ has a initial model that is reachable. $\square$

Corollary 60. *Any set of sentences in* $\mathbf{HFOLSA}_i$ *has a initial reachable model, where* $\mathbf{HFOLSA}_i$ *is the restriction of* **HFOLSA** *to sentences constructed from* $\mathbf{HFOLSA}_b$ *formulae by applying logical implication.*

Proof. By Theorem 56, any set of sentences in $\mathbf{HFOLSA}_b$ is epi-basic and the basic models are reachable. $\mathbf{HFOLSA}_b$ is semantically closed to @ as for all nominals k and sentences ρ of $\mathbf{HFOLSA}_b$, we have $\rho \models\!\mid @_k\rho$. By Theorem 58, any set of sentences in $\mathbf{HFOLSA}_i$ has a initial model that is reachable. $\square$

6.2. Store

Assume two sub-functors $\mathtt{Sen}^{\mathtt{HI}_s} \subseteq \mathtt{Sen}^{\mathtt{HI}}$ and $\mathtt{Sen}^{\mathtt{HI}_*} \subseteq \mathtt{Sen}^{\mathtt{HI}}$ such that all sentences of $\mathtt{HI}_s = (\mathtt{Sig}^{\mathtt{HI}}, \mathtt{F}^{\mathtt{HI}}, \mathtt{Sen}^{\mathtt{HI}_s}, \mathtt{Mod}^{\mathtt{HI}}, \mathtt{K}^{\mathtt{HI}}, \models^{\mathtt{HI}})$ are constructed from the sentences of $\mathtt{HI}_* = (\mathtt{Sig}^{\mathtt{HI}}, \mathtt{F}^{\mathtt{HI}}, \mathtt{Sen}^{\mathtt{HI}_*}, \mathtt{Mod}^{\mathtt{HI}}, \mathtt{K}^{\mathtt{HI}}, \models^{\mathtt{HI}})$ by applying store.

In order to prove that initiality is closed to store we need the following assumption, which is easily satisfied in all examples of hybrid institutions that we are aware of.

Assumption 61. $\mathtt{HI}$ *range over hybrid institutions satisfying another mild property: given a signature* $\Delta \in |\mathtt{Sig}^{\mathtt{HI}}|$ *and a nominal variable* j*,*

> *the* **REL** *signature morphism* $\theta^{\mathbf{REL}}_{j\leftarrow k} : (\mathtt{Nom}^\Delta \cup \{j\}, \Lambda^\Delta) \to (\mathtt{Nom}^\Delta, \Lambda^\Delta)$
>
> *which preserves* $(\mathtt{Nom}^\Delta, \Lambda^\Delta)$ *and maps* j *to* k

defines a signature morphism $\theta_{j\leftarrow k} : \Delta[j] \to \Delta$ *in* $\mathtt{HI}$
such that $\mathtt{F}(\theta_{j\leftarrow k}) = \theta^{\mathbf{REL}}_{j\leftarrow k}$ *and* $\chi_j;\theta_{j\leftarrow k} = 1_\Delta$.

Lemma 62. *We have*

(1) $M\restriction_{\theta_{j\leftarrow k}} = M^{(j,w)}$, *where* $w = \mathtt{K}_\Delta(M)_k$,

(2) $@_k(\downarrow j)\rho \models\!\mid @_k\theta_{j\leftarrow k}(\rho)$, *and*

(3) $@_k(\bigwedge H \Rightarrow C) \models\!\mid \bigwedge_{h\in H} @_k h \Rightarrow @_k C$

where $\Delta \in |\mathtt{Sig}^{\mathtt{HI}}|$, $k \in \mathtt{Nom}^\Delta$, *$j$ is a nominal variable,* $\rho \in \mathtt{Sen}^{\mathtt{HI}}(\Delta[j])$, *and* $\bigwedge H \Rightarrow C \in \mathtt{Sen}^{\mathtt{HI}}(\Delta)$.

Proof.

(1) We have $\mathtt{K}_{\Delta[j]}(M\restriction_{\theta_{j\leftarrow k}}) = \mathtt{K}_\Delta(M)\restriction_{\theta^{\mathbf{REL}}_{j\leftarrow k}} = \mathtt{K}_\Delta(M)^{(j,w)}$.

Since $\chi_j;\theta_{j\leftarrow k} = 1_\Delta$, the model $(M\restriction_{\theta_{j\leftarrow k}})$ is a χ_j-expansion of M.

Recall that $M^{(j,w)}$ is the unique χ_j-expansion of M such that $\mathtt{K}_{\Delta[j]}(M^{(j,w)}) = \mathtt{K}_\Delta(M)^{(j,w)}$. It follows that $M\restriction_{\theta_{j\leftarrow k}} = M^{(j,w)}$.

(2) Assume that $M \models^{\mathtt{HI}} @_k(\downarrow j)\rho$, where M is a Δ-model. It follows that $M \models^w (\downarrow j)\rho$, where $w = \mathtt{K}_\Delta(M)_k$. We have $M^{(j,w)} \models^w \rho$. By the local satisfaction condition, $M \models^w \theta_{j\leftarrow k}(\rho)$. Hence, $M \models^{\mathtt{HI}} @_k\theta_{j\leftarrow k}(\rho)$.

