DETECTING EXOTIC SPHERES VIA FOLD MAPS

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Abstract. In this survey article, we present two subgroup filtrations of the group of homotopy spheres whose definitions are both based on the existence of certain fold maps subject to index constraints. Both filtrations have recently been introduced and studied by the author in order to obtain new insights into global singularity theory of fold maps from high dimensional manifolds into Euclidean spaces. We discuss fundamental relations of our filtrations to other known filtrations of geometric topology. Moreover, we show how our results can be applied to compute an invariant of Saeki for the Milnor 7-sphere, as well as the value of Banagl's TFT-type aggregate invariant on certain exotic spheres including Kervaire spheres. Along the way, we raise some problems for future study.

1. Introduction

Developing powerful systematics for the classification of manifolds has always been a central issue of differential topology [25]. Major achievements like smooth bordism theories or the surgery program are usually governed by algebraic key invariants of manifold theory such as characteristic classes and the signature. A look behind the scenes, however, will often reveal a natural link to global singularity theory of differentiable maps. For instance, the construction of characteristic classes à la Stiefel [26] is motivated by the original observation that the set of points on a manifold where a given tuple of tangent vector fields is linearly dependent determines a homology cycle. In this spirit, our objective here is to present recent results of the author [31, 30, 32] concerning the study of exotic smooth structures on spheres from the perspective of global singularity theory.

Historically speaking, the modern understanding of differentiable structures on manifolds has its roots in the study of homotopy spheres, i.e., compact smooth manifolds having the homotopy type of the sphere of the same dimension. While being homeomorphic to spheres, homotopy spheres of dimension ≥ 5 do often possess non-standard ("exotic") smooth structures – this was first realized by John Milnor in 1956 with his revolutionary discovery of exotic 7-spheres [17]. Homotopy spheres appear to be fascinating objects in

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their own right: despite of their simple topology, they exhibit a rich structural diversity. Indeed, as shown by Kervaire and Milnor [12], oriented diffeomorphism classes of oriented smooth structures on the topological n-sphere ($n \geq 5$) form a finite abelian group Θ_n with group law induced by oriented connected sum. In their proof, Kervaire and Milnor reduce the study of smooth structures on spheres to questions of stable homotopy theory. A central ingredient is Smale's h-cobordism theorem (see [19]), whose proof is in turn based essentially on singularity theory of Morse functions. The work of Kervaire-Milnor initiated what is nowadays known as the surgery program, and which has been developed through the work of many others to a significant tool of high dimensional manifold theory.

The main purpose of the present article is to discuss how exotic spheres can be detected by means of singularity theory of so-called fold maps. Those can be thought of as maps between smooth manifolds that look locally like a family of Morse functions (i.e., smooth functions with only non-degenerate critical points). More precisely, a smooth map $F\colon M^n \to \mathbb{R}^p$ on a manifold M^n of dimension $n \geq p \geq 1$ is a fold map if for every singular point $x \in S(F) = \{x \in M; \operatorname{rank} d_x F < p\}$ there exist local coordinates $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_p)$ centered at x and $F(x)$, respectively, in which F takes for suitable integer i the form

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{p-1}, -x_p^2 - \cdots - x_{p+i-1}^2 + x_{p+i}^2 + \cdots + x_n^2).$$

Consequently, the singular locus of a fold map (its "folds") is a submanifold on which the fold map restricts to a codimension one immersion. Using the notion of Morse index, one can assign to every fold component an integer called (absolute) fold index. The index of a fold point $x \in S(F)$ with the above normal form is explicitly given by $\max\{i, n-p+1-i\}$. In generalization of the notion of (in)definite Morse critical points, fold points of index $n+p-1$ are called *definite*, and *indefinite* otherwise.

There is a long tradition in studying the following global problems about fold maps (see e.g. [24] for a current overview):

(a) *In what ways do fold maps reflect the topology of a space, e.g., in terms of invariants such as characteristic classes?* For instance, Levine [15] has characterized the existence of a fold map from closed manifolds into the plane in terms of the Euler characteristic of the source manifold.

(b) *Construct fold maps with desired properties such as prescribed boundary conditions, or fold index constraints.* An essential tool for the construction of fold maps is Eliashberg's folding theorem [7]. Roughly speaking, this sort of a "homotopy principle" (h-principle) produces up to homotopy from more algebraic data a fold map with prescribed singular locus. However, the h-principle cannot be used for constructing fold maps subject to constraints on the fold indices.

In the context of these problems, fold maps having only folds of definite index (so-called *special generic maps*) seem to be of special importance. Indeed, as observed in Remark 3.6 of [22], global singularity theory of special generic maps is closely related to the study of smooth structures on manifolds. More specifically, Saeki [21] has shown that the smooth standard n-sphere can be characterized among all homotopy spheres Σ^n in terms of the existence of special generic maps of Σ^n into Euclidean spaces of various target dimensions (see Theorem 2.2). Furthermore, Saeki [22] has characterized the smooth standard n-sphere in terms of bordism theory of *special generic functions* on Σ^n, that is, special generic maps $\Sigma^n \to \mathbb{R}$ (see Theorem 2.4). Beyond these results, the objective of this paper is to present cases in which fold maps subject to certain constraints on the fold index allow to detect individual exotic smooth structures on spheres. An innovative feature of our approach is that we use global singularity theory in a natural way to define subgroup filtrations of Θ_n that capture information about homotopy spheres.

The paper is organized as follows. In Section 2 we discuss several invariants of homotopy spheres, namely Milnor's λ-invariant (see Section 2.1), Saeki's invariant $\mathcal{S}$ (see Section 2.2), as well as Banagl's aggregate invariant $\mathfrak{A}$ that has shown up recently in the context of so-called positive topological field theories (see Section 2.3). Roughly speaking, Banagl's aggregate invariant detects the minimal number of closed components ("loops") that can occur in the (1-dimensional) singular locus of fold maps from bordisms bounded by homotopy spheres into the plane. In Section 3 we review several geometric-topological filtrations of Θ_n. Then, in Section 4, we introduce our singularity theoretic filtrations, and present their relation to the filtrations of Section 3. Namely, the *standard filtration*, which is introduced in terms of certain special generic maps in Section 4.1, turns out to be related to the Gromoll filtration (see Section 3.1). Moreover, the *index filtration* (see Section 4.2), which is based on bordism theory of Morse functions that are subject to certain index constraints, will be shown in Theorem 4.5 to be related to the connectivity filtration (see Section 3.2). Throughout our discussion we also pose some problems that can be the subject of future research. Finally, in Section 5, we discuss applications of our results to the computation of the invariants presented in Section 2 for some concrete exotic spheres, namely Milnor spheres (see Example 2.1) and Kervaire spheres (see Example 3.2).

