BORDISM OF CONSTRAINED MORSE FUNCTIONS

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Abstract. A Morse function f on a closed manifold is called *k-constrained* if neither f nor $-f$ has critical points of indefinite Morse index $< k$. We study bordism of k-constrained Morse functions, thus interpolating between the case $k = 1$ of bordism of Morse functions (computed by Ikegami) and the case $k \gg 1$ of bordism of special generic functions (computed by Saeki). For this purpose, we introduce the notion of *constrained generic bordism* which interpolates between the smooth bordism group and the homotopy group of spheres. By means of Stein factorization and a handle extension theorem for fold maps due to Gay-Kirby we then show that k-constrained generic bordism is strongly related to k-connective bordism.

Finally, as an application of our results we show that the bordism group of constrained Morse functions detects exotic Kervaire spheres in certain dimensions.

Contents

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1. Introduction

In 1954, bordism theory of differentiable maps was initiated by René Thom [31] with the computation of bordism groups of embedded manifolds, and has been a central issue in the field of global singularity theory ever since. The Pontrjagin-Thom construction is crucial to Thom's homotopy theoretic approach, and Rimányi and Szűcs [25] established a construction of the same type for bordism of smooth maps of m-manifolds into a fixed n-manifold with certain prescribed types of singularities in the case of positive codimension ($n-m>0$). Subsequently, Kalmár [12] gave an analogous construction for smooth maps with prescribed singular fibers in the case of negative codimension ($n-m<0$).

For the mildest types of singularities, explicit methods of geometric topology can sometimes serve as adequate tools for computing concrete bordism groups of smooth maps. In fact, bordism groups of Morse functions, which were originally introduced by Ikegami-Saeki in [11], have been entirely computed by Ikegami [10] by means of Levine's method of eliminating pairs of cusps [18] and the Kervaire semi-characteristic [13]. Furthermore, employing the technique of Stein factorization [1] as well as Cerf's pseudo-isotopy theorem [2], Saeki [27] showed that the bordism group of so-called special generic functions, i.e., Morse functions having only minima and maxima as critical points, is isomorphic to the group of homotopy spheres [14]. More generally, Sadykov [26] combined the Pontrjagin-Thom construction with Smale-Hirsch theory [8] to express bordism groups of special generic maps in terms of stable homotopy theory.

In this paper, we advance the explicit geometric-topological approach and study bordism groups of Morse functions whose critical points are subject to the following type of index constraints. For a given integer $k \geq 1$ we call a Morse function on a closed n-manifold *k-constrained* if all indefinite Morse indices of its critical points are contained in the interval $\{k, \dots, n-k\}$. Thus, on closed n-manifolds, the notion of a k-constrained Morse function interpolates between ordinary Morse functions ($k=1$) and special generic functions ($k>n/2$). From the viewpoint of Morse theory [21], it is a fundamental observation that a high-dimensional manifold admits a k-constrained Morse function if it is $(k-1)$-connected (with the converse being true in any dimension). This observation suggests that bordism groups of constrained Morse functions should be strongly related to so-called connective bordism groups (see Section 2), which will in fact be manifest in our Theorem 1.1. In the context of a generalization of the Madsen-Weiss theorem, Morse index constraints of the above form were recently imposed by Perlmutter [23] on Morse functions of bordisms seen as morphisms of the bordism category.

The key notion of k-constrained n-bordism (see Definition 3.2) involves so-called fold maps of $(n+1)$-bordisms into the plane. By definition, these are smooth maps all of whose singular points are determined by map germs of the form

$$(t, x_1, \dots, x_n) \mapsto (t, -x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2).$$

Thus, fold maps into the plane can locally be thought of as one-parameter families of Morse functions, and the *absolute index* $\max\{i, n-i\}$ turns out to be a locally constant invariant of singular points. Therefore, index constraints imposed on Morse functions induce constraints on the absolute index of fold maps in a natural way. In Definition 3.2 we will introduce a suitable notion of constrained bordism between constrained Morse functions.

Kitazawa [15, Remark 1, pp. 392f] constructed explicit examples of so-called round fold maps with connected indefinite fold locus when the source manifold is the total space of a fiber bundle over the sphere with fiber a twisted sphere. The general existence problem for fold maps in the presence of index constraints has been posed by Saeki in [28, Problem 5.13, p. 200]. Technical difficulties in approaching this problem arise from the fact that Eliashberg's h-principle [5] fails when fold maps are exposed to index constraints. Our results can be considered as a partial solution to Saeki's problem in the case of fold maps into $\mathbb{R}^2$ that are subject to our type of index constraints.

In the following discussion of our main results the focus lies on oriented bordism groups; unoriented versions of the results hold in an analogous way, and details are pointed out in Remark 4.4 and Remark 5.10.

Besides the n-bordism group of k-constrained Morse functions, which will be denoted by $\widetilde{\mathcal{M}}_n^k$ (see Definition 3.2), our two main results below involve the k-connective n-bordism group $\widetilde{\mathcal{C}}_n^k$ as reviewed in Section 2, as well as the group $\widetilde{\mathcal{G}}_n^k$ of k-constrained generic n-bordism (see Definition 3.1). The latter can be shown to interpolate between the smooth oriented bordism group and the group of homotopy spheres (see Remark 3.4).

The first main result of this paper slightly generalizes one of the main results of the author's thesis [32, Theorem 10.1.3, p. 243] (compare Remark 6.5) by relating the constrained generic bordism group to the group of connective bordism as follows.

Theorem 1.1. *Let $n \geq 5$ and $1 \leq k < n$ be integers. Then, there exist homomorphisms as follows:*

(i) $\varepsilon_n^k \colon \widetilde{\mathcal{C}}_n^k \to \widetilde{\mathcal{G}}_n^k$, $[M^n] \mapsto [f]$, where $f \colon M^n \to \mathbb{R}$ denotes an arbitrarily chosen k-constrained Morse function, and

(ii) for $k > 1$, $\delta_n^k \colon \widetilde{\mathcal{G}}_n^k \to \widetilde{\mathcal{C}}_n^{k-1}$, $[f \colon M^n \to \mathbb{R}] \mapsto [\sharp(M^n)]$, where $\sharp(M^n)$ denotes the oriented connected sum of the connected components of M^n. (We use the convention that $\sharp(\emptyset) = S^n$.)

Moreover, for $1 < k < n$, the natural homomorphism $\widetilde{\mathcal{C}}_n^k \to \widetilde{\mathcal{C}}_n^{k-1}$, $[M^n] \mapsto [M^n]$, factors as the composition $\delta_n^k \circ \varepsilon_n^k$, and the natural homomorphism $\widetilde{\mathcal{G}}_n^k \to \widetilde{\mathcal{G}}_n^{k-1}$, $[f \colon M^n \to \mathbb{R}] \mapsto [f]$, factors as the composition $\varepsilon_n^{k-1} \circ \delta_n^k$.

The two-index theorem of Hatcher and Wagoner [7] (see Section 3.2) will serve as an essential tool for showing that the homomorphism of part (i) is well-defined in the case $1 < k < n/2$ (see the proof of Proposition 4.3). The two-index theorem is based on a parametrized implementation of the Smale trick, by which one may trade a Morse critical point of index i for one of index $i + 2$ by creating a pair of critical points of successive indices $i + 1$ and $i + 2$, and then cancelling the Morse critical point of index i with that of index $i + 1$. Under stronger assumptions the Smale trick has been used by Cerf in his proof of the pseudo-isotopy theorem (see [2, Lemma 0, p. 101]). Furthermore, we will exploit a handle extension theorem for fold maps that has recently been established by Gay and Kirby [6] in the context of symplectic geometry (see Section 3.3). In order to show that the homomorphism of part (ii) is well-defined, we use Stein factorization for generic maps into the plane with certain fold index constraints (see Section 3.4) to prove Proposition 4.2 by generalizing the proof of [27, Lemma 3.3, p. 293].

While the structure of $\widetilde{\mathcal{M}}_n^k$ turns out to be very similar to that of bordism groups of Morse functions [10], a somewhat surprising phenomenon arises for $n \equiv 3 \bmod 4$ in our second main result (see parts (*iii*) and (*iv*) of Theorem 1.2 below). Namely, the size of the group $\widetilde{\mathcal{M}}_n^k$ is governed by an integer $\kappa_{(n+1)/4}$ that measures the existence of closed constrained $(n+1)$-bordisms having odd Euler characteristic (see Definition 5.6). The techniques that are used in the proof of Theorem 1.1 allow us to relate the sequence $\kappa_1, \kappa_2, \dots$ to another sequence $\gamma_1, \gamma_2, \dots$ of positive integers measuring the existence of highly-connected closed manifolds with odd Euler characteristic (see Definition 2.3).

