

SINGULAR PATTERNS OF GENERIC MAPS OF SURFACES WITH BOUNDARY INTO THE PLANE

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ABSTRACT. For generic maps from compact surfaces with boundary into the plane we develop an explicit algorithm for minimizing both the number of cusps and the number of components of the singular locus. More precisely, we minimize among maps with fixed boundary conditions and prescribed *singular pattern*, by which we mean the combinatorial information of how the 1-dimensional singular locus meets the boundary. Each step of our algorithm modifies the given map only locally by either creating or eliminating a pair of cusps. We show that the number of cusps is an invariant modulo 2 and can be reduced to at most one, and we compute the minimal number of components of the singular locus in terms of the prescribed data. Applications include a discussion of pseudo-immersions as well as the computation of state sums in Banagl's positive topological field theory.

1. INTRODUCTION

By a classical result of Whitney [13], any smooth map $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ (“smooth” always means differentiable of class C^∞) can be approximated arbitrarily well by a smooth map $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that for every point $p \in \mathbb{R}^2$, the map germ of F at p is smoothly right-left equivalent to one of the following map germs $(\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$:

$$(x, y) \mapsto \begin{cases} (x, y), & p \text{ is a regular point of } F, \\ (x, y^2), & p \text{ is a fold point of } F, \\ (x, xy + y^3), & p \text{ is a cusp of } F. \end{cases}$$

In general, a smooth map $F: W \rightarrow V$ between smooth 2-manifolds W and V is called *generic* if its singular locus

$$S(F) = \{p \in W; \text{ the differential } dF_p: T_p W \rightarrow T_{F(p)} V \text{ has rank } < 2\}$$

consists only of fold points and cusps. If $F: W \rightarrow V$ is generic, then $S(F) \subset W$ is a 1-dimensional submanifold, and the cusps of F form a discrete subset of W . Moreover, a generic map $F: W \rightarrow V$ is called fold map if $S(F)$ contains no cusps. Note that any fold map $F: W \rightarrow V$ restricts to an immersion $S(F) \rightarrow V$.

The problem of classifying generic maps between surfaces in terms of suitable notions of complexity related to their singular locus has attracted the attention of several authors; see e.g. [10, 15] for counting cusps and components of the singular locus, [16] for the study of apparent contours of stable maps (compare Remark 8.1), and [9] for the combinatorics of certain graphs associated to stable maps.

Date: December 21, 2018.

Key words and phrases. Generic smooth map, Elimination of cusps, swallowtail, winding number, Euler characteristic, pseudo-immersion.

Our paper contributes to this field of study by determining the possible number of cusps and of singular components (i.e., components of the singular locus) for generic maps of compact surfaces with boundary into the plane. For the generic maps under consideration, we impose reasonable boundary conditions, and also fix their *singular pattern* (see Definition 5.1), namely the combinatorial information of how the 1-dimensional singular locus runs into the boundary.

Initial motivation for our setting comes from the requirement that the number of singular components of a generic map should be detectable in a way that behaves nicely under gluing of cobordisms – a property that is desirable from the viewpoint of topological field theory (TFT). In fact, suppose that a 2-dimensional cobordism W is the result of gluing a cobordism W_1 from M to N and a cobordism W_2 from N to P along their common boundary part N , i.e., $W = W_1 \cup_N W_2$. Consider a generic map $F: W \rightarrow \mathbb{R}^2$ whose singular locus $S(F)$ is transverse to M , N and P . Then, the number of singular components of F cannot directly be computed from the number of singular components of $F|_{W_1}$ and $F|_{W_2}$. The remedy is to record also the additional information that is provided by singular patterns. It is exactly this gluing principle that has been exploited by Banagl [2, Section 10] in order to construct an explicit example of a so-called positive TFT which is based on certain fold maps of cobordisms into the plane (see Remark 9.5). The spirit of TFT is also visible in the proof of Proposition 4.2, where a relative version of the winding number for arcs in the plane is used.

Let us introduce some terminology to prepare the discussion of our main results. Suppose that W is a connected compact smooth 2-manifold with boundary $P = \partial W$, and fix a collar $[0, \infty) \times P \subset W$. A *boundary condition* is a fold map $f: (-\varepsilon, \varepsilon) \times P \rightarrow \mathbb{R}^2$ whose singular locus $S(f)$ is transverse to $\{0\} \times P$. By a *(singular) pattern* (see Definition 5.1) on W we mean a pair (f, φ) consisting of a boundary condition f , and a partition φ of the finite set $P_f := (\{0\} \times P) \cap S(f)$ into subsets of cardinality 2. Finally, a *realization* of a pattern (f, φ) is a generic extension $F: W \rightarrow \mathbb{R}^2$ of the germ of $f|_{[0, \varepsilon) \times P}$ at $\{0\} \times P$ (such F do always exist) such that a subset $A \subset P_f$ belongs to the partition φ if and only if A is the boundary of a component of $S(F)$. In Proposition 5.4 we will provide a combinatorial criterion for deciding whether a realization of a given pattern exists.

The proofs of our main results will provide an explicit algorithm for minimizing the number of cusps (Theorem 1.1) and the number of components (Theorem 1.2) of realizations of a given pattern on W . Our algorithm relies merely on two local modifications of generic maps which can be realized by homotopies supported in $W \setminus \partial W$. Namely, we use Levine's homotopy [11] for eliminating pairs of cusps of generic maps between surfaces, as well as the complementary process of creating a pair of cusps on a fold line by means of the swallow-tail homotopy. These two modifications, say (E) and (C), serve as elementary building blocks for more complicated moves (see Section 2) that will be used to build up our algorithm. Note that the moves (E) and (C) correspond to two out of 10 types of codimension one local strata of the discriminant hypersurface $\Gamma \subset C^\infty(W, \mathbb{R}^2)$ consisting of smooth maps $W \rightarrow \mathbb{R}^2$ that have non-generic apparent contours [1].

Theorem 1.1. *Let (f, φ) be a pattern on W .*

- (a) *All realizations of (f, φ) have the same number of cusps modulo 2.*
- (b) *Every realization of (f, φ) can be modified on $W \setminus \partial W$ by a finite sequence of (E) and (C) moves to obtain a realization with at most one cusp.*

In particular, part (a) yields a $\mathbb{Z}/2$ -valued invariant of certain patterns on W . This invariant turns out to be independent of φ , and can be considered as a relative Thom polynomial for cusps of generic maps of surfaces into the plane. In Proposition 6.1 we provide a formula for computing this invariant in terms of f . It involves the *boundary turning invariant* of f , which will be defined in Section 3 by means of winding numbers of certain immersions of the circle into the plane. Our formula resembles the formula of Fukuda-Ishikawa [7] for maps between compact surfaces with boundary (see Remark 6.2). In the context of Proposition 6.1 we define an integer Γ^σ that depends on an orientation σ of P . We will use Γ^σ in the definition of another integer Δ^σ (see Section 7) that will appear in our second main result concerning the minimal number of loops (i.e., singular components diffeomorphic to the circle) that can occur for realizations of a given pattern on W .

Theorem 1.2. *Let (f, φ) be a pattern on W such that $P_f \neq \emptyset$. Suppose that $F: W \rightarrow \mathbb{R}^2$ is a realization of (f, φ) without cusps.*

- (a) *Let W be non-orientable. Then, given an integer $l \geq 0$, F can be modified on $W \setminus \partial W$ by a finite sequence of (E) and (C) moves to a realization of (f, φ) which has no cusps and l loops.*
- (b) *Let W be orientable, and let σ denote an orientation of $P = \partial W$ which is induced by one of the two orientations of W . Then, the following statements are equivalent for any integer l :*
 - (i) *F can be modified on $W \setminus \partial W$ by a finite sequence of (E) and (C) moves to a realization of (f, φ) which has no cusps and l loops.*
 - (ii) *$l \in \mathbb{N} \cap (\Delta^\sigma + 2\mathbb{N}) \cap (\Delta^{-\sigma} + 2\mathbb{N})$.*

In Section 9 we use the gluing principle for singular patterns to generalize Theorem 1.2 to the case $P_f = \emptyset$ of pseudo-immersions (see Example 9.2), as well as to the case that F has cusps (see Proposition 9.4).

The paper is structured as follows. In Section 2 we present the local moves that will be essential for our approach. The concept of boundary turning invariant will be introduced in Section 3. The proofs of our main results will cover Section 4 to Section 8. Section 9 concludes with some applications.

Notation. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ denote the set of non-negative integers. Let W denote a 2-dimensional connected compact smooth manifold. In case of a non-empty boundary $P = \partial W$ we also fix a collar neighborhood $[0, \infty) \times P \subset W$. Let $\chi(W)$ denote the Euler characteristic of W . The plane \mathbb{R}^2 is equipped with the standard orientation. We use the following convention for the induced orientation on the boundary of an oriented surface. If the first vector of an oriented frame points out of the surface at a boundary point, then the second vector defines the orientation of the boundary at that point.

Acknowledgements. The material presented in this paper is based on Chapter 5 of the author's PhD thesis [14]. The author is grateful to his advisor Professor Markus Banagl for constant encouragement and many fruitful discussions. The author would like to thank Professor Takahiro Yamamoto, Professor Minoru Yamamoto and Professor Osamu Saeki for discussions and invaluable comments on an early version of the manuscript. Also, the author thanks the anonymous referee for reading the paper very carefully, which led to several improvements.

The author is grateful to the German National Merit Foundation (Studienstiftung des deutschen Volkes) for financial support. Moreover, the author has been supported by JSPS KAKENHI Grant Number JP17H06128.

2. LOCAL MODIFICATIONS

Throughout this section, let $F: W \rightarrow \mathbb{R}^2$ denote a realization of some pattern (f, φ) on W as defined in the introduction (see also Definition 5.1).

The discussion in this section will be supported by figures showing the singular locus of F in open neighborhoods $U \subset W \setminus \partial W$. These neighborhoods U will be colored in grey. The intersection $S(F) \cap U$ will be represented by massive black lines, whereas $S(F) \cap (W \setminus U)$ will be indicated by spotted lines in the figures. The cusps of F will be marked by discrete points on the massive lines. Every cusp c is furthermore equipped with a vector in $T_x W$ that points *downward* in the sense of Levine (see the definition on p. 284 of [11]). (A vector points downward at a cusp if its image in the plane points in the direction of the cusp. Intuitively, the vector itself points into the half plane into which the cusp can propagate.) The direction of a downward pointing tangent vector will be referred to as the direction into which the cusp *points*. Finally, *joining curves* (see Section (4.4) in [11, p. 285]) between cusps will be symbolized by dashed lines (see the left of Figure 1). (A joining curve between two cusps c_0 and c_1 is an embedding $\alpha: [0, 1] \rightarrow W \setminus \partial W$ such that $\alpha^{-1}(S(F)) = \{0, 1\}$, $\alpha(i) = c_i$, $i = 0, 1$, and $(-1)^i \cdot \alpha'(i) \in T_{c_i} W$ points downward at c_i .)