Assume that $M \models^{\mathtt{HI}} @_k\theta_{j\leftarrow k}(\rho)$, where M is a Δ-model. It follows that $M \models^w \theta_{j\leftarrow k}(\rho)$, where $w = \mathtt{K}_\Delta(M)_k$. By the local satisfaction condition, $M\restriction_{\theta_{j\leftarrow k}} = M^{(j,w)} \models^w \rho$. We have $M \models^w (\downarrow j)\rho$. Hence, $M \models^{\mathtt{HI}} @_k(\downarrow j)\rho$.

(3) Straightforward.

□

The following result provide conditions for proving that initiality is closed to store.

Theorem 63. *Any set of sentences in* $\mathtt{HI}_s$ *has a initial reachable model if any set of sentences in* $\mathtt{HI}_*$ *has a initial reachable model and* $\mathtt{HI}_*$ *is semantically closed to retrieve.*

Proof. Let Γ be a set of Δ-sentences in $\mathtt{HI}_s$, where Δ is a signature. We define the set of sentences $\Gamma_* = \{\rho \in \mathtt{Sen}^{\mathtt{HI}_*}(\Delta) \mid \Gamma \models^{\mathtt{HI}} \rho\}$. Let M^{Γ_*} be the initial model of Γ_*. Notice that for all Δ-models M such that $M \models^{\mathtt{HI}} \Gamma$ there exists a unique arrow $M^{\Gamma_*} \to M$ as $M \models^{\mathtt{HI}} \Gamma_*$ and M^{Γ_*} is the initial model of Γ_*. If we prove that $M^{\Gamma_*} \models \Gamma$ then M^{Γ_*} is the initial model of Γ.

Let $(\downarrow j)\gamma \in \Gamma$ and $w \in |\mathtt{K}_\Delta(M^{\Gamma_*})|$. We show that $M^{\Gamma_*} \models^w (\downarrow j)\gamma$. Since M^* is reachable, by Proposition 51, there exists $k \in \mathtt{Nom}$ such that $K_\Delta(M^{\Gamma_*})_k = w$. We have $(\downarrow j)\gamma \models^{\mathtt{HI}} @_k(\downarrow j)\gamma$. By Lemma 62(2), $(\downarrow j)\gamma \models^{\mathtt{HI}} @_k\theta_{j\leftarrow k}(\gamma)$. Since $\mathtt{HI}_*$ is semantically closed to retrieve, there exists $\rho \in \mathtt{Sen}^{\mathtt{HI}_*}(\Delta)$ such that we have $\rho \models\!\mid @_k\theta_{j\leftarrow k}(\gamma)$. It follows that $\Gamma \models^{\mathtt{HI}} \rho$, which implies $\rho \in \Gamma_*$. We have $M^{\Gamma_*} \models^{\mathtt{HI}} \rho$, which implies $M^{\Gamma_*} \models^{\mathtt{HI}} @_k\theta_{j\leftarrow k}(\gamma)$. By the local satisfaction condition, $M^{\Gamma_*}\restriction_{\theta_{j\leftarrow k}} = (M^{\Gamma_*})^{(j,w)} \models^w \gamma$. Hence, $M^{\Gamma_*} \models^w (\downarrow j)\gamma$. □

The following results are corollaries of Theorem 63.

Corollary 64. *Any set of sentences in* $\mathbf{HPL}_s$ *has a initial reachable model, where* $\mathbf{HPL}_s$ *is the restriction of* $\mathbf{HPL}$ *to sentences constructed from* $\mathbf{HPL}_i$ *formulae by applying store.*

Proof. By Corollary 59, any set of sentences in $\mathbf{HPL}_i$ has a initial reachable model. Since $\mathbf{HPL}_b$ is semantically closed to retrieve, by Lemma 62(3), $\mathbf{HPL}_i$ is semantically closed to retrieve. By Theorem 63, any set of sentences in $\mathbf{HPL}_s$ has a initial reachable model.

□

Corollary 65. *Any set of sentences in* $\mathbf{HFOLSA}_s$ *has a initial reachable model, where* $\mathbf{HFOLSA}_s$ *is the restriction of* $\mathbf{HFOLSA}$ *to sentences constructed from* $\mathbf{HFOLSA}_i$ *formulae by applying store.*

Proof. By Corollary 60, any set of sentences in $\mathbf{HFOLSA}_i$ has a initial reachable model. Since $\mathbf{HFOLSA}_b$ is semantically closed to retrieve, by Lemma 62(3), $\mathbf{HFOLSA}_i$ is semantically closed to retrieve. By Theorem 63, any set of sentences in $\mathbf{HFOLSA}_s$ has a initial reachable model. □

6.3. Quantification

Consider two sub-functors $\mathtt{Sen}^{\mathtt{HI}_q} \subseteq \mathtt{Sen}^{\mathtt{HI}}$ and $\mathtt{Sen}^{\mathtt{HI}_*} \subseteq \mathtt{Sen}^{\mathtt{HI}_g}$ and a broad subcategory $\mathtt{Q}^{\mathtt{HI}} \subseteq \mathtt{D}^{\mathtt{HI}}$ of signature morphisms such that all sentences of $\mathtt{HI}_q = (\mathtt{Sig}^{\mathtt{HI}}, \mathtt{F}^{\mathtt{HI}}, \mathtt{Sen}^{\mathtt{HI}_q}, \mathtt{Mod}^{\mathtt{HI}}, \mathtt{K}^{\mathtt{HI}}, \models^{\mathtt{HI}})$ are constructed from the formulae of $\mathtt{HI}_* = (\mathtt{Sig}^{\mathtt{HI}}, \mathtt{F}^{\mathtt{HI}}, \mathtt{Sen}^{\mathtt{HI}_*}, \mathtt{Mod}^{\mathtt{HI}}, \mathtt{K}^{\mathtt{HI}}, \models^{\mathtt{HI}})$ by applying universal quantification over the signature morphisms in $\mathtt{Q}^{\mathtt{HI}}$.

