TOPOLOGY OF SINGULAR FIBERS OF GENERIC MAPS

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Dedicated to Professor Yukio Matsumoto on the occasion of his 60th birthday

ABSTRACT. We classify singular fibers of C^∞ stable maps of orientable 4-manifolds into 3-manifolds up to right-left equivalence. Furthermore, we obtain some results on the co-existence of singular fibers of such maps, and as a consequence, we show that the Euler characteristic of the source 4-manifold of such a stable map has the same parity as the number of singular fibers of a certain type. We construct some explicit examples which indicate that the study of such singular fibers is essential for a topological study of C^∞ stable maps of negative codimension from a global viewpoint. In fact, for a generic map of negative codimension, a similar consideration enables us to obtain a stratification of the target manifold according to the fibers, which leads us naturally to the notion of universal complexes of singular fibers similar to Vassiliev's universal complexes of multi-singularities. In this paper, we develop a rather detailed theory of such universal complexes of singular fibers in a general setting in order to apply it to several explicit situations. We also show that the cohomology groups of such universal complexes of singular fibers give rise to cobordism invariants of smooth maps with a given set of local and global singularities.

CONTENTS

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1. Introduction

Let $f : M \to N$ be a proper smooth map of an n-dimensional manifold M into a p-dimensional manifold N. When the codimension $p - n$ is nonnegative, for any point y in the target N, the inverse image $f^{-1}(y)$ consists of a finite number of points, provided that f is generic enough. Hence, in order to study the semi-local behavior of a generic map f around (the inverse image of) a point $y \in N$, we have only to consider the multi-germ $f : (M, f^{-1}(y)) \to (N, y)$. Therefore, we can use the well-developed theory of multi-jet spaces and their sections in order to study such semi-local behaviors of generic maps.

However, if the codimension $p - n$ is strictly negative, then the inverse image $f^{-1}(y)$ is no longer a discrete set. In general, $f^{-1}(y)$ forms a complex of positive dimension $n - p$. Hence, we have to study the map germ $f : (M, f^{-1}(y)) \to (N, y)$ along a set $f^{-1}(y)$ of positive dimension and the theory of multi-jet spaces is not sufficient any more. Surprizingly enough, there has been no systematic study of such map germs in the literature, as long as the author knows, although we can find some studies of the multi-germ of f at the singular points of f contained in $f^{-1}(y)$.

In this paper, we consider the codimension -1 case, i.e. the case with $n - p = 1$, and classify the right-left equivalence classes of generic map germs $f : (M, f^{-1}(y)) \to (N, y)$ for $n = 2, 3, 4$. For the case $n = 3$, Kushner, Levine and Porto [23, 25] classified the singular fibers of C^∞ stable maps of 3-manifolds into surfaces up to *diffeomorphism*; however, they did not mention a classification up to *right-left equivalence* (for details, see Definition 2.1 (2) in §2). In this paper, we clarify the difference between the classification up to diffeomorphism and that up to right-left equivalence by completely classifying the singular fibers up to these two equivalences.

Given a generic map $f : M \to N$ of negative codimension, the target manifold N is naturally stratified according to the right-left equivalence classes of f-fibers. By carefully investigating how the strata are incident to each other, we get some information on the homology class represented by a set of the points in the target

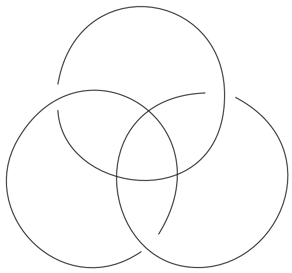

FIGURE 1. The singular fiber whose number has the same parity as the Euler characteristic of the source 4-manifold M

whose associated fibers are of certain types. This leads to some limitations on the co-existence of singular fibers. For example, we show that for a C^∞ stable map of a closed orientable 4-manifold into a 3-manifold, the number of singular fibers containing both a cusp point and a fold point is always even.

As an interesting and very important consequence of such co-existence results, we show that for a C^∞ stable map $f : M \to N$ of a closed orientable 4-manifold M into a 3-manifold N, the Euler characteristic of the source manifold M has the same parity as the number of singular fibers as depicted in Fig. 1 (Theorem 6.1). Note that this type of result would be impossible if we used the multi-germs of a given map at the singular points contained in a fiber instead of considering the topology of the fibers. In other words, our idea of essentially using the topology of singular fibers leads to new information on the global structure of generic maps.

