# 確率論と幾何学

(Geometry and Probability)

**日程** 2016 年 10 月 24 日 (月)〜28 日 (金)
(24–28 October 2016)

**会場** 数理解析研究所 420 室
(Room 420, Research Institute for Mathematical Science)

## Program

### 10 月 24 日 (月) Oct. 24 (Mon.)

13:00–13:50 Nicola Gigli (La Scuola Internazionale Superiore di Studi Avanzati)
Nonsmooth differential geometry I

14:00–14:50 田中 勝 Masaru Tanaka (福岡大学 Fukuoka University)
$\tau$-Information geometry endowed with a contact structure

15:10–16:00 Young-Heon Kim (University of British Colombia)
Optimal martingale transport in general dimensions I

16:10–17:00 Short communications I

### 10 月 25 日 (火) Oct. 25 (Tue.)

9:50–10:40 Nicola Gigli (La Scuola Internazionale Superiore di Studi Avanzati)
Nonsmooth differential geometry II

10:50–11:40 白井 朋之 Tomoyuki Shirai (九州大学 Kyushu University)
Persistent homology of random complexes built over point processes

13:00–13:50 Young-Heon Kim (University of British Colombia)
Optimal martingale transport in general dimensions II

14:00–14:50 稲濱 譲 Yuzuru Inahama (九州大学 Kyushu University)
Short time full asymptotic expansion of hypoelliptic heat kernel at the cut locus

15:10–16:00 Paul Woon Yin Lee (Chinese University of Hong Kong)
The Harnack inequality for the Jordan-Kinderlehrer-Otto/minimizing movement scheme of the heat equation

16:10–17:00 Short communications II

**10月26日 (水) Oct. 26 (Wed.)**

9:50–10:40 Nicol Gigli (La Scuola Internazionale Superiore di Studi Avanzati)
Nonsmooth differential geometry III

10:50–11:40 Laurent Saloff-Coste (Cornell University)
The isoperimetric profile for random walks on finitely generated groups I

**10月27日 (木) Oct. 27 (Thu.)**

9:50–10:40 白石 大典 Daisuke Shiraishi (京都大学 Kyoto University)
Geometry of Brownian motion

10:50–11:40 Laurent Saloff-Coste (Cornell University)
The isoperimetric profile for random walks on finitely generated groups II

13:00–13:50 利根川 吉廣 Yoshihiro Tonegawa (東京工業大学 Tokyo Institute of Technology)
Long-time existence of mean curvature flow of grain boundaries

14:00–14:50 Paul Woon Yin Lee (Chinese University of Hong Kong)
Ricci curvature type lower bounds on Sasakian manifolds

15:10–16:00 加須栄 篤 Atsushi Kasue (金沢大学 Kanazawa University)
Ideal boundaries of open Riemannian manifolds and convergence of induced Dirichlet forms

16:10–17:00 Short communications III

**10月28日 (金) Oct. 28 (Fri.)**

9:50–10:40 正宗 淳 Jun Masamune (北海道大学 Hokkaido University)
On the $L^1$ Liouville property of a manifold with ends

10:50–11:40 Laurent Saloff-Coste (Cornell University)
The isoperimetric profile for random walks on finitely generated groups III

世話人: (Organizers)
石渡聡 Satoshi Ishiwata (山形大学 Yamagata University)
桑田 和正 Kazumasa Kuwada (東京工業大学 Tokyo Institute of Technology)
桑江 一洋 Kazuhiro Kuwae (福岡大学 Fukuoka University)
太田 慎一 Shin-ichi Ohta (京都大学 Kyoto University)
塩谷 隆 Takashi Shioya (東北大学 Tohoku University)

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# Abstracts

**Nicola Gigli (La Scuola Internazionale Superiore di Studi Avanzati)**

*Nonsmooth differential geometry I–III*

Abstract: Aim of the lectures is to give an overview about differential calculus on metric measure spaces: we shall see in which sense general metric measure spaces posses a weak first order differential structure and how on spaces with Ricci curvature bounded from below a second order calculus is possible.