Theorem 66. *All sets of sentences in* $\mathtt{HI}_q$ *have a initial reachable model if all sentences in* $\mathtt{HI}_*$ *have a initial reachable model.*

Proof. Let Γ be a set of Δ-sentences in $\mathtt{HI}_q$, where Δ is a signature. We define $\Gamma_* = \{\rho \in \mathtt{Sen}^{\mathtt{HI}_*} \mid \Gamma \models^{\mathtt{HI}} \rho\}$. Let M^{Γ_*} be the initial model of Γ_*. Notice that for all Δ-models M such that $M \models^{\mathtt{HI}} \Gamma$ there exists a unique arrow $M^{\Gamma_*} \to M$ as $M \models^{\mathtt{HI}} \Gamma_*$ and M^{Γ_*} is the initial model of Γ_*. In what follows we prove that $M^{\Gamma_*} \models^{\mathtt{HI}} \Gamma$.

Consider a sentence $(\forall)\rho \in \Gamma$, where $\Delta \xrightarrow{\chi} \Delta' \in \mathtt{Q}^{\mathtt{HI}}$ and $\rho \in \mathtt{Sen}^{\mathtt{HI}_*}(\Delta')$. Let w be a state of M^{Γ_*} and M' be a χ-expansion of M^{Γ_*}. We prove that $M' \models^w \rho$. Since M^{Γ_*} is reachable, there exists a hybrid substitution $\chi \xrightarrow{\theta} 1_\Delta \in \mathtt{HSt}^{\mathtt{HI}}(\Delta)$ such that $M^{\Gamma_*}\restriction_\theta = M'$. Note that $(\forall\chi)\rho \models^{\mathtt{HI}} \theta(\rho)$, which implies $\theta(\rho) \in \Gamma_*$. It follows that $M^{\Gamma_*} \models^w \theta(\rho)$. By the local satisfaction condition, $M' \models^w \rho$. □

The following results are corollaries of Theorem 66.

Corollary 67. *Any set of sentences in* $\mathbf{HPL}_q$ *has a initial reachable model, where* $\mathbf{HPL}_q$ *is the restriction of* $\mathbf{HPL}$ *to sentences constructed from* $\mathbf{HPL}_s$ *formulae by applying universal quantification over nominal variables.*

Proof. By Corollary 64, any set of sentences in $\mathbf{HPL}_s$ has a initial reachable model. By Theorem 66, any set of sentences in $\mathbf{HPL}_q$ has a initial reachable model. $\square$

Corollary 68. *Any set of sentences in* $\mathbf{HFOLSA}_q$ *has a initial reachable model, where* $\mathbf{HFOLSA}_q$ *is the restriction of* $\mathbf{HFOLSA}$ *to sentences constructed from* $\mathbf{HFOLSA}_s$ *formulae by applying universal quantification over nominal and rigid variables.*

Proof. By Corollary 65, any set of sentences in $\mathbf{HFOLSA}_s$ has a initial reachable model. By Theorem 66, any set of sentences in $\mathbf{HFOLSA}_q$ has a initial reachable model. $\square$

6.4. Necessity

Consider two sub-functors $\mathtt{Sen}^{\mathtt{HI}_n} \subseteq \mathtt{Sen}^{\mathtt{HI}}$ and $\mathtt{Sen}^{\mathtt{HI}_*} \subseteq \mathtt{Sen}^{\mathtt{HI}_n}$ such that all sentences of $\mathtt{HI}_n = (\mathtt{Sig}^{\mathtt{HI}}, \mathtt{F}^{\mathtt{HI}}, \mathtt{Sen}^{\mathtt{HI}_n}, \mathtt{Mod}^{\mathtt{HI}}, \mathtt{K}^{\mathtt{HI}}, \models^{\mathtt{HI}})$ are constructed from the sentences of $\mathtt{HI}_*$ by applying (a) necessity over binary modalities, and (b) retrieve.

The following result show that initiality is closed to necessity restricted to binary modalities.

Theorem 69. *Any set of sentences in* $\mathtt{HI}_n$ *has a initial reachable if any set of sentences in* $\mathtt{HI}_*$ *has a initial reachable and* $\mathtt{HI}_*$ *is semantically closed to retrieve.*

Proof. Let Γ be a set of Δ-sentences in $\mathtt{HI}_n$, where Δ is a signature. We define $\Gamma_* = \{\rho \in \mathtt{Sen}^{\mathtt{HI}_*}(\Delta) \mid \Gamma \models^{\mathtt{HI}} \rho\}$. Let M^{Γ_*} be a initial reachable model of Γ_*. Since for all models M that satisfies Γ there exists a unique arrow $M^{\Gamma_*} \to M$, we only need to prove that $M^{\Gamma_*} \models^{\mathtt{HI}} \Gamma$.