Our approach is based on the following two local modifications (see Figure 1).

- (E) **Elimination of cusp pairs** (see Figure 1(a)). A pair of cusps of F can be eliminated if the cusps point into a common component of $W \setminus S(F)$ (see Fig. 3 in [11, p. 286]). In fact, it follows from the proofs of Lemma (1) and (2) in Section (4.4) of [11, p. 285] that such a pair of cusps can be connected by a joining curve. As W is a 2-dimensional cobordism, the pair of cusps of F is a matching pair in the sense of [11] and hence *removable* (as defined in Section (4.5) of [11, p. 285]). As explained in [11, p. 286], the pair of cusps can therefore be eliminated by a homotopy of F that modifies F only in a tubular neighborhood of the joining curve.
- (C) **Creation of cusp pairs** (see Figure 1(b)). In a small neighborhood of a fold point of F a pair of two new cusps can be introduced on the fold line in such a way that the cusps point into opposite directions of the singular curve. This modification can be realized by a homotopy of F that modifies F only in this neighborhood and is based on the swallow-tail homotopy (for details, see for instance Section 4.7 in [14, p. 110]).

Definition 2.1. By a *loop* of F we mean a component of $S(F)$ that is diffeomorphic to the circle. A loop C of F is called

- *pure* if C contains no cusps of F .
- *trivial* if the normal bundle of C in W is trivial.
- *contractible* if there is an open disc $D \subset W \setminus \partial W$ such that $D \cap S(F) = C$.

Proposition 2.2 (Loop generation). *(i) Let $p \in W \setminus \partial W$ be a fold point of F . Given a neighborhood $U \subset W$ of p , F can be modified on U by two (C) moves followed by two (E) moves in such a way that the modified map $\tilde{F}: W \rightarrow \mathbb{R}^2$ has two loops more than F .*

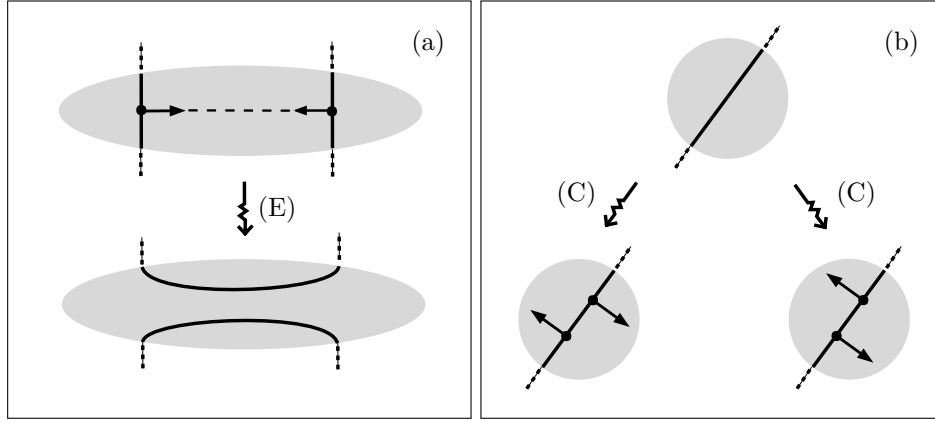


FIGURE 1. (a): Levine's homotopy [11] for **eliminating pairs of cusps** of generic maps between surfaces. (b): The swallow-tail homotopy for **creating a pair of cusps** on a fold line.

- (ii) Let c_0, c_1 be two cusps of F . Suppose that $\alpha: [0, 1] \rightarrow W \setminus \partial W$ is a joining curve (see Section (4.4) in [11, p. 285]) between $\alpha(0) = c_0$ and $\alpha(1) = c_1$. Fix a neighborhood V of $\alpha([0, 1])$. Let $\gamma_i: [-1, 1] \rightarrow S(F) \cap V$, $i = 0, 1$, be embeddings such that $\gamma_i(0) = c_i$ and $\gamma_0([-1, 1]) \cap \gamma_1([-1, 1]) = \emptyset$. Suppose that $U \subset V$ is a neighborhood of $\alpha([0, 1])$ which does not contain the points $\gamma_i(\pm 1)$, $i = 0, 1$. Then, F can be modified on U by one (E) move along the joining curve α , followed by two (C) moves, followed by two (E) moves in such a way that the modified map $\tilde{F}: W \rightarrow \mathbb{R}^2$ has one loop more than F (namely, a contractible loop on U), and there exist embeddings $\tilde{\gamma}_i: [-1, 1] \rightarrow S(\tilde{F}) \cap V$, $i = 0, 1$, such that $\tilde{\gamma}_i(j) = \gamma_i(j)$ for $j = \pm 1$.

Definition 2.3. The canonical orientation of $S(F)$ is uniquely determined by requiring that, at every fold point of F , W is “folded to the left” into the plane, by which we mean the following. Recall that every fold point $p \in S(F)$ admits a neighborhood $U \subset W$ such that F restricts to an embedding $S(F) \cap U \rightarrow \mathbb{R}^2$, and $F(U) \subset \mathbb{R}^2$ is a codimension 0 submanifold with boundary $F(S(F) \cap U)$. Then the canonical orientation at $p \in S(F)$ is such that the induced orientation at $F(p) \in F(S(F) \cap U)$ coincides with the orientation of $F(S(F) \cap U)$ induced by $F(U) \subset \mathbb{R}^2$. Note that the resulting orientation of the fold locus of F extends unambiguously over the cusps of F .

Proposition 2.4 (General elimination of cusps). *Let c_0, c_1 be two cusps of F . Suppose that $\alpha: [0, 1] \rightarrow W \setminus \partial W$ is an embedding such that $\alpha(i) = c_i$, $i = 0, 1$, $(-1)^i \cdot \alpha'(i)$ is a downward pointing tangent vector of c_i , $i = 0, 1$, and $\alpha|_{(0,1)}$ intersects $S(F)$ transversely and only in fold points of F . Fix a neighborhood U of $\alpha([0, 1])$ and orientation preserving embeddings $\gamma_0, \gamma_1: [0, 1] \rightarrow S(F)$ (where $S(F)$ is equipped with the canonical orientation of Definition 2.3) such that $\gamma_0(0) \notin U$, $\gamma_0(1) = c_0$, and $\gamma_1(0) = c_1$, $\gamma_1(1) \notin U$. Then, F can be modified on U by a finite sequence of (E) and (C) moves in such a way that the modified map $\tilde{F}: W \rightarrow \mathbb{R}^2$ has two cusps less than F , and there exists an embedding $\gamma: [0, 1] \rightarrow S(\tilde{F})$ such that $\gamma(i) = \gamma_i(i)$, $i = 0, 1$.*

Proof. We proceed by induction over the cardinality of $\alpha^{-1}(S(F)) = \{t_0, \dots, t_r\}$, where $0 = t_r < \dots < t_0 = 1$. If $r = 1$, then the claim follows by applying (E) to the joining curve α , where note that the existence of γ is automatic. If $r > 1$, then apply one of the local modifications indicated in Figure 2 depending on the position of $\gamma_0(0)$, and modify α on $[0, t_{r-2})$ by reducing the number of cusps by one. \square

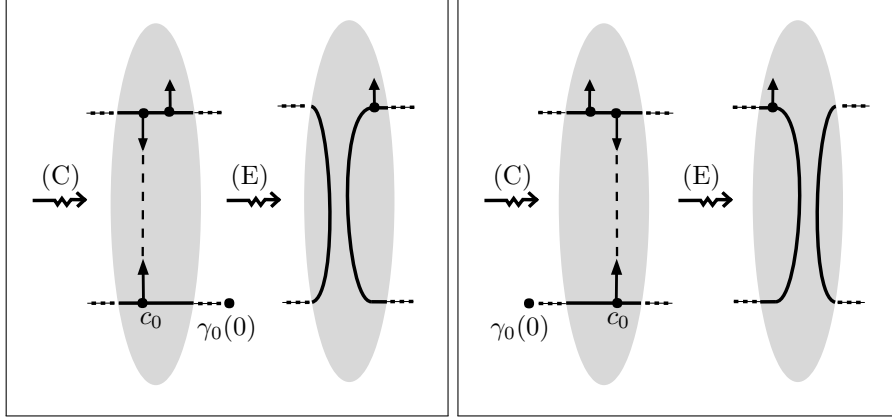


FIGURE 2. Induction step for general elimination of cusps.

Remark 2.5. An explicit version of Proposition 2.4 (not claiming the existence of the embedding γ) is originally due to Kálmán (see Lemma 1.4 of [10, p. 308]).

Proposition 2.6 (Loop simplification, see Figure 3). *Let C be a loop of F . Suppose that $\alpha: [0, 1] \rightarrow W \setminus \partial W$ is an embedding such that $\alpha^{-1}(S(F)) = \{0, 1\}$, where $\alpha(0) \in C$ and $\alpha(1) \notin C$ are fold points of F . Fix a neighborhood U of $\alpha([0, 1])$. Then, F can be modified on U by a finite sequence of (E) and (C) moves in such a way that the modified map $\tilde{F}: W \rightarrow \mathbb{R}^2$ has a contractible loop on U , and F and \tilde{F} have the same number of loops and the same number of cusps.*

Proposition 2.7 (Loop reduction, see Figure 4). *Let C_0, C_1 be two loops of F . Suppose that $\alpha: [0, 1] \rightarrow W \setminus \partial W$ is an embedding such that $\alpha^{-1}(S(F)) = \{0, t, 1\}$ for some $t \in (0, 1)$, where $\alpha(i) \in C_i$, $i = 0, 1$, are fold points of F , and α intersects $S(F)$ transversely at t in a fold point $\alpha(t) \notin C_0 \cup C_1$ of F . Fix a neighborhood U of $\alpha([0, 1])$. Then, F can be modified by a finite sequence of (E) and (C) moves in such a way that the modified map $\tilde{F}: W \rightarrow \mathbb{R}^2$ has two loops less than F , and F and \tilde{F} have the same number of cusps.*

Proposition 2.8 (Tunneling, see Figure 5). *Let C be a trivial loop of F . Suppose that $\alpha: [0, 1] \rightarrow W \setminus \partial W$ is an embedding such that $\alpha^{-1}(S(F)) = \{0, t_0, t_1\}$ for some $0 < t_0 < t_1 < 1$, where $\alpha(0) \in C$, and α intersects $S(F)$ at t_i , $i = 0, 1$, transversely in fold points $\alpha(t_i) \notin C$ of F . Fix a neighborhood V of $C \cup \alpha([0, 1])$. Let $C' \subset V \setminus S(F)$ be an embedded circle which is contained in a tubular neighborhood of $C \subset W$ such that $\alpha^{-1}(C') = \{s\}$ for some $0 < s < t_0$. Let $\gamma_i: [-1, 1] \rightarrow S(F) \cap V$, $i = 0, 1$, be embeddings such that $\gamma_i(0) = \alpha(t_i)$ and $\gamma_0([-1, 1]) \cap \gamma_1([-1, 1]) = \emptyset$. Suppose that $U \subset V$ is a neighborhood of $C' \cup \alpha([s, t_1])$ which does neither contain C nor any of the points $\alpha(1)$ and $\gamma_i(\pm 1)$, $i = 0, 1$. Then, F can be modified*

on U by a finite sequence of (E) and (C) moves in such a way that the modified map $\tilde{F}: W \rightarrow \mathbb{R}^2$ has the same number of loops and the same number of cusps as F , and there exists an embedding $\tilde{\alpha}: [0, 1] \rightarrow V$ such that $\tilde{\alpha}^{-1}(S(\tilde{F})) = \{0\}$ and $\tilde{\alpha}(i) = \alpha(i)$, $i = 0, 1$, and embeddings $\tilde{\gamma}_i: [-1, 1] \rightarrow S(\tilde{F}) \cap V$, $i = 0, 1$, such that $\tilde{\gamma}_i(j) = \gamma_i(j)$ for $j = 0, 1$.