Theorem 1.2. *Let $n \geq 4$ and $1 < k \leq n/2$ be integers. The oriented n-bordism group of k-constrained Morse functions $\widetilde{\mathcal{M}}_n^k$ fits into a short exact sequence of abelian groups*

$$0 \to A_n^k \xrightarrow{\alpha_n^k} \widetilde{\mathcal{M}}_n^k \xrightarrow{\beta_n^k} \widetilde{\mathcal{G}}_n^k \oplus \mathbb{Z}^{\lfloor n/2 \rfloor - k} \to 0,$$

where the homomorphisms α_n^k and β_n^k are defined in Lemma 5.3 and Lemma 5.1, respectively. We have either $A_n^k = 0$ or $A_n^k = \mathbb{Z}/2$, depending on the following cases:

(*i*) *If n is even, then $A_n^k = 0$, so β_n^k is an isomorphism.*
(*ii*) *If $n \equiv 1 \bmod 4$, then $A_n^k = \mathbb{Z}/2$, and α_n^k admits a splitting (see Lemma 5.5).*
(*iii*) *If $n \equiv 3 \bmod 4$ and $k \leq \kappa_{(n+1)/4}$, then $A_n^k = 0$, so β_n^k is an isomorphism.*
(*iv*) *If $n \equiv 3 \bmod 4$ and $k > \kappa_{(n+1)/4}$, then $A_n^k = \mathbb{Z}/2$.*

Furthermore, the sequences $\kappa_1, \kappa_2, \dots$ and $\gamma_1, \gamma_2, \dots$ (see Definition 5.6 and Definition 2.3, respectively) are for all integers $i \geq 1$ related by $\gamma_i \leq \kappa_i \leq \gamma_i + 1$.

In view of the fact [27] that $\widetilde{\mathcal{M}}_n^1$ is isomorphic to the group of homotopy spheres, the following natural question arises. For which k does a given homotopy sphere admit a k-constrained Morse function that represents $0 \in \widetilde{\mathcal{M}}_n^k$? For $k = 1$ this is only possible for the standard n-sphere. As an application of our results we show in Theorem 6.2 that for certain $n \equiv 1 \mod 4$ and $k = (n-1)/2$, the exotic Kervaire n-sphere can be characterized among all exotic spheres by the property that it admits a k-constrained Morse function that represents $0 \in \widetilde{\mathcal{M}}_n^k$. This type of result has recently become relevant in the context of a concrete positive TFTs constructed by Banagl (see [32]).

Notation. All manifolds and maps considered in this paper will be smooth of class C^∞. For an oriented closed manifold M^n the manifold equipped with the opposite orientation will be denoted by $-M^n$. The symbol $\cong$ will either mean orientation preserving diffeomorphism of manifolds or isomorphism of groups.

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2. Connective Bordism

Theorem 1.1 relates constrained generic bordism groups (see Definition 3.1) to connective bordism groups defined as follows (compare [30, Example 17, p. 51]).

Definition 2.1. Fix integers $n \geq 1$ and $1 \leq k < n$. Let M^n and N^n be non-empty k-connected oriented closed n-manifolds. An *oriented k-connective bordism from M^n to N^n* is a k-connected oriented compact manifold W^{n+1} with boundary $\partial W^{n+1} = M^n \sqcup -N^n$.

The *oriented k-connective n-bordism group* $\widetilde{\mathcal{C}}_n^k$ is the set of equivalence classes $[M^n]$ of non-empty k-connected oriented closed n-manifolds M^n subject to the equivalence relation of oriented k-connective bordism.

By definition, $\widetilde{\mathcal{C}}_n^k$ coincides for $k > (n-1)/2$ with the group of homotopy spheres Θ_n as defined in [14], and it can be shown analogously that $\widetilde{\mathcal{C}}_n^k$ is for any $1 \leq k < n$ an abelian group with group law induced by oriented connected sum, $[M^n]+[N^n] := [\sharp(M^n \sqcup N^n)]$, identity element represented by the standard sphere S^n, and inverses induced by reversing the orientation, $-[M^n] = [-M^n]$.

Proposition 2.2. *Let $1 < k \leq (n-1)/2$. The natural homomorphism $\widetilde{\mathcal{C}}_n^k \to \widetilde{\mathcal{C}}_n^{k-1}$, $[M^n] \mapsto [M^n]$, is injective for $k \equiv 3, 5, 6, 7 \bmod 8$.*

Proof. If M^n represents $0 \in \widetilde{\mathcal{C}}_n^{k-1}$, then there exists an oriented $(k-1)$-connected compact manifold W^{n+1} with boundary $\partial W^{n+1} = M^n$. Note that any chosen triangulation of W^{n+1} is $(k-1)$-parallelizable (i.e., parallelizable over the $(k-1)$-skeleton, see [20, Section 5, p. 49]). The obstruction for being k-parallelizable vanishes since $\pi_{k-1}(SO(n)) = 0$ for $k \equiv 3, 5, 6, 7 \bmod 8$ (see the proof of [14, Theorem 3.1, p. 508]). Therefore, by [20, Theorem 3, p. 49] W can be made $\min\{k, \lfloor \dim W^{n+1}/2 - 1 \rfloor\} = k$-connected by a finite sequence of surgeries without changing $M^n = \partial W^{n+1}$. Hence, if M^n happens to be k-connected, then M^n represents $0 \in \widetilde{\mathcal{C}}_n^k$. □

Poincaré duality implies that orientable closed manifolds with odd Euler characteristic can only exist in dimensions which are a multiple of 4. For instance, $\mathbb{C}P^{2i}$ is for any integer $i \geq 1$ a simply connected closed $4i$-manifold with odd Euler characteristic. We define a sequence $\gamma_1, \gamma_2, \ldots$ of positive integers as follows (compare [4, Problem 2.6, p. 151]).

Definition 2.3. For every integer $i \geq 1$ let γ_i be the greatest integer $k \geq 1$ for which there exists a k-connected closed manifold V^{4i} with odd Euler characteristic (or, equivalently, odd signature).

Note that $\gamma_i < 2i$ because $2i$-connected closed $4i$-manifolds are homotopy spheres. For odd i, we always have $\gamma_i = 1$ because 2-connected closed $4i$-manifolds V^{4i} are spinable, which implies that their signature is a multiple of 16 according to Ochanine's generalization of Rochlin's theorem (see [22, p. 133]). For even i, we have $\gamma_i \geq 3$ because the quaternionic projective space $\mathbb{H}P^i$ has odd Euler characteristic. In particular, $\gamma_2 = 3$. When $i = 4j$ is a multiple of 4, then $\gamma_i \geq 7$ because the j-fold power $\mathbb{O}P^2 \times \cdots \times \mathbb{O}P^2$ of the octonionic projective plane $\mathbb{O}P^2$ is a 7-connected closed manifold with odd Euler characteristic. In particular, $\gamma_4 = 7$.

By an argument analogous to the proof of Proposition 2.2, we can show that $\gamma_i \not\equiv 2, 4, 5, 6 \bmod 8$ for all $i \geq 1$.

3. Preliminaries on Generic Maps into the Plane

In this section we collect essential techniques for constructing and studying generic maps from bordisms into the plane.

Fix an integer $n \geq 1$. Recall that any smooth map of a manifold X^{n+1} into the plane can be approximated arbitrarily well in the Whitney C^∞ topology by a smooth map $G\colon X^{n+1} \to \mathbb{R}^2$ whose singular locus $S(G) = \{x \in X^{n+1};\ \operatorname{rank} d_x G < 2\}$ consists of those $x \in X^{n+1}$ admitting coordinate charts centered at x and $G(x)$, respectively, in which G has one of the following normal forms:

(1) $(t, x_1, \dots, x_n) \mapsto (t, x_1 x_2 + x_1^3 \pm x_3^2 \pm \dots \pm x_n^2)$, i.e., x is a *cusp* of G.
(2) $(t, x_1, \dots, x_n) \mapsto (t, x_1^2 \pm \dots \pm x_n^2)$, i.e., x is a *fold point* of G.

Definition 3.1. Let $f\colon M^n \to \mathbb{R}$ and $g\colon N^n \to \mathbb{R}$ be k-constrained Morse functions on oriented closed n-manifolds. An *oriented k-constrained generic bordism from f to g* is an oriented bordism W^{n+1} from M^n to $-N^n$ equipped with a generic map $G\colon W^{n+1} \to \mathbb{R} \times [0,1]$ such that

(i) there exist tubular neighbourhoods $M^n \times [0, \varepsilon) \subset W^{n+1}$ of $M^n \times \{0\} = M^n \subset W^{n+1}$ and $N^n \times (1-\varepsilon, 1] \subset W^{n+1}$ of $N^n \times \{1\} = N^n \subset W^{n+1}$ such that
$$G|_{M^n \times [0,\varepsilon)} = f \times \mathrm{id}_{[0,\varepsilon)}, \qquad G|_{N^n \times (1-\varepsilon,1]} = g \times \mathrm{id}_{(1-\varepsilon,1]}\,.$$
(ii) all absolute indices of fold points of G are contained in $\{\lceil n/2 \rceil, \dots, n-k\} \cup \{n\}$.