Consider a sentence $[\lambda]\gamma \in \Gamma$ and let $w, w' \in |\mathtt{K}_\Delta(M^{\Gamma_*})|$ such that $(w, w') \in K_\Delta(M^{\Gamma_*})_\lambda$. We show that $M^{\Gamma_*} \models^{w'} \gamma$. Since M^{Γ_*} is reachable, by Proposition 51, there exists $k, k' \in \mathtt{Nom}$ such that $\mathtt{K}_\Delta(M^{\Gamma_*})_k = w$ and $\mathtt{K}_\Delta(M^{\Gamma_*})_{k'} = w'$. Note that $\Gamma \models^{\mathtt{HI}} @_{k'}\gamma$:

> Indeed for any Δ-model M that satisfies Γ, we have $M \models^{\mathtt{HI}} \Gamma^*$.
>
> Since M^{Γ_*} is initial, there exists an arrow $\mathtt{K}_\Delta(M^{\Gamma_*}) \to \mathtt{K}_\Delta(M)$ in $\mathbf{REL}$.
>
> It follows that $(s, s') \in \mathtt{K}_\Delta(M)_\lambda$, where $s = \mathtt{K}_\Delta(M)_k$ and $s' = \mathtt{K}_\Delta(M)_{k'}$.
>
> Since $M \models^{\mathtt{HI}} [\lambda]\gamma$, we have $M \models^{s'} \gamma$. Therefore, $M \models^{\mathtt{HI}} @_{k'}\gamma$.

Since $\mathtt{HI}_*$ is semantically closed to retrieve, there exists $\rho \in \mathtt{Sen}^{\mathtt{HI}_*}(\Delta)$ such that $\rho \models\!\!| @_{k'}\gamma$. We have $\Gamma \models^{\mathtt{HI}} \rho$, where $\rho \in \mathtt{Sen}^{\mathtt{HI}_*}(\Delta)$, which implies $\rho \in \Gamma_*$. It follows that $M^{\Gamma_*} \models^{\mathtt{HI}} \rho$, which is equivalent to $M^{\Gamma_*} \models^{\mathtt{HI}} @_{k'}\gamma$. Hence, $M^{\Gamma_*} \models^{w'} \gamma$.

The case $@_k[\lambda]\gamma \in \Gamma$ is similar. $\square$

The following result is useful to show that the instances of $\mathtt{HI}_*$ in this subsection are semantically closed to retrieve.

Lemma 70. *We have* $@_k(\forall\chi)\rho \models\!\mid (\forall\chi)@_{\mathtt{F}(\chi)(k)}\rho$ *for all sentences of the form* $(\forall\chi)\rho$*, where* $\Delta \xrightarrow{\chi} \Delta' \in \mathtt{Sig}^{\mathtt{HI}}$ *and* $\rho \in \mathtt{Sen}^{\mathtt{HI}}(\Delta')$*.*

Proof. For any Δ'-model M' we have $K_{\Delta'}(M')_{F(\chi)(k)} = (K_{\Delta'}(M')\upharpoonright_{F(\chi)})_k = K_\Delta(M'\upharpoonright_\chi)_k$. The following are equivalent:
$M \models^w @_k(\forall\chi)\rho$ iff $M \models^{K_\Delta(M)_k} (\forall\chi)\rho$ iff
$M' \models^{K_\Delta(M)_k} \rho$ for all χ-expansions M' of M iff
$M' \models^{K_{\Delta'}(M')_{F(\chi)(k)}} \rho$ for all χ-expansions M' of M iff
$M' \models^w @_{F(\chi)(k)}\rho$ for all χ-expansions M' of M iff $M \models^w (\forall\chi)@_{F(\chi)(k)}\rho$.
Hence, $@_k(\forall\chi)\rho \models\!\mid (\forall\chi)@_{F(\chi)(k)}\rho$ □

The following results are corollaries of Theorem 69.

Corollary 71. *Any set of sentences in* $\mathbf{HPL}_n$ *has a initial reachable model, where* $\mathbf{HPL}_n$ *is the restriction of* **HPL** *to sentences constructed from* $\mathbf{HPL}_q$ *formulae by applying (a) necessity over binary modalities, and (b) retrieve.*

Proof. By Corollary 67, any set of sentences in $\mathbf{HPL}_q$ has a initial reachable model. By Lemma 62(2), $\mathbf{HPL}_i$ is semantically closed to retrieve. By Lemma 62(3), $\mathbf{HPL}_s$ is semantically closed to retrieve. By Lemma 70, $\mathbf{HPL}_q$ is semantically closed to retrieve. By Theorem 69, any set of sentences in $\mathbf{HPL}_n$ has a initial reachable model. □

Corollary 72. *Any set of sentences of* $\mathbf{HFOLSA}_n$ *has a initial reachable model, where* $\mathbf{HFOLSA}_n$ *is the restriction of* **HFOLSA** *to sentences constructed from* $\mathbf{HFOLSA}_q$ *formulae by applying (a) necessity over binary modalities, and (b) retrieve.*

Proof. By Corollary 68, any set of sentences in $\mathbf{HFOLSA}_q$ has a initial reachable model. By Lemma 62(2), $\mathbf{HFOLSA}_i$ is semantically closed to retrieve. By Lemma 62(3), $\mathbf{HFOLSA}_s$ is semantically closed to retrieve. By Lemma 70, $\mathbf{HFOLSA}_q$ is semantically closed to retrieve. By Theorem 69, any set of sentences in $\mathbf{HFOLSA}_n$ has a initial reachable model. □

We have shown that in hybrid institutions such as **HPL** and **HFOLSA** the sentences constructed from some basic sentences by applying logical implication, store, universal quantification, necessity over binary modalities and retrieve, have an initial model. The sentence operators store, quantification, necessity, and retrieve may be applied in a different order by a finite number of times and initiality is still preserved. Moreover, the results may be applied in different contexts such as preorder algebras.