Proposition 2.9 (Balancing, see Figure 6). *Let C_0, C_1 be two loops of F . Suppose that $\alpha: [0, 1] \rightarrow W \setminus \partial W$ is an embedding such that $\alpha^{-1}(S(F)) = \{0, t_0, t_1, 1\}$ for some $0 < t_0 < t_1 < 1$, where $\alpha(i) \in C_i$, $i = 0, 1$, are fold points of F , and α intersects $S(F)$ at t_i , $i = 0, 1$, transversely in fold points $\alpha(t_i) \notin C_0 \cup C_1$ of F . Furthermore, suppose that $H \subset W \setminus S(F)$ is an embedded circle that intersects α transversely, and such that $\alpha^{-1}(H) = \{s\}$ for some $s \in (t_0, t_1)$. Fix a neighborhood V of $H \cup \alpha([0, 1])$. Let $\gamma_i: [-1, 1] \rightarrow S(F) \cap V$, $i = 0, 1$, be embeddings such that $\gamma_i(0) = \alpha(t_i)$ and $\gamma_0([-1, 1]) \cap \gamma_1([-1, 1]) = \emptyset$. Suppose that $U \subset V$ is a neighborhood of $H \cup \alpha([0, 1])$ which does not contain the points $\gamma_i(\pm 1)$, $i = 0, 1$. Then, F can be modified on U by a finite sequence of (E) and (C) moves in such a way that the modified map $\tilde{F}: W \rightarrow \mathbb{R}^2$ has two loops less than F , whereas F and \tilde{F} have the same number of cusps, and there exist embeddings $\tilde{\gamma}_i: [-1, 1] \rightarrow S(\tilde{F}) \cap V$, $i = 0, 1$, such that $\tilde{\gamma}_i(j) = \gamma_i(j)$ for $j = 0, 1$.*

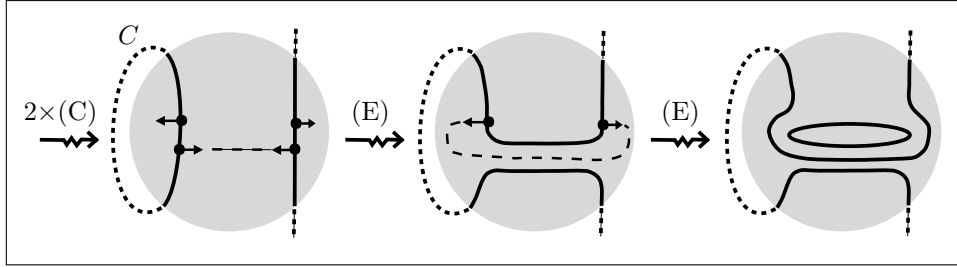


FIGURE 3. Loop simplification. The modification is performed as indicated on an embedded open disc in $W \setminus \partial W$ containing $\alpha([0, 1])$.

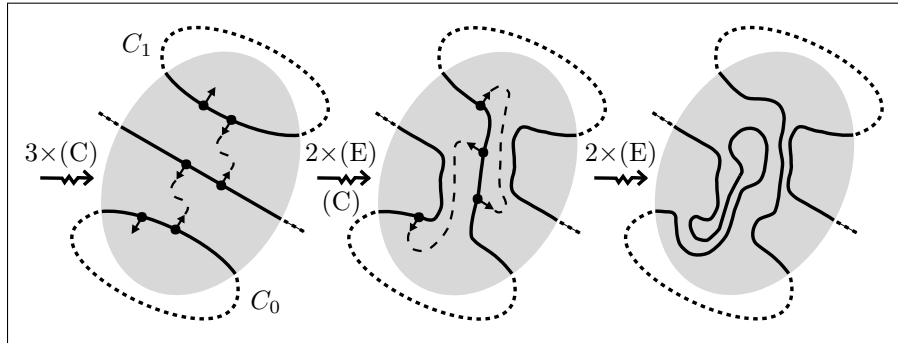


FIGURE 4. Loop reduction. The modification is performed as indicated on an embedded open disc in $W \setminus \partial W$ containing $\alpha([0, 1])$.

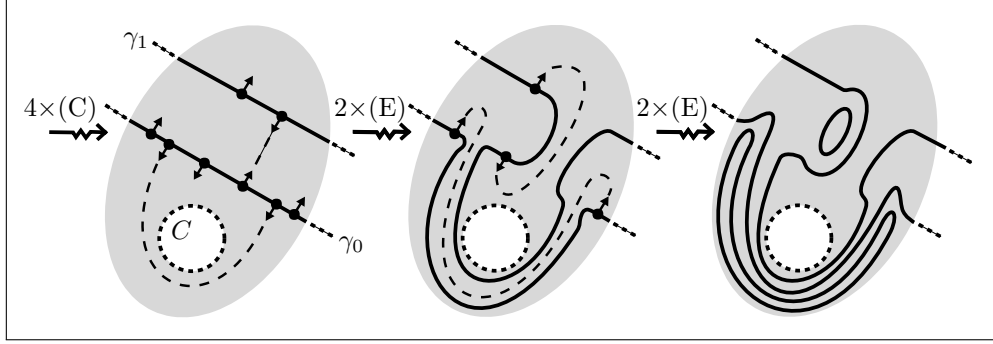


FIGURE 5. Tunneling. The modification is performed as indicated on an embedded open annulus in U containing $C' \cup \alpha([s, t_1])$. The construction is completed by using loop reduction (Proposition 2.7) to eliminate the two contractible loops in the figure on the right.

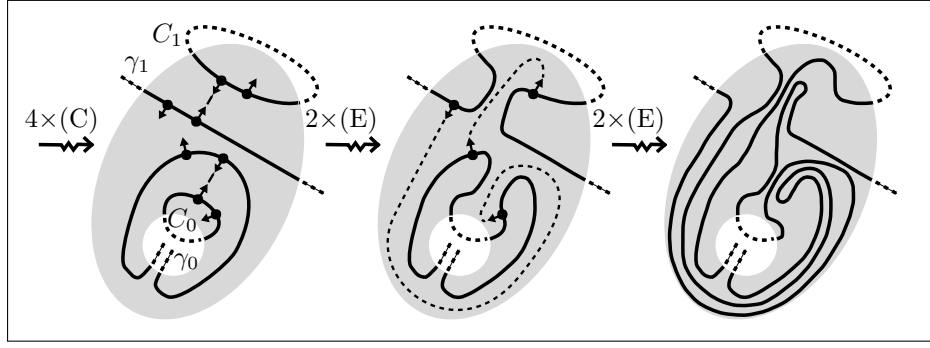


FIGURE 6. Balancing. The modification is performed as indicated on an embedded open annulus in U containing $H \cup \alpha([0, 1])$.

3. BOUNDARY TURNING INVARIANT

Throughout this section let P denote a closed 1-dimensional manifold equipped with an orientation σ . Furthermore, let f be a boundary condition on P . That is, $f: (-\varepsilon, \varepsilon) \times P \rightarrow \mathbb{R}^2$, $\varepsilon > 0$, is a fold map whose singular locus $S(f) \subset (-\varepsilon, \varepsilon) \times P$ is transverse to $\{0\} \times P$. In particular, the set $P_f := S(f) \cap \{0\} \times P$ is finite. To these data we will assign in Definition 3.5 an integer $\omega_\sigma(f)$ called the boundary turning invariant.

Lemma 3.1. (a) *There exists $\varepsilon' \in (0, \varepsilon)$ such that the projection to the first factor $(-\varepsilon', \varepsilon') \times P \rightarrow (-\varepsilon', \varepsilon')$ restricts to a local diffeomorphism $S(f) \cap (-\varepsilon', \varepsilon') \times P \rightarrow (-\varepsilon', \varepsilon')$.*
 (b) *For every component C of P , the (possibly empty) set $C_f := S(f) \cap \{0\} \times C$ is finite, and has even cardinality.*

Definition 3.2. Let C be a component of P . An embedding $\alpha: C \rightarrow (-\varepsilon, \varepsilon) \times C$ is called (σ, f) -adapted if the following properties are satisfied:

- (i) There exists $\varepsilon' \in (0, \varepsilon)$ as in Lemma 3.1(a) such that $\alpha(C) \subset (-\varepsilon', \varepsilon') \times C$.
- (ii) α is transverse to $S(f)$, and $\alpha(C) \cap S(f) = C_f$ (see Lemma 3.1(b)).

- (iii) The composition of α with the projection to the second factor $(-\varepsilon, \varepsilon) \times C \rightarrow C$ has degree $+1$.
- (iv) The composition $f \circ \alpha: C \rightarrow \mathbb{R}^2$ is an immersion.

Figure 7 illustrates the behavior of a (σ, f) -adapted embedding $\alpha: C \rightarrow (-\varepsilon, \varepsilon) \times C$ around a point $x \in C_f$. Let $u_x \in T_x S(f)$ denote a non-zero tangent vector which points into $(0, \varepsilon) \times C$. Let $v_x \in \ker d_x f$ be a non-zero vector such that the pair (u_x, v_x) gives the product orientation of $(-\varepsilon, \varepsilon) \times C$ at x . If $w_x \in T_x(\alpha(C))$ denotes a non-zero tangent vector that gives the orientation of $\alpha(C)$ induced by the orientation $\sigma|_C$, then properties (i), (ii) and (iii) of Definition 3.2 imply that the pair (u_x, w_x) is a basis of $T_x((-\varepsilon, \varepsilon) \times C)$ which gives the product orientation of $(-\varepsilon, \varepsilon) \times C$. Moreover, property (iv) implies that the pair (v_x, w_x) is a basis of $T_x((-\varepsilon, \varepsilon) \times C)$.