The *oriented k-constrained generic n-bordism group* $\widetilde{\mathcal{G}}_n^k$ is the set of equivalence classes $[f]$ of k-constrained Morse functions $f\colon M^n \to \mathbb{R}$ on oriented closed n-manifolds subject to the equivalence relation of oriented k-constrained generic bordism.

Definition 3.2. Let $f\colon M^n \to \mathbb{R}$ and $g\colon N^n \to \mathbb{R}$ be k-constrained Morse functions on oriented closed n-manifolds. An *oriented k-constrained bordism from f to g* is an oriented k-constrained generic bordism from f to g without cusps. The *oriented n-bordism group of k-constrained Morse functions* $\widetilde{\mathcal{M}}_n^k$ is the set of equivalence classes $[f]$ of k-constrained Morse functions $f\colon M^n \to \mathbb{R}$ on oriented closed n-manifolds subject to the equivalence relation of oriented k-constrained bordism.

Note that $\widetilde{\mathcal{G}}_n^k$ and $\widetilde{\mathcal{M}}_n^k$ are abelian groups. In both cases, the group law is induced by disjoint union, $[f\colon M^n \to \mathbb{R}] + [g\colon N^n \to \mathbb{R}] := [f \sqcup g\colon M^n \sqcup N^n \to \mathbb{R}]$, the identity element is represented by the unique map $f_\emptyset\colon \emptyset \to \mathbb{R}$, and the inverse of $[f\colon M^n \to \mathbb{R}]$ is given by $[-f\colon -M^n \to \mathbb{R}]$.

There are natural homomorphisms $\widetilde{\mathcal{G}}_n^l \to \widetilde{\mathcal{G}}_n^k$ and $\widetilde{\mathcal{M}}_n^l \to \widetilde{\mathcal{M}}_n^k$ whenever $l \geq k$. Moreover, there is a natural homomorphism $\widetilde{\mathcal{M}}_n^k \to \widetilde{\mathcal{G}}_n^k$ which maps the class $[f\colon M^n \to \mathbb{R}] \in \widetilde{\mathcal{M}}_n^k$ to the class $[f\colon M^n \to \mathbb{R}] \in \widetilde{\mathcal{G}}_n^k$.

Remark 3.3. By definition, the groups $\widetilde{\mathcal{G}}_n^k$ and $\widetilde{\mathcal{M}}_n^k$ coincide for $k > n/2$ both with the oriented bordism group of special generic functions on n-manifolds $\widetilde{\Gamma}(n,1)$ as defined in [27].

Remark 3.4. Varying k, the group $\widetilde{\mathcal{G}}_n^k$ interpolates between the smooth oriented bordism group Ω_n^{SO} (an isomorphism $\widetilde{\mathcal{G}}_n^1 \xrightarrow{\cong} \Omega_n^{SO}$ is given by $[f\colon M^n \to \mathbb{R}] \mapsto [M^n]$) and, by [27], the group of homotopy spheres $\Theta_n \cong \widetilde{\mathcal{G}}_n^k$ ($k > n/2$).

3.1. **Elimination of Cusps; Cusps and Euler Characteristic.** We refer to [10] for a detailed discussion of the material presented in this section.

Recall from [10, Definition 2.2, p. 213] that there exist homomorphisms

$$\widetilde{\varphi}_\lambda \colon \widetilde{\mathcal{M}}^1_n \to \mathbb{Z}, \quad [f] \mapsto C_\lambda(f) - C_{n-\lambda}(f), \qquad \lambda \in \{0, \dots, n\},$$

where $C_\mu(f)$ denotes the number of critical points of f of Morse index μ. For any integer $1 < k \leq n/2$ we use the natural homomorphism $\widetilde{\mathcal{M}}^k_n \to \widetilde{\mathcal{M}}^1_n$ to define a homomorphism (compare [10, Definition 2.3, p. 213])

$$\widetilde{\Phi}^k \colon \widetilde{\mathcal{M}}^k_n \to \mathbb{Z}^{\lfloor n/2 \rfloor - k}, \quad [f] \mapsto (\widetilde{\varphi}_{\lfloor (n+3)/2 \rfloor}([f]), \dots, \widetilde{\varphi}_{n-k}([f])).$$

Levine's technique [18] of eliminating pairs of cusps of generic maps into the plane (see also [10, Section 3, pp. 215ff]) can be used as in [10] to prove the following.

Theorem 3.5. *Suppose that $n \geq 2$. Let $G \colon W^{n+1} \to \mathbb{R}^2$ be an oriented k-constrained generic bordism from $g_0 \colon M^n_0 \to \mathbb{R}$ to $g_1 \colon M^n_1 \to \mathbb{R}$. Suppose that $\widetilde{\Phi}^k(g_0) = \widetilde{\Phi}^k(g_1)$. Moreover, if n is odd, then suppose that G has an even number of cusps. Then, $[g_0] = [g_1] \in \widetilde{\mathcal{M}}^k_n$.*

Proof. We make W^{n+1} connected by using the oriented connected sum operation, and modify G accordingly while performing the oriented connected sum along small 2-discs centered at definite fold points of G. If W^{n+1} is connected, then G is homotopic rel ∂W^{n+1} to an oriented k-constrained bordism from g_0 to g_1 by iterated elimination of cusps. For details, see [10, proof of Theorem 2.7, p. 220ff]. □

Remark 3.6. For $k = n/2 > 1$ it can be shown that any oriented $n/2$-constrained generic bordism $G \colon W^{n+1} \to \mathbb{R}^2$ is already an oriented $n/2$-constrained bordism. Indeed, the map G cannot have cusps because the occuring absolute fold indices n and $n/2$ are not consecutive integers when $n/2 > 1$. Consequently, $\widetilde{\mathcal{M}}^{n/2}_n = \widetilde{\mathcal{G}}^{n/2}_n$.

By an adaption of the proof of [10, Lemma 5.2, p. 226] we have the following.

Proposition 3.7. *Let $G \colon W^{n+1} \to \mathbb{R}^2$ be an oriented 1-constrained generic bordism from $g_0 \colon M^n_0 \to \mathbb{R}$ to $g_1 \colon M^n_1 \to \mathbb{R}$. Let c denote the number of cusps of G, and let ν denote the number of critical points of $g_0 \sqcup g_1$. Then, ν is even, and $c + \nu/2 \equiv \chi(W^{n+1}) \mod 2$, where $\chi(W^{n+1})$ denotes the Euler characteristic of W^{n+1}.*

3.2. **Two-Index Theorem.**

Theorem 3.8. *Fix integers $n \geq 5$ and $1 < k < n/2$. Suppose that $f_0, f_1 \colon M^n \to \mathbb{R}$ are k-constrained Morse functions on a closed manifold M^n. Then, there exists an oriented k-constrained generic bordism $F \colon M^n \times [0,1] \to \mathbb{R}^2$ from f_0 to f_1.*

Proof. Without loss of generality, we may assume that M^n is connected, and that $f_0(M^n) = f_1(M^n) = [0,1]$. For $i = 0, 1$ let c^0_i and c^1_i denote the unique critical points of f_i of index 0 and n, respectively. For $i, j \in \{0, 1\}$ and suitable $\varepsilon > 0$ there exist orientation preserving embeddings $\iota^j_i \colon D^n_{2\varepsilon} \to M$ such that $\iota^j_i(0) = c^j_i$ and

$$(f_i \circ \iota^j_i)(x) = e^j(||x||^2) := j + (-1)^j ||x||^2, \qquad x \in D^n_{2\varepsilon}. \qquad (*)$$

Furthermore, for possibly smaller $\varepsilon > 0$, there exists an isotopy $H \colon [0,1] \times M \to M$ of diffeomorphisms $H_t := H(t, -) \colon M \to M$ such that $H_0 = \mathrm{id}_M$ and $H_1 \circ \iota^j_0 = \iota^j_1$ for $j = 0, 1$. Therefore, after replacing f_1 by $f_1 \circ H_1$, we may without loss of generality work with the assumption that $f_0 \circ \iota^j_0 = f_1 \circ \iota^j_0$ for $j = 0, 1$. Set $U^j := \iota^j_0(D^n_\varepsilon)$ for