All manifolds considered in this note will be differentiable of class C^∞. Let S^n denote the smooth standard sphere of dimension $n \geq 0$.

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2. Invariants

In what follows, let Σ^n denote a homotopy sphere of dimension $n \geq 1$, i.e., a closed smooth n-manifold which is homotopy equivalent to S^n. We recall the following two well-known facts about homotopy spheres for later reference.

(1) From the perspective of Morse theory [19], Σ^n admits for $n \neq 4$ a Morse function with exactly two critical points (see Figure 1), namely one minimum and one maximum. (For $n = 3$ the claim is valid by Perelman's solution to the smooth Poincaré conjecture.) Recall from the introduction that such a Morse function is called *special generic function*.

(2) From the viewpoint of bordism theory, Σ^n is known to be *oriented nullbordant* in the smooth oriented bordism group, $[\Sigma^n] = 0 \in \Omega_n^{SO}$. (In fact, it suffices to convince oneself that all Stiefel-Whitney numbers and Pontrjagin numbers of Σ^n vanish.) In other words, Σ^n can be realized as the boundary of an oriented compact manifold W^{n+1} (see Figure 1).

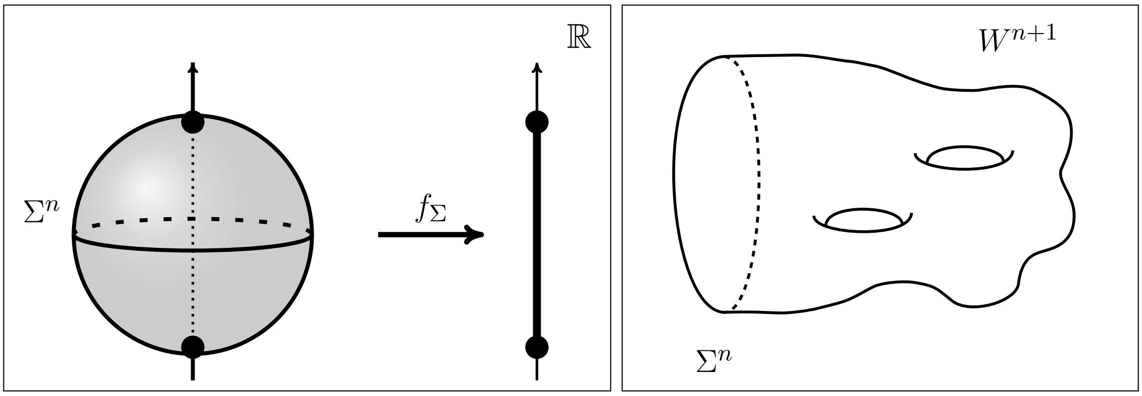

Figure 1. Left: A special generic function $f_\Sigma \colon \Sigma^n \to \mathbb{R}$ symbolized by the height function on the 2-sphere. Right: An oriented nullbordism W^{n+1} of Σ^n.

At first glance, the above facts do not help in distinguishing homotopy spheres. However, using them as a tool, interesting invariants of homotopy spheres can be obtained as explained in the following sections. Since we are concerned with high dimensional phenomena, we will generally assume that $n \geq 5$ in the following.

2.1. **Milnor's λ-invariant.** A key role in the discovery of exotic 7-spheres [17] is played by Milnor's celebrated λ-*invariant*, which is defined on a homotopy 7-sphere Σ^7 as follows. Exploiting fact (2), we realize Σ^7 as the boundary of any compact oriented manifold W^8, and consider the linear combination in $\mathbb{Z}/7$ of the Pontrjagin number $p_1^2[W^8]$ and the signature $\sigma(W^8)$ given by

$$\lambda(\Sigma^7) = 2p_1^2[W^8] - \sigma(W^8) \bmod 7.$$

It is a consequence of Novikov additivity and the Hirzebruch signature theorem that the above definition of $\lambda(\Sigma^7)$ is independent of the choice of W^8, and is thus in fact an invariant of Σ^7. Note that $\lambda(S^7) = 0$ because S^7 bounds the unit 8-disc, which is contractible. In order to prove the existence of exotic 7-spheres, Milnor proceeds to construct explicit homotopy 7-spheres with non-trivial λ-invariant by studying total spaces of certain linear 3-sphere bundles over S^4.

While the λ-invariant distinguishes certain exotic 7-spheres from the standard sphere, it is certainly not a complete invariant because it takes values in $\mathbb{Z}/7$, whereas $\Theta_7 \cong \mathbb{Z}/28$ (see [12]). Employing Hirzebruch's $\hat{A}$-genus, Eells-Kuiper [5] introduced the μ-invariant, which is a refinement of the λ-invariant that can be defined for certain closed spin nullbordant manifolds in dimensions of the form $n \equiv 3 \bmod 4$. Eells and Kuiper use their μ-invariant to decide which of the 27 exotic 7-spheres can be realized as total spaces of 3-sphere bundles over S^4 as in Milnor's construction.

Example 2.1 (Milnor spheres). *A generator of $\Theta_7 \cong \mathbb{Z}/28$ is given by the* Milnor 7-sphere *Σ^7_M, which is a concrete homotopy 7-sphere that is uniquely determined by requiring that it has an oriented nullbordism whose signature is equal to 8. The computations of Eells and Kuiper in* [5] *imply that Σ^7_M can be realized as the total spaces of a linear 3-sphere bundle over S^4.*

2.2. **Saeki's invariant $\mathcal{S}$.** Inspired by fact (1), Osamu Saeki [21] has defined an invariant $\mathcal{S}$ of homotopy spheres as follows.

Recall from the introduction that a smooth map $M^n \to \mathbb{R}^p$ on a closed n-manifold M^n is called *special generic map* if all of its critical points are *definite fold points*, i.e. singular points determined by the map germ

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{p-1}, x_{p+1}^2 + \cdots + x_n^2).$$

Thus, special generic maps generalize Morse functions with only minima and maxima as critical points in a natural way to maps with higher dimensional codomain.