The construction is based on the well-known notion of Sobolev function on a metric measure space and on the one of $L^2$-normed $L^\infty$-module, for which I've got inspiration from a work of Weaver (the more general case of $L^p$-modules can also be discussed, but we shall focus on $p = 2$ for these lectures). Such modules provide an abstraction of the notion of space of $L^2$-sections of a normed vector bundle on a smooth manifold and we shall see that in a very natural way every mm space admits a cotangent module and a differential operator acting on Sobolev functions. Starting from this, by duality we can introduce the notion of tangent module. We shall see that: - vector fields (intended as elements of the tangent module) are in 1-1 correspondence with $L^2$ derivations od Sobolev functions; - there is a tight relation between the metric speed of curves and the norm of their velocities interpreted in the sense of modules. Thus, in an appropriate weak sense, the norm of vector fields induces the distance on the base space.

Still concerning the first order structure, we shall discuss in which sense maps $f$ between metric measure spaces which are both Lipschitz and of bounded compression (i.e. $f_*m_1 \le Cm_2$ for some $C > 0$, where $m_1, m_2$ are the reference measures on the given spaces) canonically induce a pullback of 1-form and, by duality, have a differential acting on vector fields.

Then we introduce spaces with Ricci curvature bounded from below (RCD spaces in short): for what concerns the topic of the lectures, the key feature of these spaces which allows to move beyond first order calculus is the presence of the Bochner inequality. In particular, exploiting the self-improving properties of such inequality we shall see that it is possible to introduce the Sobolev space $W^{2,2}$ of functions having Hessian in $L^2$ and prove that it is dense in $W^{1,2}$. Here the Hessian is intended as element of the - naturally defined - tensor product of cotangent bundle with itself.

Once this is done, it will be easy to speak about Sobolev vector fields, i.e. with covariant derivative in $L^2$: the basic example of such vector field is given by $\nabla f$ with $f \in W^{2,2}$. Similarly, we can introduce exterior differentiation of $k$-forms and we shall verify that mimicking the constructions in the smooth setting the de Rham cohomology can be introduced and a very natural version of Hodge theory can also be developed. As it turns out, these constructions are functorial w.r.t. the Lipschitz maps of bounded compression previously discussed.

We shall conclude the lectures by discussing in which sense RCD spaces possess a measure valued Ricci tensor and how its study can give informations about the geometry of the space.

# Short time full asymptotic expansion of hypoelliptic heat kernel at the cut locus

Yuzuru Inahama *

This is a jointwork with Setsuo Taniguchi (Kyushu University) and can be found at arXiv Preprint Server (arXiv:1603.01386).

We discuss a short time asymptotic expansion of a hypoelliptic heat kernel on a Euclidean space and on a compact manifold. We study the "cut locus" case, namely, the case where energy-minimizing paths which join the two points under consideration form not a finite set, but a compact manifold. Under mild assumptions we obtain an asymptotic expansion of the heat kernel up to any order. Our approach is probabilistic and the heat kernel is regarded as the density of the law of a hypoelliptic diffusion process, which is realized as a unique solution of the corresponding stochastic differential equation (SDE). Our main tools are S. Watanabe's distributional Malliavin calculus and T. Lyons' rough path theory.

Our work has the following three features. To our knowledge, there are no works which satisfy all of these conditions simultaneously:

1. The manifold and the hypoelliptic diffusion process on it are rather general. In other words, this is not a study of special examples.

2. The "cut locus" case is studied. More precisely, we mean by this that the set of energy-minimizing paths (or controls) which connect the two points under consideration becomes a compact manifold of finite dimension.

3. The asymptotic expansion is full, that is, the polynomial part of the asymptotics is up to any order.

On a Euclidean space, however, there are two famous results which satisfy (2), (3) and the latter half of (1). Both of them are probabilistic and use generalized versions of Malliavin calculus. One is Takanobu and Watanabe [3]. They use Watanabe's distributional Malliavin calculus. The other is Kusuoka and Stroock [2]. They use their version of generalized Malliavin calculus. We use the former.

Though we basically follow Takanobu-Watanabe's argument in [3], the main difference is that we use T. Lyons' rough path theory together, which is something like a deterministic version of the SDE theory. The main advantage of using rough path theory is that while the usual Itô map i.e., the solution map of an SDE is discontinuous, the Lyons-Itô map i.e., the solution map of a rough differential equation (RDE) is continuous.

This fact enables us to do "local analysis" of the Lyons-Itô map (for instance, restricting the map on a neighborhood of its critical point and doing a Taylor-like expansion) in a somewhat similar way we do in the Fréchet calculus. Recall that in the standard SDE theory, this type of local operation is very hard and sometimes impossible, due to

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the discontinuity of the Itô map. For this reason, the localization procedure in [3] looks so complicated that it might be difficult to generalize their method if rough path theory did not exist. Of course, there is a possibility that our main result can be proved without rough path theory, but we believe that the theory is quite suitable for this problem and gives us a very clear view (in particular, in the manifold case).