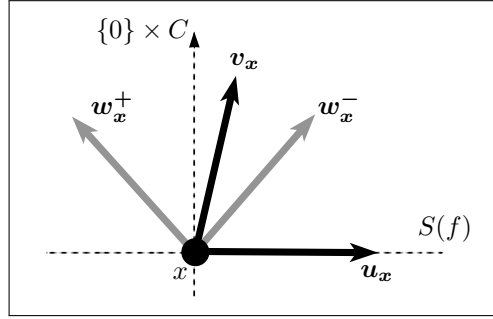


FIGURE 7. A neighborhood in $(-\varepsilon, \varepsilon) \times C$ of a point $x \in C_f$. Given a (σ, f) -adapted embedding α , the vector w_x looks like w_x^+ or w_x^- depending on whether x is positive or negative with respect to α (compare Definition 3.3). There are no requirements about the positions of v_x and w_x relative to $\{0\} \times C$.

Definition 3.3. A point $x \in C_f$ is called *positive* or *negative* with respect to a (σ, f) -adapted embedding $\alpha: C \rightarrow (-\varepsilon, \varepsilon) \times C$ (compare Figure 7) depending on whether the pair (v_x, w_x) gives the product orientation of $(-\varepsilon, \varepsilon) \times C$ or its opposite orientation, respectively. Let C_f^α denote the sets of points of C_f which are negative with respect to α . A (σ, f) -adapted embedding $\alpha: C \rightarrow (-\varepsilon, \varepsilon) \times C$ is called *positive* if $C_f^\alpha = \emptyset$.

Lemma 3.4. Let C be a component of P . There exists a positive (σ, f) -adapted embedding $\alpha: C \rightarrow (-\varepsilon, \varepsilon) \times C$. Moreover, any two such embeddings are smoothly isotopic in the space of all positive (σ, f) -adapted embeddings.

Lemma 3.4 enables us to assign to every component C of P a well-defined integer $\omega_{\sigma|_C}(f|_{(-\varepsilon, \varepsilon) \times C})$ given by the winding number $W(f \circ \alpha)$ (i.e., the degree of the Gauss map, compare [12]) of the immersion $f \circ \alpha: C \rightarrow \mathbb{R}^2$ for any positive (σ, f) -adapted embedding $\alpha: C \rightarrow (-\varepsilon, \varepsilon) \times C$.

Definition 3.5. The *boundary turning invariant* $\omega_\sigma(f)$ of f is defined as

$$\omega_\sigma(f) := \sum_C \omega_{\sigma|_C}(f|_{(-\varepsilon, \varepsilon) \times C}) \in \mathbb{Z},$$

where the sum runs through the components C of P .

Example 3.6. The embedding $g: \mathbb{R} \times S^1 \rightarrow \mathbb{R}^2$, $g(r, s) = e^r \cdot s$, defines a tubular neighborhood of the circle $S^1 \subset \mathbb{R}^2$. Let f_{fold} denote the composition of g with the fold map $(x, y) \mapsto (x, y^2)$, and let f_{cusp} denote the composition of g with the stable Whitney cusp $(x, y) \mapsto (x, xy + y^3)$. Then, for any orientation σ on the circle S^1 , one can show that $\omega_\sigma(f_{\text{fold}}) = 0$ and $\omega_\sigma(f_{\text{cusp}}) = \pm 1$.

Lemma 3.7. *Let C be a component of P . If $\alpha: C \rightarrow (-\varepsilon, \varepsilon) \times C$ is a (σ, f) -adapted embedding, then the winding number of the immersion $f \circ \alpha: C \rightarrow \mathbb{R}^2$ has the same parity as $\omega_{\sigma|_C}(f|_{(-\varepsilon, \varepsilon) \times C}) + |C_f^\alpha|$.*

Proof. Given $x \in C_f^\alpha$, we show how to modify α in a small neighborhood of $\alpha^{-1}(x)$ to obtain a (σ, f) -adapted embedding $\beta: C \rightarrow (-\varepsilon, \varepsilon) \times C$ such that $x \notin C_f^\beta$, and such that the winding numbers of the immersions $f \circ \alpha$ and $f \circ \beta$ differ by ± 1 .

Let \mathfrak{X} denote the space (equipped with the Whitney C^∞ topology) of all smooth maps $\gamma: C \rightarrow (-\varepsilon, \varepsilon) \times C$ for which the composition $f \circ \gamma$ is an immersion. Note that the winding number of $f \circ \gamma$ is invariant under homotopies of γ in \mathfrak{X} .

The given (σ, f) -adapted embedding α is homotopic in \mathfrak{X} to the embedding $\alpha_1: C \rightarrow (-\varepsilon, \varepsilon) \times C$ indicated in Figure 8(a) by means of a homotopy supported in a small neighborhood of x . Then, the winding numbers of the immersions $f \circ \alpha$ and $f \circ \alpha_1$ agree. Let $\alpha_2: C \rightarrow (-\varepsilon, \varepsilon) \times C$ denote an immersion which arises from α_1 by inserting two loops based at x as indicated in Figure 8(b). Then, the winding numbers of the immersions $f \circ \alpha_1$ and $f \circ \alpha_2$ differ by ± 1 . Finally, α_2 is homotopic in \mathfrak{X} to the desired embedding $\beta: C \rightarrow (-\varepsilon, \varepsilon) \times C$ (see Figure 8(c)). \square

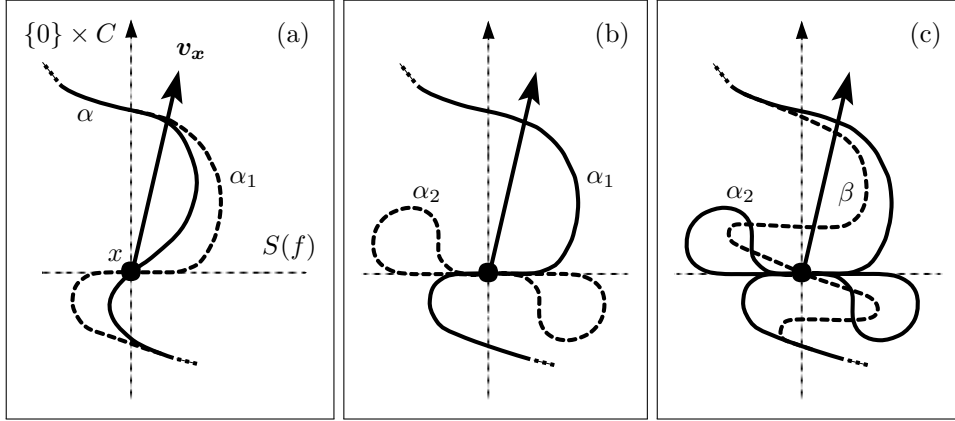


FIGURE 8. Modification of α in a small neighborhood of $\alpha^{-1}(x)$.

Proposition 3.8. *If ρ is any orientation of P , then $\omega_\sigma(f) \equiv \omega_\rho(f) \pmod{2}$.*

Proof. It suffices to show that $\omega_{\sigma|_C}(f|_{(-\varepsilon, \varepsilon) \times C}) \equiv \omega_{-\sigma|_C}(f|_{(-\varepsilon, \varepsilon) \times C}) \pmod{2}$ for every component C of P . For this purpose, note that if $\alpha: C \rightarrow (-\varepsilon, \varepsilon) \times C$ is a positive (σ, f) -adapted embedding, then $\beta := \alpha \circ \iota$ is a $(-\sigma, f)$ -adapted embedding such that $C_f^\beta = C_f$, where $\iota: C \rightarrow C$ denotes an orientation reversing automorphism. Hence, the claim follows from Lemma 3.7 because C_f has even cardinality by Lemma 3.1(b). \square

4. NUMBER OF CUSPS OF GENERIC EXTENSIONS

Let f be a boundary condition on $P = \partial W$. By an *extension of f to W* we mean in the following a generic map $F: W \rightarrow \mathbb{R}^2$ such that $F|_{[0, \varepsilon') \times \partial W} = f|_{[0, \varepsilon') \times \partial W}$ for some $\varepsilon' \in (0, \varepsilon)$. Note that any f admits an extension, and any two extensions have the same parity of number of cusps. The purpose of this section is to prepare the ingredients needed to derive a formula for this parity in terms of the boundary turning invariant (see Proposition 6.1).

Definition 4.1. Suppose that W is oriented, and let σ be the induced orientation on the boundary $P = \partial W$. For any extension $F: W \rightarrow \mathbb{R}^2$ of f let W_F^σ denote the closure in W of the union of components U of $W \setminus S(F)$ for which the local diffeomorphism $F|_U: U \rightarrow \mathbb{R}^2$ is orientation preserving.

Under the assumptions of Definition 4.1, note that W_F^σ is a compact 2-dimensional submanifold of W with corners, and the set of corners is $P_f = S(f) \cap P$. Moreover, we have $W = W_F^\sigma \cup W_F^{-\sigma}$ and $\partial W_F^\sigma \cap \partial W_F^{-\sigma} = S(F) = W_F^\sigma \cap W_F^{-\sigma}$. Note that every component of $S(F)$ is contained in the boundary of exactly two components of $W \setminus S(F)$. One of these components belongs to W_F^σ , and the other component belongs to $W_F^{-\sigma}$.

The following proposition relates the boundary turning invariant as introduced in Section 3 to the Euler characteristic.

Proposition 4.2. *Suppose that W is oriented, and let σ be the induced orientation on $P = \partial W$. Then, for any extension $F: W \rightarrow \mathbb{R}^2$ of f without cusps, we have*

$$\chi(W_F^\sigma) - \chi(W_F^{-\sigma}) = \omega_\sigma(f).$$

Consequently, it follows from $\chi(W_F^\sigma) + \chi(W_F^{-\sigma}) = \chi(W) + |P_f|/2$ that

$$\chi(W_F^\sigma) = (\chi(W) + |P_f|/2 + \omega_\sigma(f))/2.$$

Proof. Let V be a component of W_F^σ . If $V \cap \partial W = \emptyset$, then $V \subset \text{int } W$ is a submanifold with boundary $V \cap S(F)$, and we set $\tilde{V} = V$. If, however, $V \cap \partial W \neq \emptyset$, then $V \subset W$ is a submanifold with corners, and the set of corners is given by $V \cap P_f$. In this case, we define a smooth submanifold with boundary $\tilde{V} \subset V$ by smoothly cutting off the corners of V in sufficiently small neighborhoods of the corner points as indicated in Figure 9.