Given a homotopy sphere Σ^n, define $\mathcal{S}(\Sigma^n)$ to be the set of all $p \in \{1, \ldots, n\}$ for which Σ^n admits a special generic map into $\mathbb{R}^p$. Note that $1 \in \mathcal{S}(\Sigma^n)$ holds for any homotopy sphere Σ^n by fact (1). The importance of the invariant $\mathcal{S}$ (which is obviously a diffeomorphism invariant) is illustrated by the following

Theorem 2.2 (Saeki [21], 1993). *A homotopy sphere Σ^n of dimension $n \geq 5$ is diffeomorphic to the standard sphere S^n if and only if $\mathcal{S}(\Sigma^n) = \{1, \ldots, n\}$.*

To the author's knowledge, the problem of computing the set $\mathcal{S}(\Sigma^n)$ for a given exotic sphere Σ^n is in general widely open (compare [23, Problem 5.1, p. 198]). However, in Section 5.1 we discuss the solution in the case that Σ^7 is the Milnor 7-sphere.

2.3. **Banagl's aggregate invariant $\mathfrak{A}$.** The axiomatic notion of topological field theory (TFT) was coined in 1988 by Michael Atiyah in his seminal paper [1]. According to Atiyah's axioms, an $(n+1)$-dimensional TFT Z assigns to every closed n-manifold M^n a *state module* $Z(M)$ (a module over some fixed base ring), and moreover to any $(n+1)$-dimensional bordism W^{n+1} an element $Z_W \in Z(\partial W)$ called *state sum* (or partition function). (Recall that a bordism is a compact manifold whose boundary components are partitioned into an ingoing and an outgoing part.) The assignment Z is required to satisfy a list of axioms which can most efficiently be summarized by saying that Z is a monoidal functor from the bordism category (with monoidal structure given by disjoint union) into the category of vector spaces (with monoidal structure given by the tensor product). Here, we shall only recall the essential *gluing axiom*, which can be stated explicitly as follows. For any triple (M, N, P) of closed n-manifolds the gluing axiom requires the existence of a contraction product of the form

$$\langle \cdot, \cdot \rangle \colon Z(M \sqcup N) \times Z(N \sqcup P) \to Z(M \sqcup P),$$

such that, whenever W' is a bordism from M to N, W'' is a bordism from N to P, and W is the bordism from M to P obtained by gluing the bordisms W' and W'' along N, the three state sums Z_W, $Z_{W'}$, and $Z_{W''}$ are related by the formula $Z_W = \langle Z_{W'}, Z_{W''} \rangle$. Thus, the gluing axiom guarantees that the state sum of a TFT is computable to a certain extent by cutting bordisms into simpler pieces ("locality of the state sum").

Recently, Markus Banagl [2] has proposed a new framework of *positive* TFT based on *semirings* rather than on rings. Compared to a ring, elements of a semiring do not necessarily possess additive inverses (i.e., "negative" elements). The Boolean semiring $\mathbb{B} = \{0, 1\}$, whose semiring structure is uniquely determined by requiring that $1 + 1 = 1$, is a simple example of a semiring that is not a ring (in fact, 1 does not have an additive inverse). Eilenberg completeness [6] is an important concept of semiring theory that is not available for rings. Roughly speaking, a complete semiring is a semiring equipped with a summation law that satisfies distributivity, and extends the addition law to arbitrary families of elements (even for uncountable index sets). While the axiomatic system of positive TFT deviates from Atiyah's original axioms in some necessary aspects, Banagl supplies a general framework which allows to construct positive TFTs of any dimension from systems of *fields* and category-valued *actions* by means of a mathematically rigorous process called *quantization*. Inspiration is taken from the definition of Feynman's path integral of theoretical quantum physics, and Banagl exploits the concept of Eilenberg completeness for semirings to avoid measure theoretic difficulties in the definition of the state sum.

In Section 10 of [2], Banagl outlines the construction of an explicit positive TFT of any dimension $n+1 \geq 2$ defined on smooth manifolds. Certain fold maps of bordisms into the plane are employed as fields of the theory. The 1-dimensional singular locus of such a *fold field* consists of intervals and circles, and the action functional translates this "singular pattern" into a morphism of the so-called Brauer category. The Brauer category is a strict monoidal category constructed as categorification of the Brauer algebras classically known from representation theory of the orthogonal group $O(n)$.

In general, the state sum of an $(n+1)$-dimensional positive TFT takes values in an additive monoid Q, which is a complete semiring with respect to two different multiplications, and serves as the ground semiring of the theory. An innovative feature of a positive TFT is the *aggregate invariant* $\mathfrak{A}$, a (differential) topological invariant of closed n-manifolds that is derived from the partition function of the theory, and takes values in certain modules over the ground semiring Q. This is in line with Atiyah's original intention to exploit the concept of TFT as a convenient source of powerful invariants for manifolds. The aggregate invariant associated to Banagl's concrete positive TFT above is particularly interesting when evaluated on homotopy spheres, where it reduces to a map $\mathfrak{A}\colon \Theta_n \to Q$ with values in the ground semiring Q. Banagl shows in Theorem 10.3 of [2] that the aggregate invariant of his theory is able to distinguish exotic spheres from the standard sphere in high dimensions.

Theorem 2.3 (Banagl [2], 2015)**.** *A homotopy sphere Σ^n of dimension $n \geq 5$ is diffeomorphic to the standard sphere S^n if and only if $\mathfrak{A}(\Sigma^n) = \mathfrak{A}(S^n)$ in Q.*

Banagl's proof of the above result is essentially based on the technique of Stein factorization [4], an important tool of global singularity theory. More specifically, in proving Theorem 2.3, Banagl uses the following result of Saeki (see Lemma 3.3 in [22]) that characterizes the standard sphere among all homotopy spheres in terms of bordism theory of special generic functions.