A detailed proof can be found in our preprint [1]. We first reprove and generalize the main result in [3] in the Euclidean setting by using rough path theory. Then, we study the manifold case. Recall that Malliavin calculus for a manifold-valued SDE was studied by Taniguchi [4]. Even in this Euclidean setting, many parts of the proof are technically improved, thanks to rough path theory. We believe that the following are worth mentioning: (i) Large deviation upper bound. (ii) Asymptotic partition of unity. (iii) A Taylor-like expansion of the Lyons-Itô map and the uniform exponential integrability lemma for the ordinary and the remainder terms of the expansion. (iv) Quasi-sure analysis for the solution of the SDE.

# References


[1] Y. Inahama, S. Taniguchi, *Short time full asymptotic expansion of hypoelliptic heat kernel at the cut locus*, preprint, arXiv:1603.01386.

[2] S. Kusuoka, D. W. Stroock, *Precise asymptotics of certain Wiener functionals*, J. Funct. Anal. 99 (1991), no. 1, 1–74.

[3] S. Takanobu, S. Watanabe, *Asymptotic expansion formulas of the Schilder type for a class of conditional Wiener functional integrations*, Asymptotic problems in probability theory: Wiener functionals and asymptotics (Sanda/Kyoto, 1990), 194–241, Pitman Res. Notes Math. Ser., 284, Longman Sci. Tech., Harlow, 1993.

[4] S. Taniguchi, *Malliavin's stochastic calculus of variations for manifold-valued Wiener functionals and its applications*, Z. Wahrsch. Verw. Gebiete 65 (1983), no. 2, 269–290.

**Atsushi Kasue (Kanazawa University)**

*Ideal boundaries of open Riemannian manifolds and convergence of induced Dirichlet forms*

Abstract: In this talk, we consider a transient open Riemannian manifold, and discuss the convergence of the Dirichlet forms induced on the boundaries of smooth compact subdomains as they exhaust the manifold, and the limit form on a certain ideal boundary of the manifold.

**Young-Heon Kim (University of British Colombia)**

*Optimal martingale transport in general dimensions I, II*

Abstract: We discuss the optimal solutions to a transport problem where mass has to move under martingale constraint; this constraint forces the transport to split the mass. This problem was originated from mathematical finance, e.g. option pricing. There have been intensive studies on the one-dimensional case, but, rarely in higher dimensions. We present structural results in general dimensions. This is a joint work with Nassif Ghoussoub and Tongseok Lim.

**Paul Woon Yin Lee (Chinese University of Hong Kong)**

*The Harnack inequality for the Jordan-Kinderlehrer-Otto/minimizing movement scheme of the heat equation*

Abstract: The Jordan-Kinderlehrer-Otto or the minimizing movement scheme on the Wasserstein space is a scheme to produce discrete time approximations to Wasserstein gradient flows. Local $L^1$ convergence of these approximations was shown by Jordan-Kinderlehrer-Otto in the case of Fokker-Planck equation. In this talk, I will discuss how this can be improved (at least in the case of the heat equation). We also discuss a version of the Harnack inequality satisfied by these discrete time approximations.

*Ricci curvature type lower bounds on Sasakian manifolds*

Abstract: Measure contraction property is a synthetic Ricci curvature lower bound for metric measure spaces introduced by Sturm and Ohta. We consider Sasakian manifolds with non-negative Tanaka-Webster Ricci curvature equipped with a one-parameter family of Riemannian metrics which degenerate to a subriemannian one. We show that each of these Riemannian metric or the subriemannian one, when equipped with a natural measure, satisfies the measure contraction property. We will also discuss the consequences following from this condition.

**Jun Masamune (Hokkaido University)**

*On the $L^1$ Liouville property of a manifold with ends*

Abstract: In this talk we will learn how to construct nontrivial integrable harmonic functions on a complete non-compact Riemannian manifold. Joint with A. Grigor'yan and M. Murata.