Furthermore, the natural stratification of the target manifold according to the fibers enables us to generalize Vassiliev's universal complex of multi-singularities [50] to our case. In this paper, we define such universal complexes of singular fibers and compute the corresponding cohomology groups in certain cases. It turns out that cohomology classes of such complexes give rise to cobordism invariants for maps with a given set of singularities in the sense of Rimányi and Szűcs [35].

The paper is organized as follows.

In §2, we give precise definitions of certain equivalence relations among the fibers of proper smooth maps, which will play essential roles in this paper.

In §3, in order to clarify our idea, we classify the fibers of proper Morse functions on surfaces. The result itself should be folklore; however, we give a rather detailed argument, since similar arguments will be used in subsequent sections.

In §4, we classify the fibers of proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds up to right-left equivalence. Our strategy is to use a combinatorial argument, for obtaining all possible 1-dimensional complexes, together with a classification up to right equivalence of certain multi-germs due to [8, 52]. After the classification, we will see that the equivalence up to diffeomorphism and that up to right-left equivalence are almost equivalent to each other in our case. Furthermore, as another consequence of the classification, we will see that two fibers of such stable maps are C^0 right-left equivalent if and only of they are C^∞ right-left equivalent. This is an analogy of Damon's result [7] for C^∞ stable map germs in nice dimensions. Furthermore, we give similar results for proper C^∞ stable maps of (not necessarily orientable) 3-manifolds into surfaces and for proper C^∞ stable

Morse functions on surfaces. For Morse functions on surfaces, we prove the following very important result: for two proper C^∞ stable Morse functions on surfaces, they are C^0 equivalent if and only if they are C^∞ equivalent.

In §5, we investigate the stratification of the target 3-manifold of a C^∞ stable map of a closed orientable 4-manifold as mentioned above and obtain certain relations among the numbers (modulo two) of certain singular fibers.

In §6, we combine the result of §5 with the result of Fukuda [11] and the author [39] about the Euler characteristics to obtain a congruence modulo two between the Euler characteristic of the source 4-manifold and the number of singular fibers as depicted in Fig. 1.

In §7, we construct explicit examples of C^∞ stable maps of closed orientable 4-manifolds into $\mathbf{R}^3$. Since $(4,3)$ is a nice dimension pair in the sense of Mather [27], given a 4-manifold M and a 3-manifold N, we have a plenty of C^∞ stable maps of M into N. However, it is surprizingly difficult to give an *explicit* example and to give a detailed description of the structure of the fibers. Here, we carry this out, and at the same time we explicitly construct infinitely many closed orientable 4-manifolds with odd Euler characteristics which admit smooth maps into $\mathbf{R}^3$ with only fold singularities. In the subsequent sections, we will see that such explicit examples are essential and very important in the study of singular fibers of generic maps.

In §8, we generalize the idea given in §§5 and 6 in a more general setting to obtain certain results on the co-existence of singular fibers.

In §9, we define the universal complexes of singular fibers for proper Thom maps with coefficients in $\mathbf{Z}_2$, using an idea similar to Vassiliev's [50] (see also [19, 33]). Our universal complexes of singular fibers are very similar to Vassiliev's universal complexes of multi-singularities. In fact, we construct the complexes using the right-left equivalence classes of fibers instead of multi-singularities, and this corresponds to increasing the generators of each cochain group according to the topological structures of fibers. In order to use such universal complexes in several situations, we will develop a rather detailed theory of universal complexes of singular fibers. Here, given a set of generic maps and a certain equivalence relation among their fibers, we will define the corresponding universal complex of singular fibers.

In §10, we apply the general construction introduced in §9 to a more specific situation, namely in the case of proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds. For such maps, we determine the structure of the universal complex of singular fibers with respect to a certain equivalence relation among the fibers and compute its cohomology groups explicitly.

In §11, we consider co-orientable fibers and construct the corresponding universal complex of co-orientable singular fibers with integer coefficients. We also give some important problems related to the theory of universal complexes of singular fibers.