The initiality property can be borrowed from hybrid institutions with annotated syntax to their classical versions.

Proposition 73. *Any set of sentences in* $\mathbf{HFOLS}_n$ *has a initial reachable model, where* *(a)* $\mathbf{HFOLS}_n$ *is the restriction of* $\mathbf{HFOLS}$ *to sentences constructed from the formulae of* $\mathbf{HFOLS}_b$ *by applying logical implication, store, quantification over nominal and rigid variables, necessity over binary modalities and retrieve, and*

(b) $\mathbf{HFOLS}_b$ *is the restriction of* $\mathbf{HFOLS}$ *to nominal equations, nominal relations and sentences of the form* $@_k\rho$*, where* k *is a nominal and* ρ *is an atomic sentence.*

Proof. Recall that **HFOLS** and **HFOLSA** share the same signatures and models. Given a signature Δ, for each atomic sentence $\rho \in \mathtt{Sen}^{\mathbf{HFOLS}_0}(\Delta)$ and any nominal k for Δ, the sentences $@_k\rho$ and $\alpha_\Delta(\rho)[j \leftarrow k]$ are satisfied by the same models:

> By Proposition 34, ρ and $\alpha_\Delta(\rho)$ are satisfied by the same models, which implies that $@_k\rho$ and $@_k\alpha_\Delta(\rho)$ are satisfied by the same models. By Lemma 62(2), $@_k\alpha_\Delta(\rho)$ is semantically equivalent to $@_k\alpha_\Delta(\rho)[j \leftarrow k]$. Since $\alpha_\Delta(\rho)[j \leftarrow k]$ is an atomic sentence, by Remark 33, $\alpha_\Delta(\rho)[j \leftarrow k]$ is semantically equivalent to $@_k\alpha_\Delta(\rho)[j \leftarrow k]$. Hence, $@_k\rho$ and $\alpha_\Delta(\rho)[j \leftarrow k]$ are satisfied by the same models.

For any sentence in $\mathbf{HFOLS}_b$ there is a semantically equivalent formula in $\mathbf{HFOLSA}_b$. It follows that for any sentence in $\mathbf{HFOLS}_n$ there is a semantically equivalent formula in $\mathbf{HFOLSA}_n$ By Corollary 72, any set of sentences in $\mathbf{HFOLSA}_n$ has a initial model, which implies that any set of sentences in $\mathbf{HFOLS}_n$ has a initial model. $\square$

7. Herbrand's theorem

We prove a version of Herbrand's theorem at the abstract level of hybrid institutions.

Theorem 74. *Let* $\mathtt{HI}$ *be a hybrid institutions equipped with a hybrid substitution functor* $\mathtt{HSt}^{\mathtt{HI}} : \mathtt{D}^{\mathtt{HI}} \to \mathbb{CAT}^{op}$ *and a sub-functor* $\mathtt{Sen}^{\mathtt{HI}_b} \subseteq \mathtt{Sen}^{\mathtt{HI}}$*. Consider*

- *a sentence* $(\exists\chi)\bigwedge E$ *such that* $\Delta \xrightarrow{\chi} \Delta' \in \mathtt{D}^{\mathtt{HI}}$ *and* $E \subseteq \mathtt{Sen}^{\mathtt{HI}_b}(\Delta')$*,*
- *a set* $\Gamma \subseteq \mathtt{Sen}^{\mathtt{HI}}(\Delta)$ *that has a initial* $\mathtt{HSt}^{\mathtt{HI}}$*-reachable model* M^Γ*, and*
- *a nominal* k *of* Δ*.*

If $\mathtt{HI}_b$ *is semantically closed to retrieve and all sets of sentences in* $\mathtt{HI}_b$ *are basic then the following are equivalent:*

(1) $\Gamma \models^{\mathtt{HI}} @_k(\exists\chi)\bigwedge E$*,*

(2) $M^\Gamma \models^{\mathtt{HI}} @_k(\exists\chi)\bigwedge E$*, and*

(3) $\Gamma \models^{\mathtt{HI}} @_k(\forall\varphi)\bigwedge\theta(E)$ *for some substitution* $\chi \xrightarrow{\theta} \varphi \in \mathtt{HSt}^{\mathtt{HI}}(\Delta)$ *such that* φ *is conservative.*

Proof.

(1) $\Rightarrow$ (2) Obvious since $M^\Gamma \models^{\mathtt{HI}} \Gamma$.

(2) $\Rightarrow$ (3) Since $M^\Gamma \models^{\mathtt{HI}} @_k(\exists\chi) \bigwedge E$, there exists a χ-expansion M' of M^Γ such that $M' \models^w E$, where $w = K_\Delta(M^\Gamma)_k$. By the reachability of M^Γ, there exists a substitution $\theta : \chi \to 1_\Delta$ in $\mathtt{HSt}^{\mathtt{HI}}(\Delta)$ such that $M^\Gamma \upharpoonright_\theta = M'$. We show that $\Gamma \models^{\mathtt{HI}} @_k \bigwedge \theta(E)$.