**Laurent Saloff-Coste (Cornell University)**

*The isoperimetric profile for random walks on finitely generated groups I–III*

Abstract: I will discuss techniques to bound the isoperimetric profile of symmetric measures on finitely generated groups, both for finitely supported measures and for more spread-out measures. I will also discuss the relation between the isoperimetric profile, the probability of return (or heat kernel decay) and other parameters.

**Tomoyuki Shirai (Kyushu University)**

*Persistent homology of random complexes built over point processes*

Abstract: Persistent homology appeared around 2000 as an algebraic method which measures topological features of objects or point cloud data. Recently, much attention has been paid to it in the context of Topological Data Analysis (TDA). Persistent homology describes, roughly speaking, the birth and death of topological feature (connected components, holes, voids, and so on) of an increasing sequence of topological objects. Connected components of random objects have been studied for long time in probability theory, especially, in percolation theory and random geometric graph theory. Persistent homology theory sheds new light on such topics in several ways. In this talk, we would like to discuss mainly some topics on persistent homology for random complexes built over point cloud data.

**Daisuke Shiraishi (Kyoto University)**

*Geometry of Brownian motion*

Abstract: In this talk, we will consider the nature of self-intersections of the Brownian motion in $\mathbb{R}^d$. It is well-known that the Brownian motion is a simple path if $d \geq 4$, while it has loops when $d \leq 3$. What are loops of the Brownian motion? How are those loops distributed in space? These are questions we want to address. In the talk, we give an explicit representation of such loops for $d \leq 3$ by establishing a decomposition of the Brownian path into independent simple path and a set of loops. It turns out that the simple path and the set of loops can be described by the scaling limit of the loop-erased random walk and a Poisson point process on a path space, respectively. We will also explain a relation between our results and Itô's excursion theory. This is joint work with Artem Sapozhnikov (University of Leipzig).

# $\tau$-Information geometry endowed with a contact structure

Masaru Tanaka
Department of Applied Mathematics, Faculty of Science
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$\tau$-Information geometry is an information geometry based on two affine spaces: one is an affine space $\overset{\tau=s}{\mathcal{P}}$ constructed by finite positive measures which are mutually absolutely continuous and defined on some measurable space, and the other is an affine space $\overset{\tau=1-s}{\mathcal{P}}$ given by taking "Hölder conjugate" with respect to a parameter $\tau$ which controls an affine property of the space of finite positive measures $\mathcal{P}$. $\tau$-Information geometry yields the same results from Amari-style information geometry, but it sometimes gives them different interpretations. The most prominent feature of $\tau$-Information geometry is that the magnitude of a measure is changed depending on a translation in the affine spaces $\overset{\tau=s}{\mathcal{P}}$ and $\overset{\tau=1-s}{\mathcal{P}}$, and conversely a scale transformation induces a translation in the affine spaces. Here we pay attention to an entropy. In the context of $\tau$-Information geometry, the entropy is given in the form of Tsallis-like non-extensive entropy. However, in general, an entropy is affected by a normalization/scale, a conformal entropy is naturally defined and used. Considering the change of a normalization caused by a translation as the change of a conformal entropy, we can restore the Tsallis-like non-extensive conformal entropy to being an extensive conformal entropy. This is quite important because an entropy should be extensive in any physical contexts. The principle that an entropy should be extensive leads to the AdS/CFT correspondence or the holography principle, which makes complex and difficult situations occurring to calculate physical quantities simple and easy situations by introducing an extra codimension. So, we introduce a coordinate for a scale transformation as an extra codimension, and consider the $(2r+1)$-dimensional space of natural coordinates $\left(\theta^1,\ldots,\theta^r\right)$ for characterizing a distribution, their dual coordinates $(\eta_1,\ldots,\eta_r)$, and the scale coordinate $\left(\theta^0\right)$. Then, if all of the coordinates are independent of each other, a contact structure is naturally arises. Furthermore, the corresponding Heisenberg group is also available so that the group structure of the Heisenberg group gives a Legendre transformation for potential functions.

In this talk, it is shown that how the contact structure appears in the context of $\tau$-Information geometry without regard to mathematical strictness too much

**Yoshihiro Tonegawa (Tokyo Institute of Technology)**

*Long-time existence of mean curvature flow of grain boundaries*

Abstract: Given an arbitrary closed countably rectifiable set of codimension one in the Euclidean space, we prove a time-global existence theorem for a mean curvature flow starting from this set. The flow can go through topological changes and is continuous in a certain sense to be clarified in the talk. This is a joint work with Lami Kim.