$j = 0, 1$. Set $V := M \setminus (\iota_0^0(\operatorname{int} D_\varepsilon^n) \cup \iota_0^1(\operatorname{int} D_\varepsilon^n))$ and $V^j := U^j \cap V \cong S^{n-1}$ for $j = 0, 1$. Then, f_i restricts for $i = 0, 1$ to a Morse function

$$g_i := f_i|_V \colon (V, V^0, V^1) \to ([\varepsilon^2, 1 - \varepsilon^2], \varepsilon^2, 1 - \varepsilon^2)$$

all of whose critical points have Morse index contained in the set $\{k, \ldots, n-k\}$. Choose a generic 1-parameter family g_t, $t \in [0, 1]$, as described in [3, Theorem 9.4, pp. 190f] with regular level sets $V_0 = g_t^{-1}(\varepsilon^2)$ and $V_1 = g_t^{-1}(1 - \varepsilon^2)$. Since the cardinality of the set $\{k, \ldots, n-k\}$ is at least 2 (recall that $k < n/2$), we can use the two-index theorem of Hatcher and Wagoner (see [7, Chapter V, Proposition 3.5]) in the form presented in [3, pp. 214f] to modify the family g_t rel g_0 and g_1 iteratively in such a way that the resulting generic map $G \colon V \times [0, 1] \to [\varepsilon^2, 1 - \varepsilon^2] \times [0, 1]$, $(x, t) \mapsto (g_t(x), t)$ is k-constrained.

In the following, we sketch the construction of the desired map F, which amounts to a careful extension of G over $U^j \times [0, 1]$ for $j = 0, 1$. (The construction is presented in full detail in [32, Section 8.4] using [32, Appendix B].) Without loss of generality, we may assume for $t \in [0, 1]$ that $g_t = g_0$ when t is near 0, and that $g_t = g_1$ when t is near 1. We extend $g \colon V \times [0, 1] \to [\varepsilon^2, 1 - \varepsilon^2]$ to a smooth map $\tilde{g} \colon \tilde{V} \times [0, 1] \to \mathbb{R}$ for some open neighborhood $\tilde{V}$ of V in M such that, for $t \in [0, 1]$, $\tilde{g}|_{\tilde{V} \times \{t\}} = f_0|_{\tilde{V}}$ when t is near 0, and that $\tilde{g}|_{\tilde{V} \times \{1\}} = f_1|_{\tilde{V}}$ when t is near 1. For $j = 0, 1$, we define a tubular neighborhood of $V^j \times [0, 1]$ in $\tilde{V} \times [0, 1]$ by

$$\alpha^j \colon (-\delta, \delta) \times V^j \times [0, 1] \to \tilde{V} \times [0, 1], \qquad \alpha^j(u, v, t) = (\iota_0^j(\rho(u) \cdot (\iota_0^j)^{-1}(v)), t),$$

where $\rho \colon (-1/2, 1/2) \to \mathbb{R}$ is given by $\rho(r) = \sqrt{r + 1}$. By construction, we have $\mathrm{pr}_{[0,1]} \circ \alpha^j = \mathrm{pr}_{[0,1]}$ and $(\tilde{g} \circ \alpha^j)(u, v, t) = e^j(\varepsilon^2(u + 1))$ when $t \in [0, 1]$ is near 0 or near 1. For $j = 0, 1$, we use the technique of integral curves of vector fields on manifolds with boundary (see [9, Chapter 6 §2, pp. 149ff]) to construct another tubular neighborhood of $V^j \times [0, 1]$ in $\tilde{V} \times [0, 1]$, say

$$\beta^j \colon (-\delta, \delta) \times V^j \times [0, 1] \to \tilde{V} \times [0, 1],$$

such that $\mathrm{pr}_{[0,1]} \circ \beta^j = \mathrm{pr}_{[0,1]}$ and $(\tilde{g} \circ \beta^j)(u, v, t) = e^j(\varepsilon^2(u + 1))$. By adapting the proof of [9, Theorem 5.3, p. 112] we can construct for some open neighborhood $U \subset \tilde{V} \times [0, 1]$ of $\partial V \times [0, 1]$ an isotopy rel $U \cap (\tilde{V} \times \{0, 1\})$ from the inclusion $U \hookrightarrow \tilde{V} \times [0, 1]$ to an embedding $\theta \colon U \to \tilde{V} \times [0, 1]$ such that $\theta \circ \alpha^j = \beta^j$ on a neighborhood of $V^j \times [0, 1]$ in $\tilde{V} \times [0, 1]$. A version of the isotopy extension theorem (see [9, Theorem 1.4, p. 180]) provides an ambient isotopy rel a neighborhood of $M \times \{0, 1\}$ in $M \times [0, 1]$ from $\mathrm{id}_{M \times [0,1]}$ to an automorphism Θ of $M \times [0, 1]$ such that $\Theta \circ \alpha^j = \beta^j$ for $j = 0, 1$ on a neighborhood of $V^j \times [0, 1]$ in $M \times [0, 1]$. Finally, the desired map $F \colon M \times [0, 1] \to \mathbb{R}^2$ can be defined as

$$F(x, t) = \begin{cases} (G \circ \Theta)(x, t), & \text{if } x \in V, \\ (e^j(||(\iota_0^j)^{-1}(x)||^2), t), & \text{if } x \in U^j. \end{cases}$$

$\square$

Remark 3.9. In [27, Lemma 3.1, p. 291], Cerf's pseudo-isotopy theorem [2] is used to show that the statement of Theorem 3.8 also holds for $n \geq 6$ and $k > n/2$.

3.3. **Handle Extension Theorem.** Let W^m be an oriented bordism from W_0 to W_1 of dimension $m = \dim W \geq 6$ Obviously, any given Morse functions $g_0 \colon W_0 \to \mathbb{R}$ and $g_1 \colon W_1 \to \mathbb{R}$ can be extended to an oriented 1-constrained generic bordism $G \colon W \to \mathbb{R}^2$. If g_0 and g_1 are k-constrained for $k > 1$, does there exist an oriented k-constrained generic bordism $G \colon W \to \mathbb{R}^2$ from g_0 to g_1? The "handle extension theorem" for k-constrained generic maps gives an affirmative answer to this question, provided that W admits a handle decomposition without handles of index contained in the set $\{m-k, \dots m-2\}$. More specifically, the following result due to Gay and Kirby [6] holds:

Theorem 3.10. *Let $k \geq 2$ be an integer. Suppose that W_0 is $(k-1)$-connected. Furthermore, suppose that*

$$\tau \colon (W, W_0, W_1) \to ([0,1], 0, 1)$$

is a Morse function all of whose critical points have the same Morse index $\lambda \in \{k+1, \dots, m-k-1\}$, and are contained in $\tau^{-1}(1/2)$. Then, there exists a smooth map $\sigma \colon W \to \mathbb{R}$ with the following properties:

(1) σ restricts for every $t \neq 1/2$ to an excellent k-constrained Morse function $\tau^{-1}(t) \to \mathbb{R}$.
(2) σ and τ form the components of an oriented k-constrained bordism

$$(\sigma, \tau) \colon W \to \mathbb{R} \times [0,1].$$

3.4. **Stein Factorization.** The importance of Stein factorization for the global study of singularities of smooth maps was first realized when Burlet and de Rham [1] used it as a tool to study special generic maps of 3-manifolds into the plane.

We recall the concept of Stein factorization of an arbitrary continuous map $f \colon X \to Y$ between topological spaces. Define an equivalence relation $\sim_f$ on X as follows. Two points $x_1, x_2 \in X$ are called equivalent, $x_1 \sim_f x_2$, if they are mapped by f to the same point $y := f(x_1) = f(x_2) \in Y$, and lie in the same connected component of $f^{-1}(y)$. The quotient map $\pi_f \colon X \to X/\sim_f$ gives rise to a unique set-theoretic factorization of f of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi_f"'] & Y \\ X/\sim_f \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

.

If we equip the quotient space $X_f := X/\sim_f$ with the quotient topology induced by the surjective map $\pi_f \colon X \to X_f$, then it follows that the maps π_f and $\overline{f}$ are continuous.

In the following, both the above diagram and the quotient space $X_f = X/\sim_f$ will be referred to as the *Stein factorization* of f.

In generalization of [27, Lemma 2.1, p. 290], the following result clarifies the structure of Stein factorization for k-constrained generic bordisms $W^m \to \mathbb{R}^2$ for $m \geq 3$ and $k > 1$.