Let M be a Δ-model such that $M \models^{\mathtt{HI}} \Gamma$. By the initiality of M^Γ, there exists a unique homomorphism $h : M^\Gamma \to M$. Since $\mathtt{HI}_b$ is semantically closed to retrieve, $B = \bigcup_{e\in E} @_k e$ is basic. Since $M' \models^w E$, we have $M' \models^{\mathtt{HI}} B$. There exists an arrow $g : M^B \to M'$, where M^B is the basic model of B. It follows that there exists an arrow $g;(h\upharpoonright_\theta) : M^B \to M\upharpoonright_\theta$, which implies $M\upharpoonright_\theta \models B$. Note that for all $e \in E$, we have $M\upharpoonright_\theta \models^w e$, and by the local satisfaction condition, $M \models^w \theta(e)$. We obtain $M \models^w \theta(E)$, which implies $M \models^{\mathtt{HI}} @_k \bigwedge \theta(E)$.

(3) $\Rightarrow$ (1) Let M be a Δ-model such that $M \models^{\mathtt{HI}} \Gamma$. We denote $\mathtt{K}_\Delta(M)_k$ by w. There exists a φ-expansion M'' of M as φ is conservative. Since $\Gamma \models^{\mathtt{HI}} @_k(\forall\varphi) \bigwedge \theta(E)$, we have $M'' \models^w \theta(E)$. By the local satisfaction condition, $M''\upharpoonright_\theta \models^w E$. Note that $(M''\upharpoonright_\theta)$ is a χ-expansion of M. It follows that $M \models^w (\exists\chi) \bigwedge E$. Hence, $M \models^{\mathtt{HI}} @_k(\exists\chi) \bigwedge E$. $\square$

The substitution θ from the third statement of Theorem 74 is called a *solution*. The existentially quantified formula $(\exists\chi) \bigwedge E$ is a *query*. The implication (1) $\Rightarrow$ (3) reduces the satisfiability of a query by a program (represented here by a theory) at the state denoted by k to the search of a substitution. The converse implication (3) $\Rightarrow$ (1) shows that the solutions are sound with respect to the given program. Implication (1) $\Rightarrow$ (3) is called *completeness*, while implication (3) $\Rightarrow$ (1) is called *soundness*.

We apply Theorem 74 to **HPL**. In this case, the quantification is restricted to signature extensions with a finite set of nominal variables. We call *Horn clauses* the sentences of $\mathbf{HPL}_n$. A *query* is any sentence of the form $(\exists J) \bigwedge E$, where J is a finite set of nominal variables and E is a finite set of $\mathbf{HPL}_b$ sentences.

Corollary 75. *In* **HPL**, *for every set of Horn clauses* Γ, *each query* $(\exists J) \bigwedge E$ *and any nominal* k *for a given signature* $\Delta = (\mathtt{Nom}, \Lambda, \mathtt{Prop})$, *the following are equivalent:*

(1) $\Gamma \models^{\mathbf{HPL}} @_k(\exists J) \bigwedge E$,

(2) $(W^\Gamma, M^\Gamma) \models^{\mathbf{HPL}} @_k(\exists J) \bigwedge E$, *where* (W^Γ, M^Γ) *is the initial model of* Γ,

(3) $\Gamma \models^{\mathbf{HPL}} @_k(\forall J') \bigwedge \theta(E)$ *for some hybrid substitution* $\theta : J \to \mathtt{Nom} \cup J'$, *where* J' *is a finite set of nominal variables.*

Proof. Since $\mathtt{Nom}$ is not empty (as $k \in \mathtt{Nom}$), the inclusion $\chi' : \Delta \hookrightarrow \Delta[J']$ is conservative. By Corollary 71, any set of Horn clauses has a initial reachable model. By Theorem 55, any set of sentences in $\mathbf{HPL}_b$ is epi-basic. By Theorem 74, the three statements of Corollary 75 are equivalent. $\square$

The institution **HFOLSA** falls into the framework of Theorem 74. We call *Horn clauses* the sentences of the institution $\mathbf{HFOLSA}_n$. A query is any existentially quantified conjunction of basic sentences of the form $(\exists X) \bigwedge E$, where X is a finite set of nominal and rigid variables and E is a finite set of sentences in $\mathbf{HFOLSA}_b$.

Corollary 76. *In* **HFOLSA***, for every set of Horn clauses* Γ*, each query* $(\exists X) \bigwedge E$ *and any nominal* k *for the signature* $\Delta = (\mathtt{Nom}, \Lambda, \Sigma)$*, the following are equivalent:*

(1) $\Gamma \models^{\mathbf{HFOLSA}} @_k (\exists X) \bigwedge E$*,*

(2) $(W^\Gamma, M^\Gamma) \models^{\mathbf{HFOLSA}} @_k (\exists X) \bigwedge E$*,*
where (W^Γ, M^Γ) *is the initial model of* Γ*,*