In §12, we define a homomorphism induced by a generic map of the cohomology group of the universal complex of singular fibers to that of the target manifold of the map. This corresponds to associating to a cohomology class α of the universal complex the Poincaré dual to the homology class represented by the set of those points over which lies a fiber appearing in a cocycle representing α. We will see that the homomorphisms induced by explicit generic maps will be very useful in the study of the cohomology groups of the universal complexes. This justifies the study developed in §7.

In §13, we define a cobordism of smooth maps with a given set of singular fibers. We will see that the homomorphism defined in §12 restricted to a certain subgroup is an invariant of such a cobordism. Furthermore, we will give a criterion for a certain cochain of the universal complex of singular fibers to be a cocycle in terms of the theory of such cobordisms, and apply it to finding a certain nontrivial cohomology class of a universal complex associated to stable maps of 5-dimensional manifolds into 4-dimensional manifolds.

In §14, we consider cobordisms of smooth maps with a given set of local singularities in the sense of [35]. We explain how a cohomology class of a universal complex of singular fibers gives rise to a cobordism invariant for such maps. Note that such cobordism relations have been thoroughly studied in [35] in the nonnegative codimension case. Our idea provides a systematic and new method to construct cobordism invariants for negative codimension cases.

In §15, we give explicit examples of cobordism invariants constructed by using the method introduced in the previous sections. In particular, we show that this method provides a complete invariant of fold cobordisms of Morse functions on closed oriented surfaces.

In §16, we give explicit applications of the general idea given in §8 to the topology of certain generic maps.

Throughout this paper, all manifolds and maps are differentiable of class C^∞. The symbol "$\cong$" denotes a diffeomorphism between manifolds or an appropriate isomorphism between algebraic objects. For a space X, the symbol "id_X" denotes the identity map of X.

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2. Preliminaries

In this section, we give some fundamental definitions, which will be essential for the classification of singular fibers of generic maps of negative codimensions.

Definition 2.1. (1) Let M_i be smooth manifolds and $A_i \subset M_i$ be subsets, $i = 0, 1$. A continuous map $g : A_0 \to A_1$ is said to be *smooth* if for every point $q \in A_0$, there exists a smooth map $\widetilde{g} : V \to M_1$ defined on a neighborhood V of q in M_0 such that $\widetilde{g}|_{V \cap A_0} = g|_{V \cap A_0}$. Furthermore, a smooth map $g : A_0 \to A_1$ is a *diffeomorphism* if it is a homeomorphism and its inverse is also smooth.

(2) Let $f_i : M_i \to N_i$ be smooth maps, $i = 0, 1$. For $y_i \in N_i$, we say that the fibers over y_0 and y_1 are *diffeomorphic* (or *homeomorphic*) if $(f_0)^{-1}(y_0) \subset M_0$ and $(f_1)^{-1}(y_1) \subset M_1$ are diffeomorphic in the above sense (resp. homeomorphic in the usual sense). Furthermore, we say that the fibers over y_0 and y_1 are C^∞ *equivalent* (or C^0 *equivalent*), if for some open neighborhoods U_i of y_i in N_i, there exist diffeomorphisms (resp. homeomorphisms) $\widetilde{\varphi} : (f_0)^{-1}(U_0) \to (f_1)^{-1}(U_1)$ and $\varphi : U_0 \to U_1$ with $\varphi(y_0) = y_1$ which make the following diagram commutative:

$$\begin{array}{ccc} ((f_0)^{-1}(U_0), (f_0)^{-1}(y_0)) & \xrightarrow{\widetilde{\varphi}} & ((f_1)^{-1}(U_1), (f_1)^{-1}(y_1)) \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ (U_0, y_0) & \xrightarrow{\varphi} & (U_1, y_1). \end{array} \tag{2.1}$$

When the fibers over y_0 and y_1 are C^∞ (or C^0) equivalent, we also say that the map germs $f_0 : (M_0, (f_0)^{-1}(y_0)) \to (N_0, y_0)$ and $f_1 : (M_1, (f_1)^{-1}(y_1)) \to (N_1, y_1)$ are smoothly (or topologically) *right-left equivalent*. Note that then $(f_0)^{-1}(y_0)$ and $(f_1)^{-1}(y_1)$ are diffeomorphic (resp. homeomorphic) to each other in the above sense.