In any case, a theorem of Haefliger [8] implies that the winding number of the immersion $F|_{\partial \tilde{V}}: \partial \tilde{V} \rightarrow \mathbb{R}^2$ equals $\chi(V)$, where $\partial \tilde{V}$ is equipped with the orientation induced by that of $\tilde{V} \subset W$. Analogously, while keeping the same orientation on W , we can identify the Euler characteristic $\chi(V)$ of any component V of $W_F^{-\sigma}$ with the negative of the winding number of the immersion $F|_{\partial \tilde{V}}: \partial \tilde{V} \rightarrow \mathbb{R}^2$, where $\tilde{V} \subset V$ is a suitably chosen smooth submanifold with boundary. Hence,

$$\chi(W_F^\sigma) - \chi(W_F^{-\sigma}) = \sum_{V \subset W_F^\sigma \sqcup W_F^{-\sigma}} \text{winding number of } \partial \tilde{V},$$

where the sum runs through all components of $W_F^\sigma \sqcup W_F^{-\sigma}$. We observe that the contributions to the sum of winding numbers by adjacent components will cancel each other along common parts of their boundaries. Thus, only contributions of boundary parts contained in a neighborhood of $[0, \varepsilon') \times P$ for sufficiently small $\varepsilon' \in (0, \varepsilon)$ remain (compare Figure 9). (Our argument requires an extension of

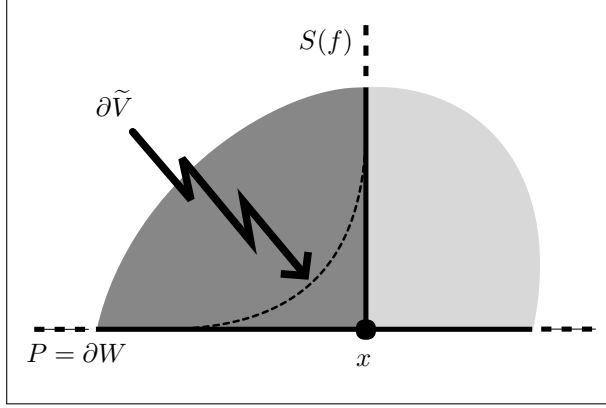


FIGURE 9. A neighborhood (grey) in $[0, \varepsilon') \times P$ of a corner point $x \in V \cap P_f$ of a component V of W_F^g (dark grey) for sufficiently small $\varepsilon' \in (0, \varepsilon)$. The boundary of \tilde{V} deviates from that of V as indicated only in small neighborhoods of the corners of V .

the concept of winding number to immersions of arcs into \mathbb{R}^2 . This generalized winding number for immersions arbitrary compact 1-manifolds turns out to behave additively under disjoint union and gluing of 1-manifolds along boundaries. By definition, the winding number of an immersion $\gamma: [a, b] \rightarrow \mathbb{R}^2$ is the real number $\lambda(b) - \lambda(a)$, where $\lambda: [a, b] \rightarrow \mathbb{R}$ denotes a lift of the Gauss map $[a, b] \rightarrow S^1$ of γ with respect to the universal cover $\mathbb{R} \rightarrow S^1, t \mapsto e^{2\pi t}$. We refer to Chapter 1 of [14] for details in the context of positive TFTs.) Finally, an argument in the spirit of the proof of Lemma 3.7 identifies the resulting sum of (generalized) winding numbers with $\omega_\sigma(f)$. \square

5. PATTERNS AND THEIR REALIZATIONS

Definition 5.1. A (singular) pattern on W is a pattern (f, φ) consisting of

- a fold map $f: (-\varepsilon, \varepsilon) \times P \rightarrow \mathbb{R}^2$ for some $\varepsilon > 0$ whose singular locus $S(f) \subset (-\varepsilon, \varepsilon) \times P$ (which is well-known to be a 1-dimensional submanifold) is transverse to $\{0\} \times P$, and
- a fixed-point free involution $\varphi: P_f \rightarrow P_f$ of the finite set $P_f := S(f) \cap (\{0\} \times P)$. (Note that φ can equivalently be considered as a partition of P_f into subsets of cardinality 2.)

Definition 5.2. A 1-dimensional submanifold $S \subset W$ satisfying $S \pitchfork \partial W$ and $\partial S = S \cap \partial W$ is said to be *adapted to a pattern* (f, φ) if for every $x \in P_f$ there exists a component S_x of S with boundary $\partial S_x = \{x, \varphi(x)\}$. By a *realization* of the pattern (f, φ) we mean a generic extension $F: W \rightarrow \mathbb{R}^2$ of $f|_{[0, \varepsilon') \times \partial W}$ for some $\varepsilon' \in (0, \varepsilon)$ such that the singular locus $S(F)$ is non-empty and adapted to φ .

Remark 5.3. Requiring that $S(F) \neq \emptyset$ for realizations F of the pattern (f, φ) excludes the case that F is an immersion. The extension problem for stable immersions $\partial W \rightarrow \mathbb{R}^2$ to immersions $W \rightarrow \mathbb{R}^2$ has been characterized combinatorially in [6].

Fix a pattern (f, φ) . A map $\iota: P_f \rightarrow \{-1, +1\}$ is defined by assigning to every point $x \in P_f$ a positive or a negative sign according to whether a non-zero tangent vector $u_x \in T_x S(f)$ which points into $(0, \varepsilon) \times \partial W$ determines the canonical orientation of $S(F)$ at x (see Definition 2.3) or its opposite orientation. If F is a realization of (f, φ) , then $S(F)$ is adapted to φ , and $\iota(x) \neq \iota(\varphi(x))$ for all $x \in P_f$.

Proposition 5.4. *The pattern (f, φ) admits a realization if and only if*

- (i) $\iota(x) \neq \iota(\varphi(x))$ for all $x \in P_f$, and
- (ii) *there exists a submanifold $S \subset W$ which is adapted to (f, φ) .*

Proof. It suffices to show that a realization F of (f, φ) exists under the given conditions. First, we produce a realization F' of some pattern (f, φ') by extending the germ of f at ∂W generically over W in an arbitrary way. Then, for every point $x \in P_f$, we use (C) once to create a new pair of cusps sufficiently close to x on $S(F) \setminus \partial W$. Next, for every pair $\{x, y\} \subset P_f$ of the partition φ we eliminate the cusp of F which is closest to x and the cusp of F which is closest to y by applying general cusp elimination (Proposition 2.4) along a path in $W \setminus \partial W$ that connects these two cusps, and whose interior intersects $S(F)$ transversely and only in fold points of F . As a result, in view of assumption (i), the singular locus of the modified generic map \tilde{F} will contain a component with boundary $\{x, y\}$, and which contains no fold points. Finally, note that assumption (ii) makes sure that we can perform all these cusp eliminations independently of each other along pairwise disjoint paths in $W \setminus \partial W$. □

6. PROOF OF THEOREM 1.1

Part (a) follows immediately from the following

Proposition 6.1. *Given an orientation σ of $P = \partial W$, the number of cusps of any realization $F: W \rightarrow \mathbb{R}^2$ of a given pattern (f, φ) on W has the same parity as the quantity*

$$\Gamma^\sigma := \chi(W) + |P_f|/2 + \omega_\sigma(f).$$

Proof. Let $F: W \rightarrow \mathbb{R}^2$ be an extension of f . First we show in part (1) of the proof that Γ^σ is even when F has an even number of cusps. This will then be used in part (2) of the proof to show that Γ^σ is odd when F has an odd number of cusps.

(1). Suppose that F has an even number of cusps.

First suppose that W is orientable. Then, we may assume by Proposition 3.8 that W is oriented in such a way that σ is the induced orientation of P . As F has an even number of cusps, we may assume by Proposition 2.4 that F has no cusps. Then, the claim that Γ^σ is even follows from Proposition 4.2.

Now suppose that W is non-orientable. Let $C \subset \text{int } W$ be an embedded circle whose normal line bundle $\nu: N \rightarrow C$ is non-trivial, where $N \subset W$ denotes a tubular neighborhood of C in W . Without loss of generality we may assume that C is transverse to $S(F)$, and avoids the cusps of F . If we cut W along C , then the resulting manifold W_0 has one additional boundary component C_0 . By construction there is a quotient map $\pi: W_0 \rightarrow W$ that restricts to a diffeomorphism $W_0 \setminus C_0 \xrightarrow{\cong} W \setminus C$, and to a covering map $\gamma: C_0 \rightarrow C$ of degree 2. The pullback bundle of $\nu: N \rightarrow C$ under $\gamma: C_0 \rightarrow C$ is the trivial line bundle over C_0 , and γ can be covered by a canonical bundle map $\tilde{\gamma}: \mathbb{R} \times C_0 \rightarrow N$ such that $\tilde{\gamma}(r, -x) = -\tilde{\gamma}(r, x)$,

where $-x$ denotes the unique element in the fiber $\gamma^{-1}(\gamma(x))$ that is distinct from $x \in C_0$. Moreover, there exists a collar $[0, \infty) \times C_0 \subset W_0$ of C_0 in W_0 such that $\pi: W_0 \rightarrow W$ and $\tilde{\gamma}: \mathbb{R} \times C_0 \rightarrow N$ restrict to the same map $[0, \infty) \times C_0 \rightarrow N$. Let ρ denote an orientation of C_0 . Then, writing $g := F \circ \tilde{\gamma}: \mathbb{R} \times C_0 \rightarrow \mathbb{R}^2$, it follows from Lemma 3.7 that $\omega_\rho(g)$ has the same parity as the cardinality of $C \cap S(F)$. (In fact, this can be shown by choosing an embedding $\alpha: C \rightarrow N$ for which the lift of $\alpha \circ \gamma$ to a map $\tilde{\alpha}: C_0 \rightarrow \mathbb{R} \times C_0$ is a (ρ, g) -adapted embedding in the sense of Definition 3.2.) The quotient map $\pi: W_0 \rightarrow W$ restricts to a diffeomorphism $\pi_0: P_0 = \pi^{-1}(P) \xrightarrow{\cong} P$. Let σ_0 denote the orientation of P_0 induced by σ via π_0 . The collar $[0, \infty) \times P \subset W$ induces a collar $[0, \infty) \times P_0 \subset W_0$ by means of $(t, x) \mapsto \pi^{-1}(t, \pi_0(x))$. Let $f_0 = f \circ \pi|_{[0, \infty) \times P_0}$. Then,

$$\omega_{\sigma_0 \sqcup \rho}(f_0 \sqcup g) = \omega_{\sigma_0}(f_0) + \omega_\rho(g) \equiv \omega_\sigma(f) + |C \cap S(F)| \pmod{2}.$$

From now on, we assume that C is chosen in such a way that W_0 is orientable. Then, since $F_0 := F \circ \pi$ is an extension of $f_0 \sqcup g$ to W_0 with an even number of cusps, we have already shown above that

$$\chi(W_0) + |(P_0 \sqcup C_0)_{f_0 \sqcup g}|/2 + \omega_{\sigma_0 \sqcup \rho}(f_0 \sqcup g) \equiv 0 \pmod{2}.$$

By construction, $\chi(W_0) = \chi(W)$. Moreover, note that we have

$$|(P_0 \sqcup C_0)_{f_0 \sqcup g}| = |P_f| + 2|C \cap S(F)|.$$

All in all, this completes the proof of part (1).