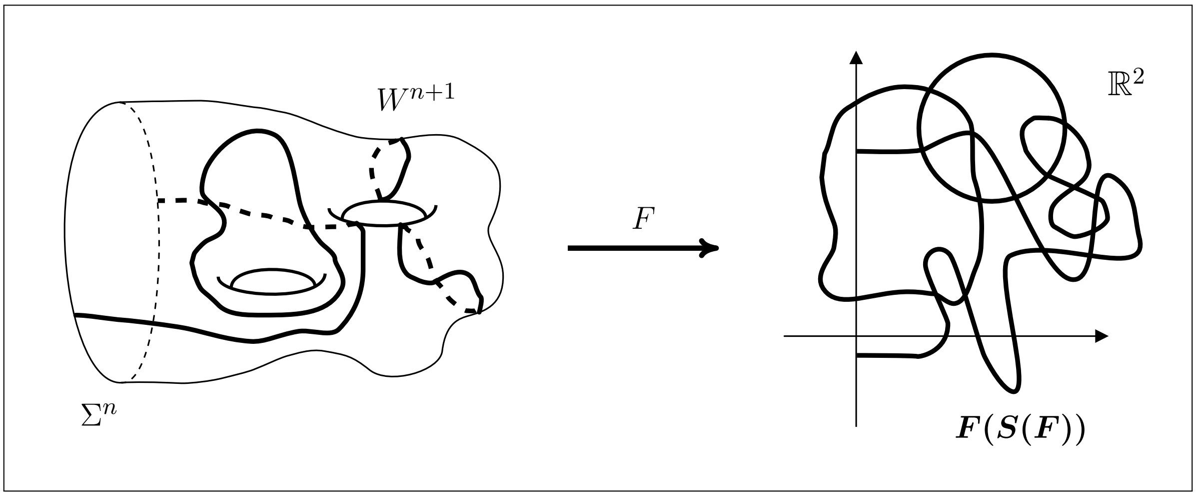

FIGURE 2. For a pair (W^{n+1}, F) with the properties of Theorem 2.4, the singular locus $S(F) \subset W$ (bold lines) is an embedded compact 1-dimensional submanifold consisting of a single interval component, and a finite number of closed components ("loops"). All components of $S(F)$ have definite fold index. Note that F restricts to an immersion $S(F) \to \mathbb{R}^2$.

Theorem 2.4 (Saeki [22], 2002)**.** *A homotopy sphere Σ^n of dimension $n \geq 5$ is diffeomorphic to the standard sphere if and only if there exists a pair (W^{n+1}, F) with the following properties (see Figure 2):*

- *W^{n+1} is a compact oriented $(n+1)$-manifold with boundary Σ^n (see fact (2)), and*
- *$F\colon W^{n+1} \to \mathbb{R}^2$ is a smooth map with the following properties:*
  - *(i) F is a special generic map, i.e., for every singular point $x \in S(F)$ there exist local coordinates $(x_1, \ldots, x_{n+1})$ and (y_1, y_2) centered at x and $F(x)$, respectively, in which F takes the form*

$$(x_1, \ldots, x_{n+1}) \mapsto (x_1, x_2^2 + x_3^2 + \cdots + x_{n+1}^2).$$

  - *(ii) There exists a collar neighborhood $[0, \varepsilon) \times \Sigma^n \subset W^{n+1}$ of $\{0\} \times \Sigma^n = \Sigma^n \subset W^{n+1}$, and a special generic function $f_\Sigma\colon \Sigma^n \to \mathbb{R}$ (compare fact (1)) such that $F|_{[0,\varepsilon)\times\Sigma^n} = \mathrm{id}_{[0,\varepsilon)} \times f_\Sigma$.*

Recall that the fields of Banagl's concrete positive TFT are certain fold maps $F\colon W \to \mathbb{R}^2$ of bordisms into the plane. In order to satisfy the essential gluing axiom, Banagl needs to impose a subtle technical condition on the fields (see Definition 10.1 in [2]) which involves the interaction of the immersion $F|\colon S(F) \to \mathbb{R}^2$ with a certain smooth map $W \to [0, 1]$ called time function. By eliminating this technical issue, the author shows in Proposition 10.1.5 of [30, p. 244] that the informational content of Banagl's aggregate invariant $\mathfrak{A}\colon \Theta_n \to Q$ is encoded in the *aggregate filtration*, namely a map

$$\mathfrak{a}\colon \Theta_n \to \{0, 1, 2, \ldots\} \cup \{\infty\}$$

which can be defined explicitly as follows. Given a homotopy sphere Σ^n of dimension $n \geq 5$, consider pairs (W^{n+1}, F) as in Theorem 2.4, but with the weaker requirement that $F\colon W^{n+1} \to \mathbb{R}^2$ is a fold map instead of a special generic map. In other words, we drop the assumption that all fold points of F are definite. As indicated in Figure 2, the singular locus $S(F) \subset W$ is an embedded compact 1-dimensional submanifold which is transverse to $\partial W = \Sigma^n$, and satisfies $S(F) \cap \partial W = \partial S(F)$. Thus, $S(F)$ consists of a single component diffeomorphic to the interval, and a finite number $a(W^{n+1}, F) \geq 0$ of closed components ("loops"). Note that, in contrast to the pairs (W^{n+1}, F) considered in Theorem 2.4, $S(F)$ may now possibly have loops of indefinite fold index. Varying over all pairs (W^{n+1}, F), we define the aggregate filtration $\mathfrak{a}(\Sigma^n)$ of Σ^n to be the minimum of the set of occuring integers $a(W^{n+1}, F)$. (If no pair (W^{n+1}, F) exists, then it is understood that $\mathfrak{a}(\Sigma^n) = \infty$.)

Similarly to the proof of Theorem 2.3, we can use Theorem 2.4 to show the following

Corollary 2.5. *A homotopy sphere Σ^n of dimension $n \geq 5$ is diffeomorphic to the standard sphere S^n if and only if $\mathfrak{a}(\Sigma^n) = 0$.*

Given an exotic sphere Σ^n of dimension $n \geq 5$, the number $\mathfrak{a}(\Sigma^n)$ is in general very hard to compute. The results discussed in Section 4.2 will shed some light on the nature of the aggregate filtration $\mathfrak{a}$.

3. Filtrations of Θ_n

Information about homotopy spheres is sometimes organized in terms of natural filtrations of Θ_n. In this section we discuss two filtrations that arise naturally in the study of exotic spheres. Let $n \geq 5$ be an integer.