In what follows, if we just say "equivalent", or "right-left equivalent", then we mean "C^∞ equivalent" or "smoothly right-left equivalent", respectively.

When $y \in N$ is a regular value of a smooth map $f : M \to N$ between smooth manifolds, we call $f^{-1}(y)$ a *regular fiber*; otherwise, a *singular fiber*.

Example 2.2. If $f : M \to N$ is a proper submersion, then every fiber is regular. Furthermore, by Ehresmann's fibration theorem [10] (see also [4, §8.12]), the fibers over two points y_0 and $y_1 \in N$ are equivalent, provided that y_0 and y_1 belong to the same connected component of N. Thus each equivalence class corresponds to a union of connected components of N.

Example 2.3. Suppose that $f : M \to N$ is a Thom map, which is a stratified map with respect to Whitney regular stratifications of M and N such that it is a submersion on each stratum and satisfies a certain regularity condition (for more details, refer to [12, Chapter I, §3], [9, §2.5], [6, §2], [46], for example).

Let Σ be a stratum of N of codimension κ. Take a point $y \in \Sigma$ and let B_y be a small κ-dimensional open disk in N centered at y which intersects Σ transversely at the unique point y and is transverse to all the strata of N. Then by Thom's second isotopy lemma (for example, see [12, Chapter II, §5]), we see that the fiber of f over y is C^0 equivalent to the fiber of $(f|_{f^{-1}(B_y)}) \times \mathrm{id}_{\mathbf{R}^{p-\kappa}}$ over $y \times 0$, where $p = \dim N$. Thus, again by Thom's second isotopy lemma, we see that *the fibers over any two points belonging to the same stratum Σ of N are C^0 equivalent to each other*. Thus, each C^0 equivalence class corresponds to a union of strata of N.

3. Singular fibers of Morse functions on surfaces

Let us begin by the simplest case; namely, that of Morse functions on surfaces.

Let M be a smooth surface and $f : M \to \mathbf{R}$ a proper Morse function. For its critical points $c_1, c_2, \ldots \in M$, we assume that $f(c_i) \neq f(c_j)$ for $i \neq j$: i.e., we assume that each fiber of f contains at most one critical point. This is equivalent to saying that f is C^∞ stable (see, for example, [9, §4.3], [13, Chapter III, §2B]), so we often call such an f a *stable* Morse function.

By the Morse Lemma, at each critical point c_i, f is C^∞ right equivalent to the function germ of the form

$$(x, y) \mapsto \pm x^2 \pm y^2 + f(c_i)$$

at the origin. In particular, each singular fiber contains exactly one of the following two:

(1) a component consisting of just one point (corresponding to a local minimum or maximum),
(2) a "crossing point" which has a neighborhood diffeomorphic to

$$X = \{(x, y) \in \mathbf{R}^2 \,:\, x^2 - y^2 = 0, x^2 + y^2 < 1\}$$

(corresponding to a saddle point).

Since f is proper, each fiber of f is compact. Furthermore, for each regular point $q \in M$, the fiber through q is a regular 1-dimensional submanifold near the point. Hence the component of a singular fiber of f containing a critical point should be diffeomorphic to one of the three figures as depicted in Fig. 2 by a combinatorial reason.

More precisely, we can show the following.

Theorem 3.1. *Let $f : M \to \mathbf{R}$ be a proper stable Morse function on a surface M. Then the fiber over each critical value in $\mathbf{R}$ is equivalent to one of the three types of fibers as depicted in Fig. 3.*

Note that the source manifolds depicted in Fig. 3 are all open and have finitely many connected components. In particular, the source manifold of Fig. 3 (3) is diffeomorphic to the union of the once punctured open Möbius band and some copies of $S^1 \times \mathbf{R}$.