Theorem 3.11. *Let (W^m, M_1, M_2) be a smooth manifold triad of dimension $m := \dim W \geq 3$. Suppose that $F \colon W^m \to \mathbb{R}^2$ is an oriented k-constrained generic bordism, and that F is a stable map.*

If $k > 1$, then the Stein factorization $W_F = W/\sim_F$ of F can be given the structure of a compact smooth manifold of dimension 2 with corners in such a way that $\pi_F\colon W \to W_F$ is a generic smooth map and $\overline{F}\colon W_F \to \mathbb{R}^2$ is an immersion. Furthermore, if $D(F)$ denotes the union of the definite fold lines of F, then the boundary of W_F decomposes as

$$\partial W_F = \pi_F(\partial W) \cup \pi_F(D(F)),$$

where $\pi_F(\partial W) \cap \pi_F(D(F)) = \pi_F(\partial W \cap D(F))$ is the set of corners of W_F, and π_F restricts to an embedding $D(F) \to \partial W_F$.

Proof. The claims follow from [16, Theorem 2.2, p. 2609]. Note that the only local neighborhoods of points in W_F that can occur are those of types (a), $(b1)$, $(b3)$, $(c2)$ and $(d2)$ in [16, Figure 1, p. 2610] because we have excluded fold points of absolute index $m-2$. This implies that W_F is a topological 2-manifold. Finally, the desired smooth structure on W_F is induced by requiring $\overline{F}$ to be a local diffeomorphism. □

4. Proof of Theorem 1.1

Fix integers $n \geq 5$ and $1 \leq k < n$.

The following straightforward generalization of [27, Lemma 3.2, p. 292] will frequently be used. Note that it can be considered as a version of Theorem 3.10 for the case that τ has only critical points of Morse index $\lambda \in \{1, m-1\}$.

Lemma 4.1. *Let $f\colon M^n \to \mathbb{R}$ be a k-constrained Morse function. Then, there exist a k-constrained Morse function $g\colon \sharp(M^n) \to \mathbb{R}$ and an oriented k-constrained bordism $G\colon W^{n+1} \to \mathbb{R}^2$ from g to f.*

Proposition 4.2 below (compare [27, Lemma 3.3, p. 293]) shows that the assignment $[f\colon M^n \to \mathbb{R}] \mapsto [\sharp(M^n)]$ defines a well-defined map $\delta_n^k\colon \widetilde{\mathcal{G}}_n^k \to \widetilde{\mathcal{C}}_n^{k-1}$ for any integer $1 < k < n$. Then, it follows that $\delta_n^k\colon \widetilde{\mathcal{G}}_n^k \to \widetilde{\mathcal{C}}_n^{k-1}$ is a homomorphism because the existence of an oriented diffeomorphism $\sharp(M \sqcup N) \cong \sharp(\sharp(M) \sqcup \sharp(N))$ implies that $\delta_n^k([f]+[g]) = [\sharp(M \sqcup N)] = [\sharp(\sharp(M) \sqcup \sharp(N))] = \delta_n^k([f]) + \delta_n^k([g])$ for any k-constrained Morse functions $f\colon M^n \to \mathbb{R}$ and $g\colon N^n \to \mathbb{R}$.

Proposition 4.2. *Suppose that $1 < k < n$. Let $f\colon M^n \to \mathbb{R}$ and $g\colon N^n \to \mathbb{R}$ be k-constrained Morse functions, and let $G\colon W^{n+1} \to \mathbb{R}^2$ be an oriented k-constrained generic bordism from f to g. Then, there exists an oriented $(k-1)$-connective bordism V^{n+1} from M^n to N^n.*

Proof. By assumption there exists an oriented k-constrained generic bordism from $f \sqcup -g\colon M^n \sqcup -N^n \to \mathbb{R}$ to $\emptyset \to \mathbb{R}$. Lemma 4.1 implies that there exists a k-constrained Morse function $h\colon \sharp(M^n \sqcup N^n) \to \mathbb{R}$, and an oriented k-constrained bordism $H\colon \overline{V}^{n+1} \to \mathbb{R}^2$ from h to $\emptyset \to \mathbb{R}$. By Theorem 3.11 the Stein factorization $\overline{V}_H$ of H is a connected oriented compact 2-manifold with two corners, where we have assumed without loss of generality that H is stable. (In fact, H can always achieved to be stable by first choosing h to have a single critical point on each level set, and then perturbing H in the interior of $\overline{V}^{n+1}$.) Then, by the classification of surfaces, we may think of $\overline{V}_H$ as a half disc with a finite number of 1-handles attached. We choose an oriented compact submanifold V^{n+1} of $\overline{V}^{n+1}$ with boundary $\partial V = \sharp(M^n \sqcup N^n) \sqcup -P^n$, where P^n is an oriented closed n-manifold whose image under π_H looks like the dotted line L in Fig. . . . , where we assume that L

avoids images of cusps of π_H, and is transverse to $\pi_H(S(\pi_H))$. Let $L_1, \ldots, L_r$ be the lines indicated in Fig. .... Then, we have an orientation preserving diffeomorphism $P^n \cong \sharp(P_1 \sqcup (-P_1) \sqcup \cdots \sqcup P_r \sqcup (-P_r))$, where $P_i := \pi_H^{-1}(L_i)$. By construction, the Stein factorization of the restriction $F := H|_V$ is diffeomorphic to a rectangle, $V_F \cong [0,1] \times [0,1]$, in such a way that $\{0,1\} \times [0,1]$ corresponds to $\pi_F(\partial V)$ and $[0,1] \times \{0,1\}$ corresponds to $\pi_F(S)$, where S denotes the definite fold locus of F. A generic projection to one direction gives us a k-constrained Morse function $V \to \mathbb{R}$ with regular level sets $\sharp(M^n \sqcup N^n)$ and P^n. Therefore, V is a $(k-1)$-connective bordism from $\sharp(M^n \sqcup N^n)$ to P^n, and the claim follows. □

Let us show that the choice of k-constrained Morse functions $f\colon M^n \to \mathbb{R}$ on given k-connected oriented closed manifolds M^n induces a well-defined map $\varepsilon_n^k\colon \widetilde{\mathcal{C}}_n^k \to \widetilde{\mathcal{G}}_n^k$, $[M^n] \mapsto [f\colon M^n \to \mathbb{R}]$. Note that ε_n^k will be a homomorphism by Lemma 4.1. We distinguish between the cases that $k > n/2$, $k = n/2$ and $k < n/2$.

For $k > n/2$, note that $\widetilde{\mathcal{C}}_n^k = \Theta_n$, the group of homotopy spheres, and $\widetilde{\mathcal{G}}_n^k = \widetilde{\Gamma}(n,1)$, the bordism group of special generic functions. Then, ε_n^k is the homomorphism $\widetilde{\Phi}\colon \Theta_n \to \widetilde{\Gamma}(n,1)$ introduced in the proof of [27, Theorem 1.1, p. 288] (compare [27, p. 294]), and is well-defined by [27, Lemma 3.1, p. 291].

For $k = n/2$ we have already noted that $\widetilde{\mathcal{C}}_n^{n/2} = \Theta_n$, the group of homotopy spheres. All indefinite critical points of an $n/2$-constrained Morse function $f\colon M^n \to \mathbb{R}$ are of index $n/2$, so there are none if M^n is a homotopy sphere. Hence, the desired homomorphism $\varepsilon_n^{n/2}\colon \widetilde{\mathcal{C}}_n^{n/2} \to \widetilde{\mathcal{G}}_n^{n/2}$ is the composition of the homomorphism $\widetilde{\mathcal{C}}_n^{n/2} = \Theta_n \xrightarrow{\widetilde{\Phi}} \widetilde{\Gamma}(n,1) = \widetilde{\mathcal{G}}_n^{n/2+1}$ mentioned above with the homomorphism $\widetilde{\mathcal{G}}_n^{n/2+1} \to \widetilde{\mathcal{G}}_n^{n/2}$.

Let $k < n/2$. For $k = 1$, the map ε_n^1 is just the composition of the map $\widetilde{\mathcal{C}}_n^1 \to \Omega_n^{SO}$, $[M^n] \to [M^n]$, with the inverse of the isomorphism $\widetilde{\mathcal{G}}_n^1 \xrightarrow{\cong} \Omega_n^{SO}$ described in Remark 3.4. For $1 < k < n/2$, evaluation of the map $\varepsilon_n^k\colon \widetilde{\mathcal{C}}_n^k \to \widetilde{\mathcal{G}}_n^k$ on a given element of $\widetilde{\mathcal{C}}_n^k$ depends a priori on the choice of a representative M^n, and on the choice of a k-constrained Morse function $f\colon M^n \to \mathbb{R}$, and independence of choices follows from the following result, which generalizes [27, Lemma 3.1, p. 291].