3.1. **Gromoll filtration.** The first filtration we shall discuss is the *Gromoll filtration* – a subgroup filtration of Θ_n of the form

$$0 = \Gamma^n_{n-1} \subset \cdots \subset \Gamma^n_1 = \Theta_n.$$

It has been introduced by Detlev Gromoll in the 1960's [9] with the purpose to sharpen the classical sphere theorem of Rauch [20], Berger [3] and Klingenberg [13] from Riemannian geometry.

Let us recall the definition of the Gromoll filtration of Θ_n, which uses the perspective of twisted spheres. By fact (1) any homotopy sphere Σ^n admits a Morse function with exactly two critical points, and one can therefore think of Σ^n as being obtained by gluing two copies of the unit n-disc along their boundaries by means of an orientation preserving automorphism γ of S^{n-1}. Furthermore, one can up to isotopy achieve that the diffeomorphism γ has support in some chart of S^{n-1}. Then, γ can be considered as a diffeomorphism $g\colon \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ with compact support. We say that $[\Sigma^n] \in \Theta_n$ has Gromoll filtration strictly greater than p (or lies in Γ^n_{p+1}) if g can be chosen in such a way that $\pi^{n-1}_p \circ g = \pi^{n-1}_p$, where $\pi^{n-1}_p \colon \mathbb{R}^{n-1} \to \mathbb{R}^p$ denotes the projection to the last p coordinates.

In general, the Gromoll filtration is far from being completely understood, and its computation is expected to be very hard. One partial result is the following

Theorem 3.1 (Weiss [29], 1993)**.** *If Σ^n is a homotopy sphere of dimension $n \geq 7$, then*

$$(\textit{Gromoll filtration of } \Sigma^n) - 1 \leq \textit{Morse perfection of } \Sigma^n.$$

Here, the Morse perfection of Σ^n is by definition the greatest integer $k \geq 0$ for which there exists a family $\eta\colon \Sigma^n \times S^k \to \mathbb{R}$ of special generic functions $\eta_s = \eta(\cdot, s)\colon \Sigma^n \to \mathbb{R}$ smoothly parametrized by $s \in S^k$ such that the additional symmetry condition $\eta_{-s} = -\eta_s$ is satisfied for all $s \in S^k$. Note that the Morse perfection of a homotopy sphere is always ≥ 0 by fact (1).

3.2. **Connectivity filtration.** Another natural filtration of Θ_n is the *connectivity filtration* – a subgroup filtration of Θ_n of the form

$$C^n_{\lfloor n/2 \rfloor} \subset \cdots \subset C^n_1 \subset (C^n_0 =)\ \Theta_n.$$

This filtration is obtained by considering the maximal connectivity of bordisms bounded by Σ^n. Explicitly, Σ^n is said to have connectivity filtration $\geq k$ (or lies in C^n_k) if it can be realized as the boundary of an oriented compact manifold W^{n+1} (see fact (2)) such that W^{n+1} is k-connected. For instance, the smooth standard n-sphere has connectivity filtration $\geq k$ for any k since it bounds a contractible bordism (namely, the $(n+1)$-disc). On the other hand, results by Stolz [27] imply that the connectivity filtration is non-trivial in general as we shall explain next. Let $bP_{n+1} \subset \Theta_n$ denote the subgroup of those homotopy n-spheres that can be realized as the boundary of a compact manifold which is parallelizable, i.e., has trivial tangent bundle (see [12, p. 510]). For instance, we have $bP_{n+1} = 0$ whenever n is even by [12, Theorem 5.1, p. 512].

Example 3.2 (Kervaire spheres)**.** *Kervaire spheres are a concrete family of homotopy spheres that can be obtained from a plumbing construction as follows (see* [14, p. 162]*). The unique Kervaire sphere Σ^n_K of dimension $n = 4m+1$ can be defined as the boundary of the parallelizable $(4m+2)$-manifold given by plumbing together two copies of the tangent disc bundle of S^{2m+1}. It follows from the construction that $[\Sigma^n_K] \in bP_{n+1}$, and the classification theorem of homotopy spheres (see Theorem 6.1 in* [16, pp. 123f]*) implies that $bP_{n+1} = \{[S^n], [\Sigma^n_K]\}$. Moreover, it is known that $bP_{n+1} \cong \mathbb{Z}/2$ whenever $n+3 \notin \{2^1, 2^2, 2^3, \ldots\}$, whereas $bP_{n+1} = 0$ for $n \in \{5, 13, 29, 61\}$.*

For odd n we observe that $bP_{n+1} \subset C^n_{\lfloor n/2 \rfloor}$ holds because by Theorem 3 of [18, p. 49] any parallelizable compact smooth manifold W^{n+1} can be made $\lfloor n/2 \rfloor$-connected by a finite sequence of surgeries without changing ∂W. Hence, in dimensions of the form $n \equiv 1 \bmod 4$, Theorem B(ii) of [27, p. XIX] implies the following

Theorem 3.3 (Stolz [27], 1985)**.** *Suppose that $n \geq 225$ and $n \equiv 1 \bmod 4$. Then,*

$$C^n_{\lfloor n/2 \rfloor} = bP_{n+1}.$$

Moreover, as shown in Remark 6.3 of [32], we have $bP_{n+1} \neq \Theta_n$ for infinitely many dimensions $n \equiv 13 \bmod 16$. Namely, this is true whenever $n = 2(p+1)(p-1) - 3$ for an odd prime p. Note that in this case we also have $bP_{n+1} \cong \mathbb{Z}/2$ provided $p \neq 3$.

4. Results

In this section we discuss two theorems of the author (see Theorem 4.1 and Theorem 4.5), which are both motivated by the invariants of Section 2 and the filtrations of Section 3. Beyond results of [21, 22] these theorems both demonstrate the important role of fold maps for the detection of exotic spheres. Both theorems involve new subgroup filtrations of Θ_n which are defined by means of global singularity theory of fold maps. In Section 5 we show how our results contribute to the computation of Saeki's invariant $\mathcal{S}$ (see Section 5.1) as well as Banagl's aggregate invariant $\mathfrak{A}$ (see Section 5.2).