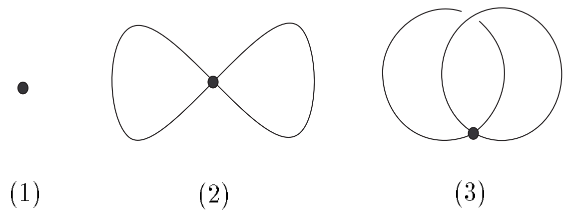

FIGURE 2. List of diffeomorphism types of singular fibers for Morse functions on surfaces

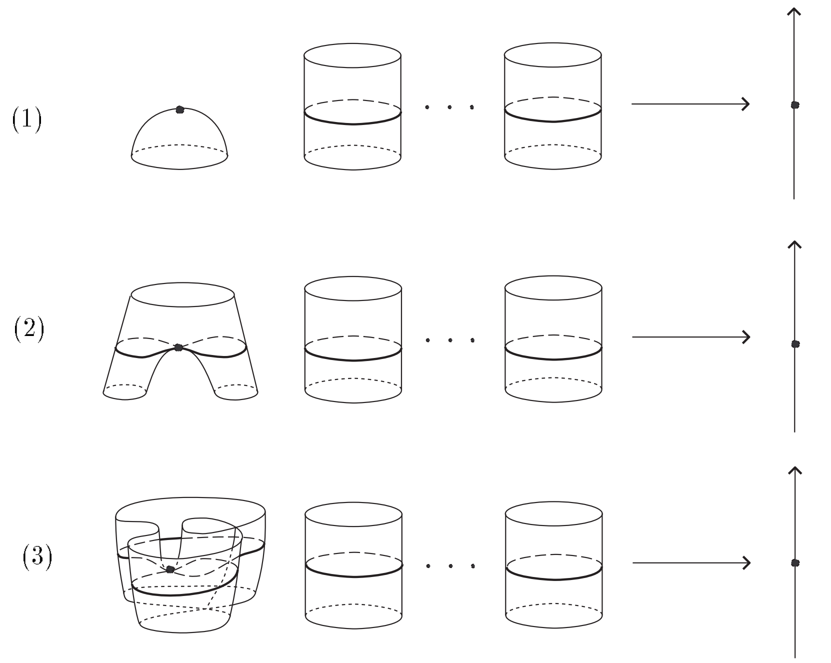

FIGURE 3. List of equivalence classes of singular fibers for Morse functions on surfaces

Proof of Theorem 3.1. If the corresponding critical point $c \in M$ is a local minimum or a local maximum, then the singular fiber is equivalent to that of Fig. 3 (1) by the Morse Lemma together with Ehresmann's fibration theorem [10].

Suppose that c is a saddle point. By the Morse Lemma, the function germ of f at c is right equivalent to the function germ of $f_1 : (x, y) \mapsto x^2 - y^2$ at the origin up to a constant: i.e., there exists a diffeomorphism $\widetilde{\varphi}_1 : V \to V_1$ such that $\widetilde{\varphi}_1(c) = (0, 0)$ and $f_1 \circ \widetilde{\varphi}_1 = f - f(c)$ on V, where V is a neighborhood of c in M and V_1 is a neighborhood of the origin in $\mathbf{R}^2$ of the form

$$V_1 = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq \varepsilon, |f_1(x, y)| < \delta\}$$

for $1 >> \exists\varepsilon >> \exists\delta > 0$. In particular, there exists a diffeomorphism $\widetilde{\varphi}_0 : V \to V_0$ such that $\widetilde{\varphi}_0(c) = c_0$ and $f_0 \circ \widetilde{\varphi}_0 = f + (f_0(c_0) - f(c))$ on V, where f_0 is the Morse function as in Fig. 3 (2) or (3), which will be chosen later, c_0 is the critical point of f_0, and V_0 is the corresponding neighborhood of c_0 (see Fig. 4). Note that the

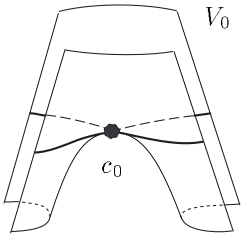

FIGURE 4. The neighborhood V_0

maps

$$(3.1)\quad f|_{\partial V\cap f^{-1}((f(c)-\delta,f(c)+\delta))} : \partial V \cap f^{-1}((f(c)-\delta, f(c)+\delta)) \to (f(c)-\delta, f(c)+\delta)$$

and

$$(3.2)\quad (f_0)|_{\partial V_0\cap(f_0)^{-1}((f_0(c_0)-\delta,f_0(c_0)+\delta))} : \partial V_0 \cap (f_0)^{-1}((f_0(c_0)-\delta, f_0(c_0)+\delta)) \to (f_0(c_0)-\delta, f_0(c_0)+\delta)$$

are proper submersions.