(2). Suppose that F has an odd number of cusps, and let c denote a cusp of F . Let $U \subset \text{int } W$ be a small open disc centered at c in which F looks like the stable Whitney cusp discussed in Example 3.6. Recall that $\omega_\rho(f_{\text{cusp}}) = \pm 1$, where ρ denotes an orientation of $\partial U \cong S^1$. For $W_0 = W \setminus U$ and the restriction $F_0 = F|_{W_0}$ we know by part (1) that

$$\chi(W_0) + |(\partial W_0)_{f \sqcup f_{\text{cusp}}}|/2 + \omega_{\sigma \sqcup \rho}(f \sqcup f_{\text{cusp}}) \equiv 0 \pmod{2}.$$

By construction, $\chi(W) = \chi(W_0) + 1$, $|(\partial W_0)_{f \sqcup f_{\text{cusp}}}| = |P_f| + 2$, and $\omega_{\sigma \sqcup \rho}(f \sqcup f_{\text{cusp}}) = \omega_\sigma(f) \pm 1$. Thus, the claim follows. \square

Remark 6.2. Compare Proposition 6.1 to the Fukuda-Ishikawa theorem (see Corollary 1.2 in [7, p. 377]), where the target manifold is a compact surface N , and the extension $F: W \rightarrow N$ of boundary conditions restricts to a Morse function $\partial W \rightarrow \partial N$. In this context, in place of the boundary turning invariant, one has to use the related notion of degree of a continuous map $S^1 \rightarrow S^1$.

As for the proof of part (b) of Theorem 1.1, let $F: W \rightarrow \mathbb{R}^2$ be a realization of the pattern (f, φ) . An inspection of the proof of Proposition 5.4 shows that F can be modified on $W \setminus \partial W$ by a finite sequence of (E) and (C) moves to obtain a realization F_1 of (f, φ) whose cusps lie all on loops of F_1 . Let $T \subset W$ denote the union of the components of $S(F_1)$ that are diffeomorphic to the unit interval. Using general cusp elimination (Proposition 2.4), we can modify F_1 on $W \setminus (T \cup \partial W)$ by a finite sequence of (E) and (C) moves to obtain a realization F_2 of (f, φ) such that each component of $W \setminus T$ contains at most one cusp of F_2 . If $T = \emptyset$, then F_2 has by construction at most one cusp as desired. Otherwise, fix a component J of T . Note that every cusp x of F_2 is the unique cusp on some loop of F_2 , say C_x . We may also assume that C_x is trivial (see Definition 2.1). (In fact, if the normal line bundle of $C_x \subset W$ is non-trivial, then we apply (C) once at a fold point of F_2 on

C_x to create a new pair of cusps, and then we apply (E) to a joining curve between these two cusps that goes around once in a small tubular neighborhood of C_x . The resulting new loop containing the cusp will then be trivial.) By a finite iteration of the loop tunneling move (Proposition 2.8) we can modify F_2 on $W \setminus (J \cup \partial W)$ by a finite sequence of (E) and (C) moves to obtain a realization F_3 of (f, φ) such that every cusp c of F_3 is the unique cusp on some loop of F_3 , and there exists a path $\alpha_c: [0, 1] \rightarrow W \setminus \partial W$ such that $\alpha_c^{-1}(S(F_3)) = \{0, 1\}$ and $\alpha_c(0) = c$, $\alpha_c(1) \in J$. Furthermore, we may assume that for every cusp c of F_3 the vector $\alpha'_c(0)$ is pointing downward at c (see Section 2). (In fact, this can always be achieved by one (C) move followed by one (E) move near c .) Two cusps c and d of F_3 for which $\alpha'_c(1)$ and $\alpha'_d(1)$ point to different sides of J can be eliminated by first applying (C) to generate a new cusp pair on J , and then applying (E) twice along joining curves given by suitable perturbations of α_c and α_d . Repeating this process until no such cusps pairs are left, the remaining cusps lie all in the same component of $W \setminus T$, and can thus be eliminated by means of general cusp elimination (Proposition 2.4) up to at most one cusp.

This completes the proof of Theorem 1.1.

7. REALIZATIONS AND NUMBER OF LOOPS

Throughout the present section, we suppose that W is orientable, and that σ denotes an orientation of $P = \partial W$ which is induced by one of the two orientations of W . The statement of our main result Theorem 1.2(b) involves the following quantities derived from a pattern (f, φ) on W and the fixed orientation σ of ∂W :

- $n_\sigma(f)$ denotes the number of components C of ∂W with the property that f restricts to an orientation preserving immersion $(-\varepsilon', \varepsilon') \times C \rightarrow \mathbb{R}^2$ for some $\varepsilon' \in (0, \varepsilon)$. (Here, we assume that $(-\varepsilon', \varepsilon') \times C$ is oriented in such a way that the inclusion $[0, \varepsilon') \times C \subset W$ is orientation preserving. Thus, $(-\varepsilon', \varepsilon') \times C$ is equipped with the orientation opposite to the product orientation.)
- $c_\sigma(f, \varphi)$ denotes the number of components of the closed 1-manifold $Z_\sigma(f, \varphi)$ obtained as follows. Let $\pi_\sigma(f)$ be the fixed-point free involution of $P_f = (\{0\} \times P) \cap S(f)$ that corresponds to the partition of P_f into pairs of the form $P_f \cap \bar{V}$, where V runs through the components of $(-\varepsilon, \varepsilon) \times \partial W \setminus S(f)$ on which f is an orientation preserving immersion. Now, $Z_\sigma(f, \varphi)$ is obtained from the finite set P_f (considered as a 0-dimensional CW-complex) by attaching for every member $\{x, y\}$ of the partitions of P_f induced by φ and $\pi_\sigma(f)$ a 1-cell with endpoints x and y .

The quantity $c_\sigma(f, \varphi)$ captures the combinatorial interplay between φ and $S(f)$ for given σ .

By construction we have the following

Proposition 7.1. *Let (f, φ) be a pattern on W , and let F be a realization of (f, φ) without cusps. Then, the number of boundary components of W_F^σ (see Definition 4.1) equals $c_\sigma(f, \varphi) + l_F + n_\sigma$, where l_F is the number of loops of F . Moreover, there exists an integer $h_F^\sigma \geq 0$ such that*

$$\chi(W_F^\sigma) = c_\sigma(f, \varphi) + l_F + n_\sigma - 2h_F^\sigma.$$

Furthermore, if $P_f \neq \emptyset$, then there exists a closed one-dimensional submanifold $Z \subset W_F^\sigma \setminus \partial W_F^\sigma$ such that the number of components of Z is $h_F^\sigma - n_\sigma$ (≥ 0), and such that every component of $W_F^\sigma \setminus Z$ has nonempty intersection with $S(F)$.

Proof. It follows from the definitions of $c_\sigma(f, \varphi)$ and n_σ that the number of boundary components of W_F^σ is $c_\sigma(f, \varphi) + l_F + n_\sigma$. To get the formula for $\chi(W_F^\sigma)$, choose h_F^σ to be the number of surgeries on embedded 0-spheres S^0 that are necessary to obtain W_F^σ from the disjoint union of $c_\sigma(f, \varphi) + l_F + n_\sigma$ copies of D^2 , the closed unit disc in \mathbb{R}^2 .

It remains to construct the desired submanifold $Z \subset W_F^\sigma \setminus \partial W_F^\sigma$ under the assumption that $P_f \neq \emptyset$. For this purpose, note that $c_\sigma(f, \varphi) + l_F$ is the number of boundary components of W_F^σ which have nonempty intersection with $S(F)$. Let X and Y denote the disjoint unions of $c_\sigma(f, \varphi) + l_F$ and n_σ copies of D^2 , respectively. Recall from the construction of h_F^σ that there exist pairwise disjoint embeddings

$$\varphi_i: S^0 \hookrightarrow (X \setminus \partial X) \sqcup (Y \setminus \partial Y), \quad i = 1, \dots, h_F^\sigma,$$

such that W_F^σ can be obtained from the disjoint union $X \sqcup Y$ by means of surgeries on the embeddings φ_i , and ∂X becomes the union of those boundary components of W_F^σ that have nonempty intersection with $S(F)$. Since W is connected and $P_f \neq \emptyset$ by assumption, every component of W_F^σ must contain at least one boundary component which has nonempty intersection with $S(F)$. Thus, by moving some of the handles $S^1 \times [-1, 1] \subset W_F^\sigma$ associated with the embeddings φ_i over each other, we may achieve that for every component D of Y , there is an index i_D such that the embedding φ_{i_D} maps one point of S^0 to X and the other point to D . In particular, $h_F^\sigma - n_\sigma \geq 0$, and we may take Z to be the union of the circles $S^1 \times \{0\} \subset S^1 \times [-1, 1]$ in the handles associated with the remaining $h_F^\sigma - n_\sigma$ embeddings. \square

The statement of our main result Theorem 1.2 involves the quantity

$$\Delta^\sigma := \Gamma^\sigma / 2 - c_\sigma(f, \varphi) + n_\sigma(f),$$

which also depends on Γ^σ as introduced in Proposition 6.1. Note that Proposition 4.2 and Proposition 7.1 together imply

Corollary 7.2. *If F is a realization of a pattern (f, φ) of W without cusps, then*

$$\Delta^\sigma = l_F - 2h_F^\sigma + 2n_\sigma.$$

8. PROOF OF THEOREM 1.2

First suppose that W is non-orientable. In view of loop generation (Proposition 2.2(i)) and loop reduction (Proposition 2.7) it suffices to show that F can be modified on $W \setminus \partial W$ by a finite number of (E) and (C) moves to obtain a realization \tilde{F} of (f, φ) without cusps in such a way that the number of loops of F does not have the same parity as the number of loops of \tilde{F} . For this purpose, let $C \subset W \setminus \partial W$ be an embedded circle which intersects $S(F)$ transversely, say in the points p_1, \dots, p_r . We modify F by applying (C) once to create a pair of cusps at p_1 . Next, we modify the resulting realization F_1 of (f, φ) by applying general elimination of cusps (Proposition 2.4) to a curve α that connects the two new cusps by going once around in a tubular neighborhood of C while intersecting $S(F_1)$ in $r - 1$ fold points. Note that the total number of (E) moves involved in the modification of F_1 equals r . As a result, we obtain an extension F_2 of f without cusps. However, the (E) moves we used to modify F_1 may have changed the singular pattern, so that F_2 is not necessarily a realization of the pattern (f, φ) . Nevertheless, we can modify F_2 further to obtain the desired realization \tilde{F} of the pattern (f, φ) as follows. Every (E) move we have used can be considered as the first step of loop

generation (Proposition 2.2(ii)), and we perform the remaining moves. All in all, we obtain a realization \tilde{F} of the pattern (f, φ) without cusps, but the number of loops has increased by r due to r -fold application of loop generation. Hence, our claim follows from the fact that r will be odd whenever C is chosen such that its normal line bundle is non-trivial. This completes the proof of part (a).

As for the proof of part (b), suppose that W is orientable. Let σ denote an orientation of ∂W which is induced by one of the two orientations of W . Moreover, let $P_f \neq \emptyset$.