4.1. Standard filtration.

In preparation of our first result, we explain the definition of the *standard filtration* (see [31, Remark 3.12, p. 355]), which is a subgroup filtration of Θ_n of the form

$$0 = F^n_{n-1} \subset \cdots \subset F^n_1 \subset \Theta_n.$$

By definition, Σ^n has standard filtration $\geq p$ (or lies in F_p^n) if it admits a special generic map $F\colon \Sigma^n \to \mathbb{R}^p$ with image $F(\Sigma^n) = D^p$ the unit p-disc, and such that all fibers $F^{-1}(y)$, $y \in D^p$, are connected.

In refinement of Theorem 3.1, we have the following

Theorem 4.1 ([31]). *Let Σ^n be a homotopy sphere of dimension $n \geq 7$. Then, the standard filtration is related to the Gromoll filtration of Σ^n by*

$$\textit{Gromoll filtration of } \Sigma^n \leq \textit{standard filtration of } \Sigma^n \leq (\textit{Morse perfection of } \Sigma^n) + 1.$$

In Section 5.1 we will discuss an application of our result to Milnor spheres.

Problem 4.2. *Do the Gromoll filtration Γ_p^n and the standard filtration F_p^n coincide?*

4.2. **Index filtration.** In Section 2.3 we discussed Banagl's construction of an explicit positive TFT based on certain fold maps of bordisms into the plane. Recall from Theorem 2.3 that the aggregate invariant associated to the partition function of this theory is powerful enough to distinguish exotic smooth spheres from the smooth standard sphere. Hence, the problem of whether the aggregate invariant can also distinguish between individual exotic spheres arises.

Problem 4.3. *Find two exotic spheres Σ_1^n and Σ_2^n for which Banagl's aggregate invariant $\mathfrak{A}$ discussed in Section 2.3 satisfies $\mathfrak{A}(\Sigma_1^n) \neq \mathfrak{A}(\Sigma_2^n)$ in Q.*

No such example is known, but in the following we present fundamental insights that allow to estimate the aggregate invariant (see Theorem 4.8), and imply its computation in some special cases. For example, in contrast to Milnor's λ-invariant (see Section 2.1), it will turn out that the aggregate invariant takes the same value on all exotic 7-spheres (see Corollary 4.10).

As explained in Section 2.3, the author has reduced the study of the aggregate invariant to the study of the aggregate filtration $A_n^l := \mathfrak{a}^{-1}(\{0, \dots, l\})$,

$$A_n^0 \subset \cdots \subset A_n^l \subset \cdots \subset \Theta_n.$$

Explicitly, the aggregate filtration of a homotopy n-sphere Σ^n is the minimal number $\mathfrak{a}(\Sigma^n) \in \{0, 1, 2, \dots\} \cup \{\infty\}$ of loops that can occur in the singular locus of fold map extensions of special generic functions on Σ^n to oriented nullbordisms of Σ^n.

In analogy with Morse homology, it seems natural to approach the study of the aggregate filtration by taking fold indices of loops into consideration. In fact, according to the Morse inequalities for Morse functions, the number of Morse critical points of prescribed index is bounded below by the corresponding Betti number of the source manifold. Hence, one can expect that the aggregate filtration, which minimizes the number of singular loops of certain fold maps, should be roughly connected to topological obstructions of nullbordisms of a given homotopy sphere. Indeed, by Lemma 2.3 of [12, p. 506] the standard sphere is the unique homotopy sphere that can be realized as the boundary of a contractible bordism (namely, the unit disc). Note that $A_n^0 = \{[S^n]\}$ by Corollary 2.5, which is a consequence of Saeki's characterization of the standard sphere in the context of bordism theory of special generic functions (Theorem 2.4). This leads to the following

Problem 4.4. *Study bordism theory of Morse functions on closed manifolds which are subject to prescribed constraints on the indefinite Morse index.*

For instance, Ikegami [11] has entirely computed the bordism groups of Morse functions (without imposing any index constraints). Furthermore, in [32] the author has studied the above problem systematically for the following specific type of index constraints. Given an integer $k \geq 1$, we call a Morse function on a closed n-manifold *k-constrained* if all indefinite Morse indices of its critical points are contained in the interval $\{k, \ldots, n-k\}$. Note that the notion of a k-constrained Morse function interpolates between ordinary Morse functions ($k = 1$) and special generic functions ($k > n/2$). In the present article we restrict ourselves to discussing the consequences of the results in [32] for the *index filtration* (see [32, Remark 6.4]), which is a subgroup filtration of Θ_n of the form

$$G^n_{\lfloor n/2 \rfloor} \subset \cdots \subset G^n_1 \subset \Theta_n$$

defined as follows. A homotopy sphere Σ^n of dimension $n \geq 5$ has index filtration $\geq k$ (or lies in G^n_k) if there exists a pair (W^{n+1}, F), where

- W^{n+1} is a compact oriented $(n+1)$-manifold with boundary $\partial W^{n+1} = \Sigma^n$, and
- $F\colon W^{n+1} \to \mathbb{R}^2$ is a smooth map with the following properties:
  - (i) every singular point $x \in S(F)$ is either a fold point of F whose fold index is contained in the set $\{\lceil n/2 \rceil, \ldots, n-k\} \cup \{n\}$, or a *cusp*, which means by definition that there exist local coordinates $(x_1, \ldots, x_{n+1})$ and (y_1, y_2) centered at x and $F(x)$, respectively, in which F takes the form
  $$(x_1, \ldots, x_{n+1}) \mapsto (x_1, x_1x_2 + x_2^3 \pm x_3^2 \pm \cdots \pm x_{n+1}^2).$$
  - (ii) There exists a collar neighborhood $[0, \varepsilon) \times \Sigma^n \subset W^{n+1}$ of $\{0\} \times \Sigma^n = \Sigma^n \subset W^{n+1}$, and a k-constrained Morse function $f_\Sigma\colon \Sigma^n \to \mathbb{R}$ such that $F|_{[0,\varepsilon)\times\Sigma^n} = \mathrm{id}_{[0,\varepsilon)} \times f_\Sigma$.