Since a Morse function is a submersion outside of the critical points, the closure of $f^{-1}(f(c))\smallsetminus V$ in M is a compact 1-dimensional smooth manifold whose boundary consists exactly of four points, and hence it is diffeomorphic to the disjoint union of two arcs and some circles. Therefore, $f^{-1}(f(c))$ is diffeomorphic to the disjoint union of (2) or (3) of Fig. 2 and some circles by a purely combinatorial reason. At this stage, we choose f_0 to be the Morse function as in Fig. 3 (2) (or (3)) if the component of $f^{-1}(f(c))$ containing c is diffeomorphic to (2) (resp. (3)) of Fig. 2. Furthermore, we choose the number of trivial circle bundle components appropriately.

When the component of $f^{-1}(f(c))$ containing c is diffeomorphic to (3) of Fig. 2, we see easily that the diffeomorphism

$$(3.3)\qquad \widetilde{\varphi}_0|_{f^{-1}(f(c))\cap V} : f^{-1}(f(c)) \cap V \to (f_0)^{-1}(f_0(c_0)) \cap V_0$$

between the local fibers extends to a diffeomorphism between the whole fibers $f^{-1}(f(c))$ and $(f_0)^{-1}(f_0(c_0))$. In the case of Fig. 2 (2), this is not necessarily true. If such an extension does not exist, then we modify the diffeomorphism $\widetilde{\varphi}_0$ by composing it with a self-diffeomorphism of V corresponding to the diffeomorphism $h_1 : V_1 \to V_1$ defined by $(x, y) \mapsto (y, x)$ such that $f_1 \circ h_1 = -f_1$. Note that then we have $f_0 \circ \widetilde{\varphi}_0 = r \circ f$, where $r : \mathbf{R} \to \mathbf{R}$ is the reflection defined by $x \mapsto f_0(c_0)+f(c)-x$. Then we see that the diffeomorphism (3.3) between the local fibers extends to one between the whole fibers.

Since the maps (3.1) and (3.2) are proper submersions, we see that f (resp. f_0) restricted to $f^{-1}((f(c)-\delta, f(c)+\delta)) - \operatorname{Int} V$ (resp. $(f_0)^{-1}((f_0(c_0)-\delta, f_0(c_0)+\delta)) - \operatorname{Int} V_0$) is a smooth fibration over an open interval by virtue of the relative version of Ehresmann's fibration theorem (see, for example, [24, §3]). These fibrations are clearly trivial, and hence the diffeomorphism $\widetilde{\varphi}_0 : V \to V_0$ can be extended to a fiber preserving diffeomorphism between $f^{-1}((f(c)-\delta, f(c)+\delta))$ and $(f_0)^{-1}((f_0(c_0)-\delta, f_0(c_0)+\delta))$. Hence we have the desired result. This completes the proof. $\square$

Remark 3.2. Let $c \in M$ be a critical point of a proper stable Morse function $f : M \to \mathbf{R}$ on a surface M. Then for $\delta > 0$ sufficiently small, the difference

$$b_0(f^{-1}(f(c) + \delta)) - b_0(f^{-1}(f(c) - \delta))$$

is equal to ± 1 if c is of type (1) or (2), and is equal to 0 if c is of type (3), where b_0 denotes the 0-th betti number, or equivalently, the number of connected components.

Now let us examine the relationship among the numbers of singular fibers of the above three types. For a stable Morse function $f : M \to \mathbf{R}$ on a closed surface M, let $\mathbf{0}_{\mathrm{odd}}$ denote the closure of the set

$$\{y \in \mathbf{R} : y \text{ is a regular value and } b_0(f^{-1}(y)) \text{ is odd}\}.$$

It is easy to see that $\mathbf{0}_{\mathrm{odd}}$ is a finite disjoint union of closed intervals. Furthermore, a point $y \in \mathbf{R}$ is in $\partial \mathbf{0}_{\mathrm{odd}}$ if and only if y is a critical value of type (1) or (2). Since the number of boundary points of a finite disjoint union of closed intervals is always even, we obtain the following.