(3) $\Gamma \models^{\mathtt{HI}} @_k (\forall X') \bigwedge \theta(E)$
for some hybrid substitution $\theta = (\theta_a : X_a \to \mathtt{Nom} \cup X'_a, \theta_b : X_b \to T_k^{\Delta[X'_a, X'_b]})$*, where* X_a *is the set of nominal variables of* X*,* X_b *is the set of rigid variables of* X*,* X'_a *is a finite set of nominal variables,* X'_b *is a finite set of rigid variables such that the sorts of the variables in* X'_b *are inhabited* [2] *and* $X' = X'_a \cup X'_b$*.*

Proof. Since $k \in \mathtt{Nom}$ and the sorts of the variables in X' are inhabited, the inclusion $\chi' : \Delta \hookrightarrow \Delta[X']$ is conservative. By Corollary 72, any set of Horn clauses has a initial reachable model. By Theorem 56, any set of sentences in $\mathbf{HFOLSA}_b$ is epi-basic. By Theorem 74, the three statements of Corollary 76 are equivalent. □

Theorem 74 can be instantiated in a similar preorder algebra context. In what follows, we apply Theorem 74 to ordinary institution such as **FOL**. We call *Horn clauses* the sentences of $\mathbf{FOL}_q$, the restriction of **FOL** to sentences constructed from atomic sentences by applying logical implication and universal quantification over finite sets of first-order variables. A query is any existentially quantified conjunction of atomic sentences of the form $(\exists X) \bigwedge E$, where X is a finite set of first-order variables and E is a finite set of atomic sentences.

Corollary 77. *In* **FOL***, for any set of Horn clauses* Γ *and any query* $(\exists X) \bigwedge E$ *for the signature* Σ*, the following are equivalent:*

(1) $\Gamma \models^{\mathbf{FOL}} (\exists X) \bigwedge E$*,*

(2) $M^\Gamma \models^{\mathbf{FOL}} (\exists X) \bigwedge E$*, where* M^Γ *is the initial model of* Γ*,*

[2] For each rigid sort s of Δ and any variable $x \in X'_b$ of sort s there exists a hybrid Δ-term t of sort s.

(3) $\Gamma \models^{\mathbf{FOL}} (\forall X') \bigwedge \theta(E)$ *for some substitution* $\theta : X \to T_\Sigma(X')$,

where X' *is a finite set of first-order variables for* Σ *such that the sorts of the variables in* X' *are inhabited.*

Proof. The substitution functor $\mathtt{St}^{\mathbf{FOL}}$ (see Proposition 43) generates a hybrid substitution functor $\mathtt{HSt}^{\mathcal{H}(\mathbf{FOL})}$ for $\mathcal{H}(\mathbf{FOL})$. By Fact 45, a model is $\mathtt{St}^{\mathbf{FOL}}$-reachable model iff it is $\mathtt{HSt}^{\mathcal{H}(\mathbf{FOL})}$-reachable. It follows that $\mathcal{H}(\mathbf{FOL})$ falls into the framework of Theorem 74. For any set of Horn clauses Γ and any query $(\exists X) \bigwedge E$ the following are equivalent:

$\Gamma \models^{\mathbf{FOL}} (\exists X) \bigwedge E$ iff

$\Gamma \models^{\mathcal{H}(\mathbf{FOL})} @_*(\exists X) \bigwedge E$ iff $M^\Gamma \models^{\mathcal{H}(\mathbf{FOL})} @_*(\exists X) \bigwedge E$ iff

$M^\Gamma \models^{\mathbf{FOL}} (\exists X) \bigwedge E$ iff

$\Gamma \models^{\mathcal{H}(\mathbf{FOL})} @_*(\forall X') \bigwedge \theta(E)$ for some substitution $\theta : X \to T_\Sigma(X')$ such that all sorts of the variables in X' are inhabited iff

$\Gamma \models^{\mathbf{FOL}} (\forall X') \bigwedge \theta(E)$ for some substitution $\theta : X \to T_\Sigma(X')$ such that all sorts of the variables in X' are inhabited.

□

The above result can be found also in [14], where Herbrand's theorem is proved at the level of an arbitrary institution satisfying certain conditions. The result in [14] is based on a notion of *representable signature morphisms* which is replaced here by more permissive hypotheses for concrete logical systems such as *reachable models*. For example, **PA** is an instance of Theorem 74 but it does not fall into the abstract framework of [14] as the signature extensions with "partial" variables (used for quantification) are not *representable*.

8. Conclusions

The existence of initial models of Horn clauses and Herbrand's theorem is proved for hybrid logics. The study on initiality is performed in two steps: (1) We commence by showing that the set of atomic sentences of hybrid logics given as examples here are epi-basic and the basic models are reachable. (2) Then we investigate the closure of the initiality property under certain sentence operators used for constructing Horn clauses in the general setting provided by hybrid institutions. Similar results on initiality have been formulated in the context of ordinary institutions in [29]. The present work is not based on quasi-variety concepts, and the results are not obtained within the framework of factorization systems [44, 46] or inclusion systems [13]. The hybrid logics with user-defined sharing and annotation are not instances of the hybridization process, since the definition of the atomic sentences involves nominals; therefore, the quasi-variety theorem in [17] does not cover these examples of hybrid logics. Finding an inclusion system for the models of the base institution to meet the preservation conditions of the quasi-variety theorem is the most difficult technical part of [17]. The aforementioned paper does not provide a good hint for the application of the abstract quasi-variety theorem to hybrid logics with user-defined sharing. Our approach requires less model-theoretic infrastructure and it is applicable to theories for which the corresponding class of models may not form a quasi-variety.