From the perspective of Morse theory [19] we observe that a closed manifold of dimension $n \neq 4$ admits a k-constrained Morse function if and only if it is $(k-1)$-connected (where the case $n = 3$ is based on Perelman's solution to the smooth Poincaré conjecture). This observation suggests that the index filtration should be related strongly to the connectivity filtration (see Section 3.2). Indeed, we have the following

Theorem 4.5 (Remark 6.4 of [32])**.** *For $n \geq 5$, the index filtration is related to the connectivity filtration for all $1 \leq k \leq \lfloor n/2 \rfloor$ by*

$$C^n_k \subset G^n_k \subset C^n_{k-1}.$$

The proof of Theorem 4.5 uses a palette of techniques of geometric topology. Among these are the two-index theorem of Hatcher and Wagoner [10], a handle extension theorem for constrained Morse functions that has recently been established by Gay and Kirby [8] in the context of symplectic geometry, and Stein factorization [4] for generic maps into the plane which are subject to certain constraints on the fold index.

Problem 4.6. *Do the connectivity filtration* C^n_k *and the index filtration* G^n_k *coincide?*

It is shown in Proposition 10.2.2(*iii*) of [30, p. 246] that $C^n_k = C^n_{k-1}$ (and, in particular, $G^n_k = C^n_k$ by Theorem 4.5) for all $1 \leq k \leq \lfloor n/2 \rfloor$ satisfying $k \equiv 3, 5, 6, 7 \bmod 8$. (Indeed, the proof of the nontrivial inclusion $C^n_{k-1} \subset C^n_k$ is an application of Theorem 3 of [18] using Bott periodicity as formulated in the proof of Theorem 3.1 in [12, p. 508].) Thus, Theorem 3.3 implies that $G^n_{\lfloor n/2 \rfloor} = bP_{n+1}$ for all $n \geq 237$ satisfying $n \equiv 13 \bmod 16$. Based on this result, Theorem 6.2 of [32] gives a characterization of Kervaire spheres in terms of bordism groups of constrained Morse functions. This can be considered as a natural continuation of Theorem 2.4!

How is the index filtration related to the aggregate filtration? For technical reasons we have to introduce modified versions $\overline{C}^n_k$ and $\overline{G}^n_k$ of the connectivity filtration and the index filtration, respectively, as follows. The homotopy sphere Σ^n lies in $\overline{C}^n_k$ if there exists an oriented compact manifold W^{n+1} with boundary $\partial W^{n+1} = \Sigma^n$ such that W^{n+1} is k-connected, and has odd Euler characteristic. Furthermore, the homotopy sphere Σ^n lies in $\overline{G}^n_k$ if there exists a pair (W^{n+1}, F) with properties as in the definition of G^n_k, but where it is in addition required that W^{n+1} has odd Euler characteristic, and the Morse function $f_\Sigma \colon \Sigma^n \to \mathbb{R}$ in property (*ii*) of F is required to be a special generic function. Then, the proof of the following theorem is almost identical to the proof of Theorem 4.5.

Theorem 4.7. *For* $n \geq 5$ *and* $1 \leq k \leq \lfloor n/2 \rfloor$, *we have*

$$\overline{C}^n_k \subset \overline{G}^n_k \subset \overline{C}^n_{k-1}.$$

The following result relates $\overline{G}^n_k$ to the aggregate filtration A^l_n (see [30, Proposition 10.1.5(b)(iii), p. 244]).

Theorem 4.8. *For* $n \geq 5$ *and* $1 \leq k \leq \lfloor n/2 \rfloor$, *we have*

$$\overline{G}^n_k \subset A^{\lfloor n/2 \rfloor + 1 - k}_n.$$

The proof of Theorem 4.8 combines Levine's cusp elimination technique [15] with the complementary process of creating a pair of cusps on a fold line by means of a swallowtail homotopy. Given a pair (W^{n+1}, F) with the properties listed in the definition of $\overline{G}^n_k$, the combination of these techniques allows us (see Proposition 6.1.3 of [30, p. 154f]) to eliminate all cusps of F in pairs in such a way that the resulting fold map $G \colon W^{n+1} \to \mathbb{R}^2$ has the following properties. G agrees with F near the boundary of W, every loop of G has fold index contained in the set $\{\lceil n/2 \rceil, \ldots, n-k\}$, and no two loops of G have the same fold index. Note that the assumption on the Euler characteristic imposed in the definition of $\overline{G}^n_k$ ensures that the total number of cusps of F is even.

We finish this section with some corollaries to Theorem 4.8.

Corollary 4.9. *Let $n \geq 5$. It follows from $\overline{G}^n_1 = \Theta_n$ that $A^{\lfloor n/2 \rfloor}_n = \Theta_n$. In other words, $\mathfrak{a}(\Sigma^n) \in \{0, 1, \dots, \lfloor n/2 \rfloor\}$ for any homotopy n-sphere Σ^n.*

Let $bP_{n+1} \subset \Theta_n$ denote the subgroup of those homotopy n-spheres that can be realized as the boundary of a parallelizable compact $(n+1)$-manifold. Recall from Section 3.2 that $bP_{n+1} \subset C^n_{\lfloor n/2 \rfloor}$. (Note that $bP_{n+1} = \{[S^n]\}$ when n is even.) Moreover, it is shown in Proposition 10.2.2(i) of [30, p. 246] that $C^n_{\lfloor n/2 \rfloor} = \overline{C}^n_{\lfloor n/2 \rfloor}$ for $n \geq 5$, $n \neq 15$. (Indeed, the nontrivial inclusion $C^n_{\lfloor n/2 \rfloor} \subset \overline{C}^n_{\lfloor n/2 \rfloor}$ is for $n \not\equiv 3 \bmod 4$ an easy consequence of Poincaré duality, and follows for $n \equiv 3 \bmod 4$ from Wall's work [28] on smooth highly connected almost closed manifolds of even dimension.) All in all, by means of Theorem 4.7 we conclude with the following

Corollary 4.10. *For all $n \geq 5$, $n \neq 15$, we have*

$$bP_{n+1} \subset C^n_{\lfloor n/2 \rfloor} = \overline{C}^n_{\lfloor n/2 \rfloor} \subset \overline{G}^n_{\lfloor n/2 \rfloor} \subset A^1_n.$$

In particular, $\mathfrak{a}(\Sigma^n) = 1$ for every exotic sphere Σ^n contained in bP_{n+1}. For instance, it follows from $bP_8 = \Theta_7$ [12] *that the aggregate invariant cannot distinguish individual exotic 7-spheres (compare Problem 4.3).*