Proposition 3.3. *Let $f : M \to \mathbf{R}$ be a stable Morse function on a closed surface M. Then the total number of singular fibers of types* (1) *and* (2) *is always even.*

Since the number of singular fibers is equal to the number of critical points, it has the same parity as the Euler characteristic $\chi(M)$ of the source surface M. Thus, we have the following.

Corollary 3.4. *Let $f : M \to \mathbf{R}$ be a stable Morse function on a closed surface M. Then the Euler characteristic $\chi(M)$ of M has the same parity as the number of singular fibers of type* (3).

Remark 3.5. Let M be a closed connected nonorientable surface of nonorientable genus g: i.e., M is homeomorphic to the connected sum of g copies of the real projective plane $\mathbf{R}P^2$. Then the number of singular fibers of type (3) of a stable Morse function on M is always less than or equal to g, since M can contain at most g disjointly embedded Möbius bands.

Since a neighborhood of a singular fiber of type (3) is nonorientable, we immediately obtain the following special case of the Poincaré duality, using the fact that every closed surface admits a stable Morse function.

Corollary 3.6. *Every orientable closed surface has even Euler characteristic.*

Remark 3.7. All the results in this section are valid also for maps into circles.

4. Classification of singular fibers

In this section, we consider proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds, and classify their singular fibers up to the equivalences described in Definition 2.1. We also pursue a similar classification of singular fibers for C^∞ stable maps of surfaces and 3-manifolds. We give several important consequences of these classifications as well.

4.1. Stable maps of orientable 4-manifolds into 3-manifolds.

Let M be a 4-manifold and N a 3-manifold. The following characterization of C^∞ stable maps $M \to N$ is well-known.

Proposition 4.1. *A proper smooth map $f : M \to N$ of a 4-manifold M into a 3-manifold N is C^∞ stable if and only if the following conditions are satisfied.*

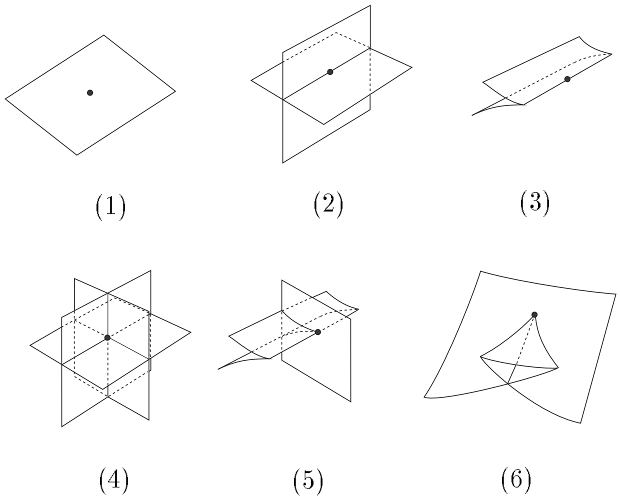

FIGURE 5. Multi-singularities of $f|_{S(f)}$

(i) *For every $q \in M$, there exist local coordinates (x, y, z, w) and (X, Y, Z) around $q \in M$ and $f(q) \in N$ respectively such that one of the following holds:*

$$
(X \circ f, Y \circ f, Z \circ f) = \begin{cases} (x, y, z), & q\text{: regular point,} \\ (x, y, z^2 + w^2), & q\text{: definite fold point,} \\ (x, y, z^2 - w^2), & q\text{: indefinite fold point,} \\ (x, y, z^3 + xz - w^2), & q\text{: cusp point,} \\ (x, y, z^4 + xz^2 + yz + w^2), & q\text{: definite swallowtail,} \\ (x, y, z^4 + xz^2 + yz - w^2), & q\text{: indefinite swallowtail.} \end{cases}
$$