(i) \Rightarrow (ii). Let F be a realization of the pattern (f, φ) which has no cusps and l_F loops. Then, Corollary 7.2 implies that $\Delta^\rho \equiv l_F \pmod{2}$ and $\Delta^\rho = l_F - 2(h_F^\rho - n_\rho) \leq l_F$ for $\rho \in \{-\sigma, \sigma\}$, where $h_F^\rho \geq n_\rho$ by Proposition 7.1. Consequently, $l_F \in \mathbb{N} \cap (\Delta^\sigma + 2\mathbb{N}) \cap (\Delta^{-\sigma} + 2\mathbb{N})$.

(ii) \Rightarrow (i). Let $l \in \mathbb{N} \cap (\Delta^\sigma + 2\mathbb{N}) \cap (\Delta^{-\sigma} + 2\mathbb{N})$. In view of loop generation (Proposition 2.2(i)) it suffices to show that F can be modified on $W \setminus \partial W$ by a finite sequence of (E) and (C) moves to a realization of (f, φ) which has no cusps and $\min(\mathbb{N} \cap (\Delta^\sigma + 2\mathbb{N}) \cap (\Delta^{-\sigma} + 2\mathbb{N}))$ loops. This can be achieved as follows. Since $P_f \neq \emptyset$, we can use loop simplification (Proposition 2.6) to achieve that all loops of F are contractible. Using tunneling (Proposition 2.8) and loop reduction (Proposition 2.7), one can in addition achieve that the remaining loops of F are boundary components of the same component of $W \setminus S(F)$, say a component V of W_F^ρ for suitable orientation $\rho \in \{\pm\sigma\}$. Since all loops of F are contractible, it follows that every loop bounds a contractible component of $W_F^{-\rho}$. In this situation, an argument analogous to the last part of the proof of Proposition 7.1 implies that

$$l_F + n_\rho \leq h_F^\rho.$$

(In fact, note that $c_\rho(f, \varphi)$ is an upper bound for the number of components of W_F^ρ because all loops of F lie in the boundary of V .) Hence, by Corollary 7.2,

$$\Delta^\rho = l_F - 2h_F^\rho + 2n_\rho \leq 2(l_F - h_F^\rho + n_\rho) \leq 0.$$

Therefore, using $\Delta^\rho \equiv l_F \equiv \Delta^{-\rho} \pmod{2}$, we obtain

$$\min(\mathbb{N} \cap (\Delta^\sigma + 2\mathbb{N}) \cap (\Delta^{-\sigma} + 2\mathbb{N})) = \min(\mathbb{N} \cap (\Delta^{-\rho} + 2\mathbb{N})).$$

Finally, we modify F by means of balancing (Proposition 2.9) to reduce the number of loops to $\min(\mathbb{N} \cap (\Delta^{-\rho} + 2\mathbb{N}))$ as follows. By Proposition 7.1, there is a closed one-dimensional submanifold $Z \subset W_F^{-\rho} \setminus \partial W_F^{-\rho}$ that consists of $b := h_F^{-\rho} - n_{-\rho} \geq 0$ components, and such that every component of $W_F^{-\rho} \setminus Z$ has nonempty intersection with $S(F)$. For every component H of Z we may then choose an embedding $\alpha_H: [0, 1] \rightarrow W \setminus \partial W$ with the following properties:

- The endpoints of α_H satisfy $\alpha_H(0), \alpha_H(1) \in W_F^\rho \setminus (\partial W \cup S(F))$.
- α_H intersects $S(F)$ transversely, and $\alpha_H^{-1}(S(F)) = \{t_0^H, t_1^H\}$ for some $0 < t_0^H < t_1^H < 1$.
- $\alpha_H(t_0^H)$ and $\alpha_H(t_1^H)$ are fold points of F that do not lie on the loops of F . (Note that the components of $W_F^{-\rho}$ that contain a loop of F in the boundary are disjoint to C because the loops of F are all contractible.)
- α_H intersects H transversely, and $\alpha_H^{-1}(H) = \{s^H\}$ for some $s^H \in (t_0^H, t_1^H)$.
- If H and H' are different components of Z , then $\alpha_H([0, 1]) \cap \alpha_{H'}([0, 1]) = \emptyset$.

If $\Delta^{-\rho} > 0$, i.e. $l_F > 2(h_F^{-\rho} - n_{-\rho}) = 2b$, then we perform b (≥ 0) balancing moves to reduce the number of loops to $\Delta^{-\rho} = \min(\mathbb{N} \cap (\Delta^{-\rho} + 2\mathbb{N}))$ as required. (More

precisely, for every component H of Z we first apply tunneling (Proposition 2.8) to move an unused pair (C_0^H, C_1^H) of loops of F to the endpoints of α_H , that is, $\alpha_H^{-1}(C_i^H) = \{i\}$ for $i = 0, 1$. This is always possible since W is connected, and all loops of F are contractible, and hence trivial. The desired balancing moves can then be performed because the assumptions of Proposition 2.9 are satisfied for our choices of H and α_H .) If, however, $\Delta^{-\rho} \leq 0$, i.e. $l_F \leq 2(h_F^{-\rho} - n_{-\rho}) = 2b$, then let $\varepsilon \in \{0, 1\}$ such that $\varepsilon \equiv l_F \pmod{2}$. Then, we can analogously perform $(l_F - \varepsilon)/2$ balancing moves to reduce the number of loops to $\varepsilon = \min(\mathbb{N} \cap (\Delta^{-\rho} + 2\mathbb{N}))$.

This completes the proof of Theorem 1.2.

Remark 8.1. Our approach ignores the complexity of the *apparent contour* $F(S(F))$ of a generic map $F: W \rightarrow \mathbb{R}^2$. In particular, when $\partial W = \emptyset$ and F is *stable* (i.e., the immersion $F|: S(F) \rightarrow \mathbb{R}^2$ self-intersects only at fold points of F , and all self-intersections are nodes, that is, transverse double points), results including the number of nodes of F can for instance be found in [16]. It might be interesting to think about a version of Theorem 1.2 for stable maps of surfaces with boundary into the plane (compare [3]) that also includes the number of nodes. Note that the number of nodes cannot directly be controlled by our algorithm since the (E) and (C) moves easily produce new nodes.

9. APPLICATIONS

Throughout this section we suppose W to be orientable (the non-orientable case could be handled similarly). Let σ denote an orientation of $P = \partial W$ which is induced by one of the two orientations of W .

In this section the spirit of the gluing principle is reflected in our applications of Theorem 1.2 to pseudo-immersions (see Proposition 9.1 and Example 9.2) and cusp counting (see Proposition 9.4).

Proposition 9.1. *Let F be a realization of the pattern (f, φ) without cusps. Then, the following statements are equivalent for any integer $l \geq 1$:*

- (i) *F can be modified on $W \setminus \partial W$ by a finite sequence of (E) and (C) moves to a realization of (f, φ) which has no cusps and l loops.*
- (ii) *$l \in 1 + \mathbb{N} \cap (\Delta^\sigma - 1 + 2\mathbb{N}) \cap (\Delta^{-\sigma} - 1 + 2\mathbb{N})$.*

Proof. We present the proof of implication (i) \Rightarrow (ii) in detail – the proof of the converse implication is similar.

Let G be a realization of (f, φ) which has no cusps and l loops. Let C be the boundary of a small open disc U centered at a fold point on a loop of G such that C intersects $S(G)$ transversely in precisely two points. Then, the surface $V := W \setminus U$ has boundary $\partial V = P \sqcup C$, which we equip with an orientation ρ such that $\rho|_{\partial W} = \sigma$, and such that ρ is induced by an orientation of V . (If $\partial W \neq \emptyset$, then ρ is uniquely determined by σ .) Now $H := G|_V$ is a fold map on V with $l - 1$ loops. If we equip C with a suitable collar neighborhood in V , then H can be considered as a realization of the pattern $(f \sqcup f_{\text{fold}}, \varphi \sqcup \varphi_2)$, where f_{fold} is taken from Example 3.6, and φ_2 denotes the unique fixed-point free involution on $C \cap S(G)$. Hence, using $\chi(W) = \chi(V) + 1$, $|(P \sqcup C)_{f \sqcup f_{\text{fold}}}| = |P_f| + 2$, and $\omega_{\pm\rho}(f \sqcup f_{\text{fold}}) = \omega_{\pm\sigma}(f)$ (see Example 3.6), we obtain

$$\Gamma_V^{\pm\rho} := \chi(V) + |(P \sqcup C)_{f \sqcup f_{\text{fold}}}|/2 + \omega_{\pm\rho}(f \sqcup f_{\text{fold}}) = \Gamma^{\pm\sigma}.$$

Moreover, $n_{\pm\rho}(f \sqcup f_{\text{fold}}) = n_{\pm\sigma}(f)$ and $c_{\pm\rho}(f \sqcup f_{\text{fold}}) = c_{\pm\sigma}(f) + 1$ imply

$$\Delta_V^{\pm\rho} := \Gamma_V^{\pm\rho}/2 - c_{\pm\rho}(f \sqcup f_{\text{fold}}) + n_{\pm\rho}(f \sqcup f_{\text{fold}}) = \Delta^{\pm\sigma} - 1.$$

Finally, the implication (i) \Rightarrow (ii) of Theorem 1.2(b) implies that $l - 1 \in \mathbb{N} \cap (\Delta^\sigma - 1 + 2\mathbb{N}) \cap (\Delta^{-\sigma} - 1 + 2\mathbb{N})$. \square

Example 9.2 (Pseudo-immersions). An important special case of Proposition 9.1 arises when f restricts to an immersion $(-\varepsilon', \varepsilon') \times \partial W \rightarrow \mathbb{R}^2$ for some $\varepsilon' \in (0, \varepsilon)$. In this case, realizations F of the pattern (f, φ_\emptyset) (where φ_\emptyset denotes the unique permutation of the empty set \emptyset) are called *pseudo-immersions* (note that $S(F) \neq \emptyset$ by Remark 5.3). The case that ∂W is connected has been considered in [15, Theorems 1.2 and 1.3], and Proposition 9.1 can be reduced to it as follows.