5. Applications

5.1. **$\mathcal{S}$-invariant of Milnor spheres.** Recall that the *Milnor n-sphere* Σ^n_M of dimension $n \equiv 3$ mod 4 is the unique homotopy sphere in Θ_n which can be realized as the boundary of a parallelizable bordism of signature 8 (see Example 2.1). A theorem of Weiss [29] (whose proof uses tools from algebraic K-theory) states for dimensions $n \geq 7$ of the form $n \equiv 3$ mod 4 that

$$\text{Gromoll filtration of } \Sigma^n_M = 2 = (\text{Morse perfection of } \Sigma^n_M) + 1.$$

Hence, Theorem 4.1 implies that the standard filtration of Σ^n_M is equal to 2. This result has an important consequence for the computation of Saeki's invariant $\mathcal{S}$ for the Milnor 7-sphere. In fact, invoking the classical Poincaré conjecture in dimension 3 as proven by Perelman, we obtain $\mathcal{S}(\Sigma^n_M) = \{1, 2, 7\}$ (see Corollary 5.4 of [31]).

5.2. **Aggregate invariant of Kervaire spheres.** It follows immediately from Example 3.2 and Corollary 4.10 that $\mathfrak{a}(\Sigma^n_K) = 1$ whenever the Kervaire sphere Σ^n_K of dimension $n = 4m + 1$, $m = 1, 2, \dots$, is exotic.

References

1. M.F. Atiyah, *Topological quantum field theory*, Publ. Math. Inst. Hautes Études Sci. **68** (1988), 175–186.
2. M. Banagl, *Positive Topological Quantum Field Theories*, Quantum Topology **6** (2015), no. 4, 609–706.
3. M. Berger, *Les variétés riemanniennes (1/4)-pincées*, Ann. Sc. Norm. Super. Pisa (3) **14** (1960), 161–170.
4. O. Burlet, G. de Rham, *Sur certaines applications génériques d'une variété close à trois dimensions dans le plan*, Enseign. Math. **20** (1974), 275–292.
5. J. Eells, N.H. Kuiper, *An invariant for certain smooth manifolds*, Ann. Mat. Pura Appl. **60** (1962), 93–110.
6. S. Eilenberg, *Automata, languages, and machines*, Pure and Applied Mathematics, vol. A, Academic Press, 1974.
7. J.M. Eliashberg, *Surgery of singularities of smooth mappings*, Math. USSR. Izv. **6** (1972), 1302–1326.
8. D.T. Gay, R. Kirby, *Indefinite Morse 2-functions; broken fibrations and generalizations*, Geom. Topol. **19** (2015), 2465–2534.
9. D. Gromoll, *Differenzierbare Strukturen und Metriken positiver Krümmung auf Sphären*, Math. Ann. **164** (1966), 353–371.
10. A. Hatcher, J. Wagoner, *Pseudo-isotopies of compact manifolds*, Astérisque **6**, Soc. Math. de France (1973).
11. K. Ikegami, *Cobordism group of Morse functions on manifolds*, Hiroshima Math. J. **34** (2004), 211–230.
12. M. Kervaire, J.W. Milnor, *Homotopy Spheres: I*, *The Annals of Mathematics*, Second Series, **77** (1963), no. 3, 504–537.
13. W. Klingenberg, *Über Riemannsche Mannigfaltigkeiten mit positiver Krümmung*, Comment. Math. Helv. **35** (1961), 47–54.
14. H.B. Lawson, M.-L. Michelsohn, *Spin Geometry*, Princeton Math. Series 38, Princeton University Press, (1989).
15. H.I. Levine, *Elimination of cusps*, Topology **3**, Suppl. 2 (1965), 263–296.
16. W. Lück, *A basic introduction to surgery theory*, version: October 27, 2004, `http://131.220.77.52/lueck/data/ictp.pdf`.
17. J.W. Milnor, *On manifolds homeomorphic to the* 7*-sphere*, Annals of Mathematics **64** (1956) no. 2, 399–405.
18. J.W. Milnor, *A procedure for killing homotopy groups of differentiable manifolds*, Symposia in Pure Math. **III**, American Mathematical Society (1961), 39–55.
19. J.W. Milnor, *Lectures on the h-Cobordism Theorem*, Math. Notes, Princeton Univ. Press, Princeton, NJ, 1965.
20. H.E. Rauch, *A contribution to differential geometry in the large*, Ann. of Math. (2) **54** (1951), 38–55.
21. O. Saeki, *Topology of special generic maps into* $\mathbb{R}^3$, Workshop on Real and Complex Singularities (São Carlos, 1992), Mat. Contemp. **5** (1993), 161–186.
22. O. Saeki, *Cobordism groups of special generic functions and groups of homotopy spheres*, Jpn. J. Math. **28** (2002), 287–297.

23. O. Saeki, *Topology of manifolds and global theory of singularities*, RIMS Kôkyûroku Bessatsu **55** (2016), 185–203.
24. K. Sakuma, *Fold dimension set of manifolds*, JP Journal of Geometry and Topology **18** (2015), 37–64.
25. S. Smale, *A survey of some recent developments in differential topology*, Bull. Amer. Math. Soc. **69**, Number 2 (1963), 131–145.
26. E. Stiefel, *Richtungsfelder und Fernparallelismus in n-dimensionalen Mannigfaltigkeiten*, Commentarii Mathematici Helvetici (1935), 305–353.
27. S. Stolz, *Hochzusammenhängende Mannigfaltigkeiten und ihre Ränder*, Lecture Notes in Mathematics **1116**, Springer-Verlag (1985).
28. C.T.C. Wall, *Classification of $(n-1)$-connected $2n$-manifolds*, Annals of Mathematics **75**, no. 1 (1962), 163–189.
29. M. Weiss, *Pinching and concordance theory*, J. Differential Geometry **38** (1993), 387–416.
30. D.J. Wrazidlo, *Fold maps and positive topological quantum field theories*, Dissertation, Heidelberg (2017), `http://nbn-resolving.de/urn:nbn:de:bsz:16-heidok-232530`.
31. D.J. Wrazidlo, *Standard special generic maps of homotopy spheres into Euclidean spaces*, Topology and its Applications **234** (2018), 348–358.
32. D.J. Wrazidlo, *Bordism of constrained Morse functions*, preprint (2018), `https://arxiv.org/abs/1803.11177`.

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