Proposition 4.3. *Suppose that $1 < k < n/2$. Let W^{n+1} be an oriented $(k-1)$-connective bordism from M_0^n to M_1^n, and let $g_0\colon M_0^n \to \mathbb{R}$ and $g_1\colon M_1^n \to \mathbb{R}$ be k-constrained Morse functions. If W^{n+1} is k-connected, then there exists an oriented k-constrained generic bordism $G\colon W^{n+1} \to \mathbb{R}^2$ from g_0 to g_1.*

Proof. By the rearrangement theorem [21, p. 44] we can modify an arbitrarily chosen Morse function $f\colon W^{n+1} \to [-1/2, n+1+1/2]$ with regular level sets $M_0^n = f^{-1}(-1/2)$ and $M_1^n = f^{-1}(n+1+1/2)$ to be self-indexing, i.e., the Morse index of every critical point c of f is $f(c)$. Following the proof of Smale's h-cobordism theorem [21, Theorem 9.1, p. 107], we then modify f in such a way that all Morse indices of critical points of f are contained in the set $\{k+1, \ldots, n-k\}$. By Theorem 3.10 there exist for every $\lambda \in \{k+1, \ldots n-k\}$ k-constrained Morse functions $g_\lambda^-\colon f^{-1}(\lambda - 1/4) \to \mathbb{R}$ and $g_\lambda^+\colon f^{-1}(\lambda + 1/4) \to \mathbb{R}$, and an oriented k-constrained generic bordism $G_\lambda\colon f^{-1}([\lambda - 1/4, \lambda + 1/4]) \to \mathbb{R}^2$ from g_λ^- to g_λ^+. Furthermore, by Theorem 3.8 there exist oriented k-constrained generic bordisms $G_{0,k+1}\colon f^{-1}([-1/2, (k+1)-1/4]) \to \mathbb{R}^2$ from g_0 to g_{k+1}^-, $G_{\lambda,\lambda+1}\colon f^{-1}([\lambda+1/4, (\lambda+$

$1)-1/4]) \to \mathbb{R}^2$ from g^+_λ to $g^-_{\lambda+1}$, $\lambda \in \{k+1, \dots n-k-1\}$, and $G_{n-k,n+1}\colon f^{-1}([(n-k)+1/4, n+1+1/2]) \to \mathbb{R}^2$ from g^+_{n-k} to g_1, and the claim follows. □

From now on, let $1 < k < n$. The composition $\delta^k_n \circ \varepsilon^k_n$ coincides by definition with the natural homomorphism $\widetilde{\mathcal{C}}^k_n \to \widetilde{\mathcal{C}}^{k-1}_n$, $[M^n] \mapsto [M^n]$. Let us show that the composition $\varepsilon^{k-1}_n \circ \delta^k_n$ coincides with the natural homomorphism $\widetilde{\mathcal{G}}^k_n \to \widetilde{\mathcal{G}}^{k-1}_n$, $[f\colon M^n \to \mathbb{R}] \mapsto [f]$. By definition, the composition $\varepsilon^{k-1}_n \circ \delta^k_n$ assigns to the class $[f] \in \widetilde{\mathcal{G}}^k_n$ of a k-constrained Morse function the class $[g] \in \widetilde{\mathcal{G}}^{k-1}_n$ represented by an arbitrarily chosen $(k-1)$-constrained Morse function $g\colon \sharp(M^n) \to \mathbb{R}$. In view of Lemma 4.1 it suffices to show that any two $(k-1)$-constrained Morse functions on $\sharp(M^n)$ are oriented $(k-1)$-constrained generic bordant. If $k-1 < n/2$, then this holds by Theorem 3.8. If $k-1 \geq n/2$, then $\widetilde{\mathcal{G}}^k_n = \widetilde{\Gamma}(n,1)$, the bordism group of special generic functions. Hence, $f\colon M^n \to \mathbb{R}$ is a special generic function, $\sharp(M^n)$ a homotopy n-sphere, and the claim follows from [27, Lemma 3.1, p. 291].

This completes the proof of Theorem 1.1.

Remark 4.4. Unoriented versions $\mathcal{C}^k_n$ and $\mathcal{G}^k_n$ of the groups $\widetilde{\mathcal{C}}^k_n$ and $\widetilde{\mathcal{G}}^k_n$ can be defined in a straightforward way by forgetting orientations in Definition 2.1 and Definition 3.1, respectively. It is easy to show that the natural epimorphisms $\widetilde{\mathcal{C}}^k_n \to \mathcal{C}^k_n$, $[M^n] \mapsto [M^n]$, has kernel $2\widetilde{\mathcal{C}}^k_n$. Moreover, for $k > 1$ we can modify the argument of the proof of Proposition 4.2 in order to show that the epimorphism $\widetilde{\mathcal{G}}^k_n \to \mathcal{G}^k_n$, $[f\colon M^n \to \mathbb{R}] \mapsto [f\colon M^n \to \mathbb{R}]$, has kernel $2\widetilde{\mathcal{G}}^k_n$ (compare [27, Remark 3.4, p. 293]). For $k=1$ we use the isomorphism $\widetilde{\mathcal{G}}^1_n \xrightarrow{\cong} \Omega^{SO}_n$ described in Remark 3.4, and the analogously defined isomorphism $\mathcal{G}^1_n \xrightarrow{\cong} \Omega^O_n$ to the unoriented smooth bordism group to show that the homomorphism $\widetilde{\mathcal{G}}^1_n \to \mathcal{G}^1_n$, $[f\colon M^n \to \mathbb{R}] \mapsto [f\colon M^n \to \mathbb{R}]$, has kernel $2\widetilde{\mathcal{G}}^1_n$. (In fact, note that the homomorphism $\Omega^{SO}_n \to \Omega^O_n$, $[M^n] \to [M^n]$, has kernel $2\Omega^{SO}_n$.) Hence, all the statements of Theorem 1.1 carry over to unoriented bordism groups.

5. Proof of Theorem 1.2

Fix integers $n \geq 4$ and $1 < k \leq n/2$. We present the proof of Theorem 1.2 in a sequence of lemmas.

Lemma 5.1. *The homomorphism*

$$\beta^k_n\colon \widetilde{\mathcal{M}}^k_n \to \widetilde{\mathcal{G}}^k_n \oplus \mathbb{Z}^{\lfloor n/2 \rfloor - k}, \quad [f\colon M^n \to \mathbb{R}] \mapsto ([f], \widetilde{\Phi}^k([f])),$$

is surjective, where $\widetilde{\Phi}^k\colon \widetilde{\mathcal{M}}^k_n \to \mathbb{Z}^{\lfloor n/2 \rfloor - k}$ *is defined in Section 3.1.*

Proof. Given an element $([g\colon M^n \to \mathbb{R}], c_1, \dots, c_{\lfloor n/2 \rfloor - k}) \in \widetilde{\mathcal{G}}^k_n \oplus \mathbb{Z}^{\lfloor n/2 \rfloor - k}$, we can use the same argument as in [10, p. 220] to modify g iteratively by introducing pairs of critical points of successive Morse indices in order to produce a k-constrained Morse function $f\colon M^n \to \mathbb{R}$ which satisfies $\widetilde{\Phi}^k([f]) = (c_1, \dots, c_{\lfloor n/2 \rfloor - k})$. Since $[f] = [g]$ in $\widetilde{\mathcal{G}}^k_n$ by means of a generic homotopy that realizes the sequence of births of critical point pairs, we obtain $\beta^k_n([f\colon M^n \to \mathbb{R}]) = ([g], c_1, \dots, c_{\lfloor n/2 \rfloor - k})$. □

The following lemma completes the proof of part (i).

Lemma 5.2. *If n is even, then β^k_n is injective.*

Proof. Suppose that $\beta_n^k([f]) = 0 \in \widetilde{\mathcal{G}}_n^k \oplus \mathbb{Z}^{\lfloor n/2 \rfloor - k}$ for some $[f\colon M^n \to \mathbb{R}] \in \widetilde{\mathcal{M}}_n^k$. Then, there exists an oriented k-constrained generic bordism $G\colon W^{n+1} \to \mathbb{R}^2$ from f to $f_\emptyset$. Since $\widetilde{\Phi}^k([f]) = 0$, Theorem 3.5 implies that $[f] = [f_\emptyset] = 0 \in \widetilde{\mathcal{M}}_n^k$. □

In order to studying the kernel of β_n^k when n is odd, we introduce a homomorphism $\alpha_n^k\colon \mathbb{Z}/2 \to \widetilde{\mathcal{M}}_n^k$ as follows.

Lemma 5.3. *Suppose that n is odd, and let $l := (n-1)/2$. If $f, g\colon S^n \to \mathbb{R}$ are Morse functions with exactly 4 critical points whose Morse indices form the set $\{0, l, l+1, n\}$, then $[f] = [g] \in \widetilde{\mathcal{M}}_n^k$. Hence, there is a well-defined homomorphism*

$$\alpha_n^k\colon \mathbb{Z}/2 \to \widetilde{\mathcal{M}}_n^k, \qquad \overline{1} \mapsto [f_\alpha\colon S^n \to \mathbb{R}],$$

where $f_\alpha\colon S^n \to \mathbb{R}$ denotes a fixed Morse function with the above properties.