Herbrand's theorem has been proved for various logical systems [32, 23, 24]. The first institution-independent version is due to [14]. The result has been refined in [11] in the context of abstract *substitution systems*. We propose a version of Herbrand's theorem for hybrid institutions where the queries are formulated for certain states of the program represented here by nominals.

Our abstract results are not applicable to hybrid institutions with user-defined sharing in their standard versions as the expansions of models that consist of interpretations of syntactic elements along signature morphisms used for quantification does not generate substitutions, in general. While initiality can be borrowed from hybrid institutions with annotated syntax to their standard variants, the solutions of queries cannot be transported in the same manner. This fact is related to the reacher syntactic expressivity provided by annotation, which is an important contribution of the present work.

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Appendix A. Exiled proofs

Proof of Lemma 30. We proceed by induction on the structure of the Δ-terms:

(1) Let $\varsigma \in F^r$, $k \in \texttt{Nom}$ and $\tau \in (T_k^\Delta)_{\texttt{ar}}$, where $\texttt{ar}$ is the arity of ς, and assume $(W', M')_{\varphi(\tau)} = ((W', M') \restriction_\varphi)_\tau$.

We have $(W', M')_{\varphi(\varsigma(\tau))} = (M'_{w'})_{\varphi(\varsigma)}((W', M')_{\varphi(\tau)})$, where $w' = W'_{\varphi^{\mathbf{REL}}(k)}$.

It follows that $(M'_{w'})_{\varphi(\varsigma)}((W', M')_{\varphi(\tau)}) = (M'_{w'} \restriction_\varphi)_\varsigma(((W', M') \restriction_\varphi)_\tau) = ((W', M') \restriction_\varphi)_{\varsigma(\tau)}$.

(2) Let $\sigma \in (F - F^r)$, $k \in \mathtt{Nom}$ and $t \in (T_k^\Delta)_a$, where a is the arity of σ, and assume $(W', M')_{\varphi(t)} = ((W', M')\upharpoonright_\varphi)_t$.

We have $(W', M')_{\varphi(\sigma_k(t))} = (M'_{w'})_{\varphi(\sigma)}((W', M')_{\varphi(t)})$, where $w' = W'_{\varphi^{\mathbf{REL}}(k)}$.

It follows that $(M'_{w'})_{\varphi(\sigma)}((W', M')_{\varphi(t)}) = (M'_{w'} \upharpoonright_\varphi)_\sigma(((W', M') \upharpoonright_\varphi)_t) = ((W', M')\upharpoonright_\varphi)_{\sigma_k(t)}$.

(3) Let $k_1 \in \mathtt{Nom}$, $\mathtt{sr} \in S^r$, $t \in (T_{k_1}^\Delta)_{\mathtt{sr}}$ and assume that $(W', M')_{\varphi(t)} = ((W', M')\upharpoonright_\varphi)_t$. We can conclude that for $t \in (T_k^\Delta)_s$ we have $(W', M')_{\varphi(t)} = ((W', M')\upharpoonright_\varphi)_t$.

$\square$

Proof of Corollary 31. In order to show that **HFOLSA** is an institution, we need to show the local satisfaction condition holds. We prove that the local satisfaction condition holds for atomic sentences.

(a) We have $(W', M') \upharpoonright_\varphi \models (t = t')$ iff $((W', M') \upharpoonright_\varphi)_t = ((W', M') \upharpoonright_\varphi)_{t'}$ for all $t, t' \in (T_k^\Delta)$; by Lemma 30, we have $((W', M')\upharpoonright_\varphi)_t = ((W', M')\upharpoonright_\varphi)_{t'}$ iff $(W', M')_{\varphi(t)} = (W', M')_{\varphi(t')}$, and we get $(W', M')_{\varphi(t)} = (W', M')_{\varphi(t')}$ iff $(W', M') \models \varphi(t = t')$.

(b) Let $\varpi \in P^r$, $k \in \mathtt{Nom}$ and $\tau \in (T_k^\Delta)_{\mathtt{ar}}$, where $\mathtt{ar}$ is the arity of ϖ: we have that $(W', M') \upharpoonright_\varphi \models \varpi(\tau)$ iff $((W', M') \upharpoonright_\varphi)_\tau \in (M'_{w'} \upharpoonright_\varphi)_\varpi$, where $w' = W'_{\varphi(k)}$, iff $(W', M')_{\varphi(\tau)} \in (M'_{w'})_{\varphi(\varpi)}$ iff $(W', M') \models \varphi(\varpi(\tau))$.

(c) Let $\pi \in (P - P^r)$, $k \in \mathtt{Nom}$ and $t \in (T_k^\Delta)_a$, where a is the arity of σ: we have $(W', M')\upharpoonright_\varphi \models \pi_k(t)$ iff $((W', M')\upharpoonright_\varphi)_t \in (M'_{w'}\upharpoonright_\varphi)_\pi$, where $w' = W'_{\varphi(k)}$, iff $(W', M')_{\varphi(t)} \in (M'_{w'})_{\varphi(\pi)}$ iff $(W', M') \models \varphi(\pi_k(t))$.

The remaining proof is straightforward and it is similar to the proof of the local satisfaction condition for hybridized institutions (see Theorem 3.2 of [17]). $\square$