The assumption on f implies that $P_f = \emptyset$ and $\omega_\sigma(f) = W(F|_{\partial W})$, the winding number (see Definition 3.5). Hence, $\Gamma^\sigma = \chi(W) + W(F|_{\partial W})$ by Proposition 6.1. Moreover, $n_\sigma(f) = 1$ (for suitable σ), and $c_\sigma(f, \varphi_\emptyset) = 0$. Thus, we obtain

$$\Delta^\sigma = (\chi(W) + W(F|_{\partial W}))/2 + 1 = ((\chi(W) + 1) + (W(F|_{\partial W}) + 1))/2.$$

On the other hand, observe that $\omega_{-\sigma}(f) = -\omega_\sigma(f) = -W(F|_{\partial W})$, $n_{-\sigma}(f) = 0$, and $c_{-\sigma}(f, \varphi_\emptyset) = 0$, which yields

$$\Delta^{-\sigma} = (\chi(W) - W(F|_{\partial W}))/2 = ((\chi(W) + 1) - (W(F|_{\partial W}) + 1))/2.$$

Altogether, $\mathbf{m} := \max\{\Delta^\sigma, \Delta^{-\sigma}\} = (\chi(W) + 1 + |W(F|_{\partial W}) + 1|)/2$, and by Proposition 9.1, in accordance with [15], the set of possible numbers of loops of F is

$$1 + \mathbb{N} \cap (\mathbf{m} - 1 + 2\mathbb{N}) = \begin{cases} \max(\mathbf{m}, 2) + 2\mathbb{N}, & \text{if } \chi(W) - W(F|_{\partial W}) \equiv 0 \pmod{4}, \\ \max(\mathbf{m}, 1) + 2\mathbb{N}, & \text{if } \chi(W) - W(F|_{\partial W}) \equiv 2 \pmod{4}. \end{cases}$$

Remark 9.3. Note that the proof of [15, Theorem 1.3] relies on a theorem due to Eliashberg [4] and Francis [5] (see [15, Theorem 3.2, p. 1330]). On the other hand, our approach is purely based on (E) and (C) moves, and we can easily deduce the Eliashberg-Francis theorem from Proposition 9.1.

Proposition 9.4. *Let F be a realization of the pattern (f, φ) . If $P_f \neq \emptyset$, then the following statements are equivalent for any integers $c > 0$ and l :*

- (i) *F can be modified on $W \setminus \partial W$ by a finite sequence of (E) and (C) moves to a realization of (f, φ) which has c cusps and l loops.*
- (ii) *$l \in \bigcup_{w \in \{-c, -c+2, \dots, c-2, c\}} \mathbb{N} \cap (\Delta^\sigma + w/2 + 2\mathbb{N}) \cap (\Delta^{-\sigma} - w/2 + 2\mathbb{N})$.*

Proof. The proof is based on similar ideas as the proof of Proposition 9.1.

(i) \Rightarrow (ii). Let G be a realization of (f, φ) which has $c \geq 1$ cusps and l loops. Let $U \subset W$ be the union of c small open discs each of which is centered at a different cusp of G . By choosing those small discs appropriately, we may assume that the boundary C of U is diffeomorphic to the disjoint union of c circles each of which intersects $S(G)$ transversely in precisely two points. Then, the surface $V := W \setminus U$ has boundary $\partial V = P \sqcup C$, which we equip with the unique orientation ρ such that $\rho|_{\partial W} = \sigma$ (where note that $\partial W \neq \emptyset$), and such that ρ is induced by an orientation of V . Now $H := G|_V$ is a fold map on V with $l - d$ loops, where d denotes the number of cusps of G that lie on a loop of G . If we equip C with a suitable collar neighborhood in V , then H is near every component of C an extension of the boundary condition f_{cusp} considered in Example 3.6. Thus, H can be considered as a realization of the pattern $(f \sqcup g, \varphi \sqcup \varphi_C)$, where g denotes the disjoint union of c

copies of f_{cusp} , and φ_C denotes the unique fixed-point free involution on $C \cap S(G)$ that interchanges for every component C_0 of C the two points in $C_0 \cap S(G)$. Hence, using $\chi(W) = \chi(V) + c$, $|(P \sqcup C)_{f \sqcup g}| = |P_f| + 2c$, and $\omega_{\pm\rho}(f \sqcup g) = \omega_{\pm\sigma}(f) \pm w$ for some integer w with the same parity as c , and such that $|w| \leq c$, we obtain

$$\Gamma_V^{\pm\rho} := \chi(V) + |(P \sqcup C)_{f \sqcup g}|/2 + \omega_{\pm\rho}(f \sqcup g) = \Gamma^{\pm\sigma} \pm w.$$

Moreover, $n_{\pm\rho}(f \sqcup g) = n_{\pm\sigma}(f)$ and $c_{\pm\rho}(f \sqcup g) = c_{\pm\sigma}(f) + d$ imply

$$\Delta_V^{\pm\rho} := \Gamma_V^{\pm\rho}/2 - c_{\pm\rho}(f \sqcup g) + n_{\pm\rho}(f \sqcup g) = \Delta^{\pm\sigma} \pm w/2 - d.$$

Finally, to conclude that (ii) holds, we apply the implication (i) \Rightarrow (ii) of Theorem 1.2(b) to the realization H of the pattern $(f \sqcup g, \varphi \sqcup \varphi_C)$ to obtain

$$l - d \in \mathbb{N} \cap (\Delta^\sigma + w/2 - d + 2\mathbb{N}) \cap (\Delta^{-\sigma} - w/2 - d + 2\mathbb{N}).$$

(ii) \Rightarrow (i). We may choose an integer $w \in \{-c, -c+2, \dots, c-2, c\}$ such that

$$l \in \mathbb{N} \cap (\Delta^\sigma + w/2 + 2\mathbb{N}) \cap (\Delta^{-\sigma} - w/2 + 2\mathbb{N}).$$

We modify F as follows. After possibly creating a number of new pairs of cusps for F by means of (C), we can choose cusps x_1, \dots, x_c of F admitting pairwise disjoint small open disc neighborhoods U_1, \dots, U_c , respectively, where the boundary circle C_i of each U_i intersects $S(F)$ transversely in precisely two points, and the following property holds. If f_i denotes the restriction of F to some fixed collar neighborhood of C_i in $V := W \setminus \bigsqcup_i U_i$, and ρ denotes the unique orientation of $\partial V = \partial W \sqcup \bigsqcup_i C_i$ that is induced by an orientation of V in such a way that $\rho|_{\partial W} = \sigma$ (where note that $\partial W \neq \emptyset$), then

$$w = \sum_{i=1}^c \omega_{\rho|_{C_i}}(f_i).$$

(Indeed, since $w \in \{-c, -c+2, \dots, c-2, c\}$ can clearly be written as the sum of c summands of the form ± 1 , we only have to apply the following observation.

If we create a new pair (x, x') of cusps for F by means of (C) and choose sufficiently small open disc neighborhoods U and U' of x and x' , respectively, whose respective boundary circles C and C' intersect $S(F)$ transversely in precisely two points, then the following property holds. If g and g' denote the restrictions of F to some fixed collar neighborhoods of C and C' in $Y = W \setminus (U \sqcup U')$, respectively, and τ denotes the unique orientation of $\partial Y = \partial W \sqcup C \sqcup C'$ that is induced by an orientation of Y in such a way that $\tau|_{\partial W} = \sigma$, then $\omega_{\tau|_C}(g), \omega_{\tau|_{C'}}(g') \in \{\pm 1\}$ and $\omega_{\tau|_C}(g) = -\omega_{\tau|_{C'}}(g')$.

In fact, the statement $\omega_{\tau|_C}(g), \omega_{\tau|_{C'}}(g') \in \{\pm 1\}$ follows because we can choose the neighborhoods U and U' so that C and C' have collar neighborhoods in $Y = W \setminus (U \sqcup U')$, respectively, in which g and g' look like copies of the boundary condition f_{cusp} considered in Example 3.6. Moreover, the statement $\omega_{\tau|_C}(g) = -\omega_{\tau|_{C'}}(g')$ can be deduced as follows. Since the pair (x, x') of cusps has been created by means of (C), we can choose the closures of U and U' to lie in a small closed disc $Z \subset W \setminus \partial W$ such that the boundary circle ∂Z intersects $S(F)$ transversely in precisely two points, and the restriction of F to some fixed collar neighborhood of ∂Z in $W \setminus Z$ looks like the boundary condition f_{fold} considered in Example 3.6. Then the claim follows by applying Proposition 4.2 to $F|_{Z \setminus (U \sqcup U')}$.

The restriction $H := F|_V$ is a generic map on V that can be considered as a realization of the pattern $(h, \psi) := (f \sqcup \bigsqcup_i f_i, \varphi \sqcup \bigsqcup_i \varphi_i)$, where φ_i denotes the

unique fixed-point free involution on $C_i \cap S(F)$. Hence, using $\chi(W) = \chi(V) + c$, $|(\partial V)_h| = |P_f| + 2c$, as well as $\omega_{\pm\rho}(h) = \omega_{\pm\sigma}(f) \pm w$, we obtain

$$\Gamma_V^{\pm\rho} := \chi(V) + |(\partial V)_h|/2 + \omega_{\pm\rho}(h) = \Gamma^{\pm\sigma} \pm w.$$

Without loss of generality, we may assume that the cusps x_1, \dots, x_c lie on components of $S(F)$ that have nonempty intersection with ∂W . Consequently, $c_{\pm\rho}(h) = c_{\pm\sigma}(f)$. Using also that $n_{\pm\rho}(h) = n_{\pm\sigma}(f)$, we obtain

$$\Delta_V^{\pm\rho} := \Gamma_V^{\pm\rho}/2 - c_{\pm\rho}(h) + n_{\pm\rho}(h) = \Delta^{\pm\sigma} \pm w/2.$$

Note that Γ_V^ρ is even because $\Delta_V^\rho = \Delta^\sigma + w/2$ is an integer by assumption. Hence, Proposition 6.1 implies that H has an even number of cusps. Thus, according to Theorem 1.1(b), H can be modified on $V \setminus \partial V$ by a finite sequence of (E) and (C) moves to obtain a realization H_0 of (h, ψ) without cusps. Since

$$l \in \mathbb{N} \cap (\Delta_V^\rho + 2\mathbb{N}) \cap (\Delta_V^{-\rho} + 2\mathbb{N}),$$

we may apply the implication $(ii) \Rightarrow (i)$ of Theorem 1.2(b) to modify the realization H_0 of the pattern (h, ψ) by a finite sequence of (E) and (C) moves to a realization of (h, ψ) which has no cusps and l loops. The above modifications of H and H_0 can be understood as modifications of F on $W \setminus \partial W$ by a finite sequence of (E) and (C) moves that turn F into a realization of (f, φ) which has c cusps and l loops. \square

Remark 9.5. The work on this paper has been motivated by Banagl's recent construction of TFT-type invariants [2]. In fact, Banagl has introduced the notion of a positive TFT as a convenient framework for constructing TFT-type invariants based on *semirings* rather than on rings. Following the essential feature of a TFT, the state sum (or partition function) of a positive TFT is required to satisfy a gluing axiom, which forces the information it detects to be local to a certain extent. As for the technical implementation of Banagl's positive TFT based on fold maps, the combinatorial information captured by singular patterns of fold maps is incorporated into the morphisms of a category. The resulting *Brauer category* is a strict monoidal category that is constructed as categorification of the Brauer algebras classically known from representation theory of the orthogonal group $O(n)$. By means of Theorem 1.2 we are able to list those integers l for which a given singular pattern on W admits a realization having no cusps and l loops. From the perspective of Banagl's positive TFT based on fold maps, this leads to the computation of the state sum invariant on all 2-dimensional cobordisms (see [14] for details).

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