Proof. By Theorem 3.8 there exists an oriented l-constrained generic bordism $G\colon S^n \times [0,1] \to \mathbb{R}^2$ from f to g. Moreover, Proposition 3.7 implies that the number of cusps of G is even. Hence, the claim that $[f] = [g] \in \widetilde{\mathcal{M}}_n^k$ follows from Theorem 3.5. As S^n admits an orientation reversing automorphism, we may choose $g = -f$, and obtain $2[f] = 0$. □

Lemma 5.4. *If n is odd, then* $\operatorname{im} \alpha_n^k = \ker \beta_n^k$.

Proof. Note that $\widetilde{\Psi}^k([f_\alpha]) = 0 \in \widetilde{\mathcal{G}}_n^k$ and $\widetilde{\Phi}^k([f_\alpha]) = 0 \in \mathbb{Z}^{\lfloor n/2 \rfloor - k}$, and thus $\operatorname{im} \alpha_n^k \subset \ker \beta_n^k$. Conversely, suppose that $\beta_n^k([f]) = 0 \in \widetilde{\mathcal{G}}_n^k \oplus \mathbb{Z}^{\lfloor n/2 \rfloor - k}$ for some $[f\colon M \to \mathbb{R}] \in \widetilde{\mathcal{M}}_n^k$. Then, there exists an oriented k-constrained generic bordism $G\colon W^{n+1} \to \mathbb{R}^2$ from f to $f_\emptyset$. Moreover, by means of Theorem 3.8, we can construct an oriented k-constrained generic bordism $G_\alpha\colon D^{n+1} \to \mathbb{R}^2$ from f_α to $f_\emptyset$. By Proposition 3.7 exactly one of the oriented k-constrained generic bordisms G and $G \sqcup G_\alpha$, say G_0, has an even number of cusps. Hence, Theorem 3.5 implies that $0 = [f_0] = [f] + m[f_\alpha] \in \widetilde{\mathcal{M}}_n^k$ for suitable $m \in \{0, 1\}$, and we conclude that $\ker \beta_n^k \subset \operatorname{im} \alpha_n^k$. □

Lemma 5.5 below completes the proof of part (*ii*). For this purpose, recall from [10, Definition 2.5, p. 214] that there is for $n \equiv 1 \bmod 4$ a well-defined homomorphism $\Lambda\colon \widetilde{\mathcal{M}}_n^1 \to \mathbb{Z}/2$ which assigns to $[f] \in \widetilde{\mathcal{M}}_n^1$ the element $\Lambda([f]) = \sigma(f) - \sigma(M^n; \mathbb{Q}) \in \mathbb{Z}/2$, where $\sigma(f) = \sum_{\lambda=0}^{(n-1)/2} C_\lambda(f)$, and $\sigma(M^n; \mathbb{Q})$ denotes the Kervaire semi-characteristic of M^n over $\mathbb{Q}$ (see [13]). Composition with the natural homomorphism $\widetilde{\mathcal{M}}_n^k \to \widetilde{\mathcal{M}}_n^1$ yields a homomorphism $\Lambda_n^k\colon \widetilde{\mathcal{M}}_n^k \to \mathbb{Z}/2$ which turns out to be a splitting of α_n^k for $n \equiv 1 \bmod 4$.

Lemma 5.5. *For $n \equiv 1 \bmod 4$ we have $\Lambda_n^k \circ \alpha_n^k = \mathrm{id}_{\mathbb{Z}/2}$.*

Proof. It suffices to note that $\Lambda^k([f_\alpha]) = \overline{1} \in \mathbb{Z}/2$, which follows from $\sigma(f_\alpha) = 2$ and $\sigma(S^n; \mathbb{Q}) = 1$. □

In Lemma 5.7 below we prove the remaining parts (*iii*) and (*iv*) by characterizing injectivity of α_n^k for $n \equiv 3 \bmod 4$. For this purpose, we introduce the required sequence $\kappa_1, \kappa_2, \dots$ of positive integers in Definition 5.6 below. Note that $\mathbb{C}P^{2i}$ is for any integer $i \geq 1$ a closed $4i$-manifold with odd Euler characteristic, and any generic map $\mathbb{C}P^{2i} \to \mathbb{R}^2$ defines an oriented 1-constrained generic bordism from $f_\emptyset\colon \emptyset \to \mathbb{R}$ to $f_\emptyset\colon \emptyset \to \mathbb{R}$.

Definition 5.6. For every integer $i \geq 1$ let κ_i be the greatest integer $k \geq 1$ for which there exists an oriented k-constrained generic bordism $V^{4i} \to \mathbb{R}^2$ from $f_\emptyset \colon \emptyset \to \mathbb{R}$ to $f_\emptyset \colon \emptyset \to \mathbb{R}$ such that V^{4i} has odd Euler characteristic (or, equivalently, odd signature).

Lemma 5.7. *Suppose that $n \equiv 3 \bmod 4$. Then, α_n^k is injective if and only if $k > \kappa_{(n+1)/4}$.*

Proof. As in the proof of Lemma 5.4 we can construct an oriented k-constrained generic bordism $G_\alpha \colon D^{n+1} \to \mathbb{R}^2$ from f_α to $f_\emptyset$.

"$\Leftarrow$". Let $k > \kappa_{(n+1)/4}$. If we suppose that $[f_\alpha] = 0 \in \widetilde{\mathcal{M}}_n^k$, then there exists an oriented k-constrained bordism $F \colon W^{n+1} \to \mathbb{R}^2$ from $f_\emptyset$ to f_α. From Proposition 3.7 we conclude that $\chi(W)$ is even because F has no cusps and f_α has 4 critical points. Hence, $V^{n+1} := W^{n+1} \cup_{S^n} D^{n+1}$ is an oriented closed manifold with odd Euler characteristic. Now F and G_α glue to an oriented k-constrained generic bordism $V^{n+1} \to \mathbb{R}^2$ from $f_\emptyset$ to $f_\emptyset$, which contradicts the assumption that $k > \kappa_{(n+1)/4}$.

"$\Rightarrow$". Suppose that $k \leq \kappa_{(n+1)/4}$, and let V^{n+1} be a closed manifold with odd Euler characteristic which admits an oriented k-constrained generic bordism $G_1 \colon V^{n+1} \to \mathbb{R}^2$ from $f_\emptyset$ to $f_\emptyset$. Note that both G_α and G_1 have an odd number of cusps by Proposition 3.7. Hence, the oriented k-constrained generic bordism $G_\alpha \sqcup G_1 \colon D^{n+1} \sqcup V^{n+1} \to \mathbb{R}^2$ from f_α to $f_\emptyset$ has an even number of cusps. Therefore, $[f_\alpha] = 0 \in \widetilde{\mathcal{M}}_n^k$ by Theorem 3.5. □

Remark 5.8. We do not know if the short exact sequence in Theorem 1.2(iv) splits.

Finally, the sequences $\kappa_1, \kappa_2, \dots$ and $\gamma_1, \gamma_2, \dots$ are related to each other as follows. The inequality $\gamma_i \leq \kappa_i$ is (for $\gamma_i > 1$) a consequence of Proposition 4.3. Moreover, the inequality $\kappa_i \leq \gamma_i + 1$ follows (for $\kappa_i > 1$) from Proposition 4.2, where one has to take care of the parity of the Euler characteristic.

This completes the proof of Theorem 1.2.

Remark 5.9. Note that $\kappa_i < 2i$ for all $i \geq 1$ (where we have used Remark 3.6 and Proposition 3.7 to exclude the case that $\kappa_i = 2i$). Hence, for $\gamma_i = 2i - 1$ (which happens (at least) for $i = 1, 2, 4$) we have $\kappa_i = \gamma_i$. Moreover, $\gamma_i \not\equiv 2, 4, 5, 6 \bmod 8$ implies that $\kappa_i \not\equiv 5, 6 \bmod 8$ because $\gamma_i \in \{\kappa_i - 1, \kappa_i\}$.

Remark 5.10. An unoriented version $\mathcal{M}_n^k$ of the group $\widetilde{\mathcal{M}}_n^k$ can be defined in a straightforward way by forgetting orientations in Definition 3.2. Then, it is possible to derive a version of Theorem 1.2 for the unoriented bordism group $\mathcal{M}_n^k$ in an analogous way.

6. Detecting Exotic Kervaire Spheres

In conclusion, we discuss in Theorem 6.2 how our results on bordism groups of constrained Morse functions can be applied to detect exotic Kervaire spheres in certain high dimensions. Recall that Kervarie spheres are a concrete family of homotopy spheres that can be obtained from a plumbing construction (see [17, p. 162]). In fact, the unique Kervaire sphere Σ_K^n of dimension $n = 4k+1$ is defined as the boundary of the parallelizable $(4k+2)$-manifold given by plumbing together two copies of the tangent disc bundle of S^{2k+1}. In Theorem 6.2 we will need to impose stronger conditions on the dimension n that originate from a result of Stolz [29] on highly connected bordisms that is used in our argument, and we do not know if they can be eliminated.

Let Θ_n denote the abelian group of homotopy n-spheres, which is known to be finite [14]. Recall that Θ_n consists of the oriented h-cobordism classes of oriented homotopy n-spheres, and its group structure is induced by taking the oriented connected sum of homotopy spheres. For $n \geq 5$, Θ_n is well-known to coincide with the group of oriented diffeomorphism classes of oriented closed n-manifolds homeomorphic to S^n, whose non-trivial elements are known as *exotic spheres*.

Let $bP_{n+1} \subset \Theta_n$ denote the subgroup of those homotopy n-spheres that can be realized as the boundary of a parallelizable cobordism (see [14, p. 510]). The following result is part of the classification theorem of homotopy spheres (see [19, Theorem 6.1, pp. 123f]).

Theorem 6.1. *Suppose that $n = 4k+1$ for some integer $k \geq 1$. Then, $bP_{n+1} = \{[S^n], [\Sigma_K^n]\}$, where Σ_K^n denotes the unique Kervaire sphere of dimension n. Moreover, $bP_{n+1} \cong \mathbb{Z}/2$ whenever $n \neq 2^j - 3$ for all integers $j \geq 0$, and $bP_{n+1} = 0$ for $n \in \{5, 13, 29, 61\}$. We have $\Theta_n / bP_{n+1} \cong \operatorname{coker} J_n$, where $J_n \colon \pi_n(SO) \to \pi_n^s$ denotes the stable J-homomorphism.*

Theorem 6.2. *Suppose that $n \geq 237$ and $n \equiv 13 \pmod{16}$. Set $l := (n-1)/2$. Then for any exotic n-sphere Σ^n the following statements are equivalent:*

- *(i) Σ^n is diffeomorphic to the Kervaire n-sphere Σ_K^n.*
- *(ii) Σ^n admits an l-constrained Morse function which represents $0 \in \widetilde{\mathcal{M}}_n^l$.*

Remark 6.3. In the setting of Theorem 6.2, any positive integer can be realized as $\sigma(f) = \sum_{\lambda=0}^{l} C_\lambda(f)$ (introduced before Lemma 5.5) for some suitable l-constrained Morse function $f \colon \Sigma^n \to \mathbb{R}$ on the exotic sphere Σ^n. Indeed, Σ^n is well-known to admit a Morse function without indefinite critical points (see the proof of [21, Theorem B, p. 109]). Hence, by introducing additional pairs of Morse critical points of subsequent indices l and $l+1$, we can realize any desired positive integer as $\sigma(f)$.

Remark 6.4. In the range $n \leq 500$ and under the assumptions of Theorem 6.2, we have $bP_{n+1} \neq \Theta_n$ at least for $n \in \{237, 285, 333, 381, 445, 461, 477, 493\}$. In fact, by Theorem 6.1 it suffices to show that $\operatorname{coker} J_n$ is non-trivial. For this purpose, note that $n \equiv 5 \pmod 8$ implies that $\pi_n(SO) = 0$ (see the proof of [14, Theorem 3.1, p. 508]). Hence, $\operatorname{coker} J_n \cong \pi_n^s$. Finally, an examination of the 5-components of π_n^s in [24, Table A3.5, pp. 365ff] shows that $\pi_n^s \neq 0$ for the desired values of n.

Proof (of Theorem 6.2). By the proof of [27, Theorem 1.1, p. 288] there is for $n \geq 6$ an isomorphism $\vartheta_n \colon \Theta_n \xrightarrow{\cong} \widetilde{\Gamma}(n,1)$ given by the assignment $[\Sigma^n] \mapsto [f]$, where $f \colon \Sigma^n \to \mathbb{R}$ denotes an arbitrarily chosen special generic function on the homotopy sphere Σ^n, i.e., a Morse function without indefinite critical points. By composition with the homomorphisms of Theorem 1.1 we can define homomorphisms

$$c_n^l \colon \Theta_n \xrightarrow{\vartheta_n} \widetilde{\Gamma}(n,1) \overset{3.3}{=} \widetilde{\mathcal{G}}_n^{l+1} \xrightarrow{\delta_n^{l+1}} \widetilde{\mathcal{C}}_n^l, \quad [\Sigma^n] \mapsto [\Sigma^n],$$
$$g_n^l \colon \Theta_n \xrightarrow{c_n^l} \widetilde{\mathcal{C}}_n^l \xrightarrow{\varepsilon_n^l} \widetilde{\mathcal{G}}_n^l, \quad [\Sigma^n] \mapsto [f],$$
$$c_n^{l-1} \colon \Theta_n \xrightarrow{g_n^l} \widetilde{\mathcal{G}}_n^l \xrightarrow{\delta_n^l} \widetilde{\mathcal{C}}_n^{l-1}, \quad [\Sigma^n] \mapsto [\Sigma^n],$$

where $f \colon \Sigma^n \to \mathbb{R}$ denotes an arbitrarily chosen l-constrained Morse function in the definition of g_n^l. Denote the kernels of g_n^l, c_n^l and c_n^{l-1} by G_n^l, C_n^l and C_n^{l-1}, respectively. Then, by construction, $C_n^l \subset G_n^l \subset C_n^{l-1}$. Note that Proposition 2.2 implies that $C_n^l = C_n^{l-1}$ because $l \equiv 6 \bmod 8$. Thus, we have shown that $G_n^l = C_n^l$. Furthermore, the inclusion $bP_{n+1} \subset C_n^l$ holds since by [20, Theorem 3, p. 49] any parallelizable compact smooth manifold W^m of dimension $m = n+1$ can be made $l = (\lfloor m/2 \rfloor - 1)$-connected by a finite sequence of surgeries without changing ∂W.

By a theorem of Stolz (see [29, Theorem B(ii), p. XIX]), the inclusion $C_n^l \subset bP_{n+1}$ holds because $m := n+1$ is by assumption of the form $m = 2k + d$ for $d = 0$ and some integer $k \geq 113$ satisfying $k \equiv 7 \pmod 8$.

All in all, we have shown that $G_n^l = C_n^l = bP_{n+1} = \{[S^n], [\Sigma_K^n]\}$, where the last equality is taken from Theorem 6.1.

Using $\sigma(\Sigma^n, \mathbb{Q}) = \overline{1} \in \mathbb{Z}/2$ as well as Remark 6.3, Theorem 1.2(ii) implies that statement (ii) is equivalent to $[\Sigma^n] = 0 \in \widetilde{\mathcal{G}}_n^l$, i.e., $[\Sigma^n] \in G_n^l = \{[S^n], [\Sigma_K^n]\}$, and the claim follows. $\square$

Remark 6.5. The groups C_n^l and G_n^l introduced in the proof of Theorem 6.2 can be generalized for any $n \geq 6$ to natural subgroup filtrations C_n^k and G_n^k of Θ_n. In fact, using Theorem 1.1, we can define for $1 \leq k \leq \lfloor n/2 \rfloor$ the homomorphisms

$$c_n^k \colon \Theta_n \xrightarrow{\vartheta_n} \widetilde{\Gamma}(n,1) \overset{3.3}{=} \widetilde{\mathcal{G}}_n^{\lfloor n/2 \rfloor + 1} \to \widetilde{\mathcal{G}}_n^{k+1} \xrightarrow{\delta_n^{k+1}} \widetilde{\mathcal{C}}_n^k, \quad [\Sigma^n] \mapsto [\Sigma^n],$$
$$g_n^k \colon \Theta_n \xrightarrow{c_n^k} \widetilde{\mathcal{C}}_n^k \xrightarrow{\varepsilon_n^k} \widetilde{\mathcal{G}}_n^k, \quad [\Sigma^n] \mapsto [f],$$

where $f \colon \Sigma^n \to \mathbb{R}$ denotes an arbitrarily chosen k-constrained Morse function in the definition of g_n^k. If C_n^k and G_n^k denotes the kernel of c_n^k and g_n^k, respectively, then $C_n^k \subset G_n^k$ for $1 \leq k \leq \lfloor n/2 \rfloor$, and $G_n^k \subset C_n^{k-1}$ for $2 \leq k \leq \lfloor n/2 \rfloor$ (compare [32, Theorem 10.1.3, p. 243]). We do not know whether the resulting filtrations C_n^k and G_n^k of Θ_n coincide or not.

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