(ii) *Set $S(f) = \{q \in M : \operatorname{rank} df_q < 3\}$, which is a regular closed 2-dimensional submanifold of M under the above condition* (i). *Then, for every $y \in f(S(f))$, $f^{-1}(y) \cap S(f)$ consists of at most three points and the multi-germ*

$$(f|_{S(f)}, f^{-1}(y) \cap S(f))$$

is right-left equivalent to one of the six multi-germs as described in Fig. 5*:* (1) *represents a single immersion germ which corresponds to a fold point,* (2) *and* (4) *represent normal crossings of two and three immersion germs, respectively, each of which corresponds to a fold point,* (3) *corresponds to a cusp point,* (5) *represents a transverse crossing of a cuspidal edge as in* (3) *and an immersion germ corresponding to a fold point, and* (6) *corresponds to a swallowtail.*

Remark 4.2. According to du Plessis and Wall [9, 51], if (n, p) is in the nice range in the sense of Mather [27], a proper smooth map between manifolds of dimensions n and p is C^∞ stable if and only if it is C^0 stable. Hence, the above proposition gives a characterization of C^0 stable maps of 4-manifolds into 3-manifolds as well, since $(4, 3)$ is in the nice range.

Let q be a singular point of a proper C^∞ stable map $f : M \to N$ of a 4-manifold M into a 3-manifold N. Then, using the above local normal forms, it is easy to describe the diffeomorphism type of a neighborhood of q in $f^{-1}(f(q))$. More precisely, we easily get the following local characterizations of singular fibers.

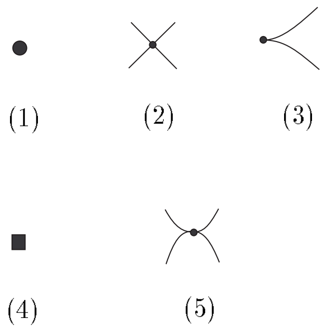

FIGURE 6. Neighborhood of a singular point in a singular fiber

Lemma 4.3. *Every singular point q of a proper C^∞ stable map $f : M \to N$ of a 4-manifold M into a 3-manifold N has one of the following neighborhoods in its corresponding singular fiber (see Fig. 6):*

1. *isolated point diffeomorphic to $\{(x, y) \in \mathbf{R}^2 : x^2 + y^2 = 0\}$, if q is a definite fold point,*
2. *union of two transverse arcs diffeomorphic to $\{(x, y) \in \mathbf{R}^2 : x^2 - y^2 = 0\}$, if q is an indefinite fold point,*
3. *cuspidal arc diffeomorphic to $\{(x, y) \in \mathbf{R}^2 : x^3 - y^2 = 0\}$, if q is a cusp point,*
4. *isolated point diffeomorphic to $\{(x, y) \in \mathbf{R}^2 : x^4 + y^2 = 0\}$, if q is a definite swallowtail,*
5. *union of two tangent arcs diffeomorphic to $\{(x, y) \in \mathbf{R}^2 : x^4 - y^2 = 0\}$, if q is an indefinite swallowtail.*

Note that in Fig. 6, both the black dot (1) and the black square (4) represent an isolated point; however, we use distinct symbols in order to distinguish them.

For the local nearby fibers, we have the following.

Lemma 4.4. *Let $f : M \to N$ be a proper C^∞ stable map of a 4-manifold M into a 3-manifold N and $q \in S(f)$ a singular point such that $f^{-1}(f(q)) \cap S(f) = \{q\}$. Then the local fibers near q are as in Fig. 7:*

1. *q is a definite fold point,*
2. *q is an indefinite fold point,*
3. *q is a cusp point,*
4. *q is a definite swallowtail,*
5. *q is an indefinite swallowtail,*

where each 0- or 1-dimensional object represents a portion of the fiber over the corresponding point in the target and each 2-dimensional object represents $f(S(f)) \subset N$ near $f(q)$.

In the following, we assume that the 4-manifold M is orientable. Then we get the following classification of singular fibers.

Theorem 4.5. *Let $f : M \to N$ be a proper C^∞ stable map of an orientable 4-manifold M into a 3-manifold N. Then, every singular fiber of f is equivalent to the disjoint union of one of the fibers as in Fig. 8 and a finite number of copies of a fiber of the trivial circle bundle.*

In Fig. 8, κ denotes the codimension of the set of points in N whose corresponding fibers are equivalent to the relevant one. For details, see Remark 4.7. Furthermore,

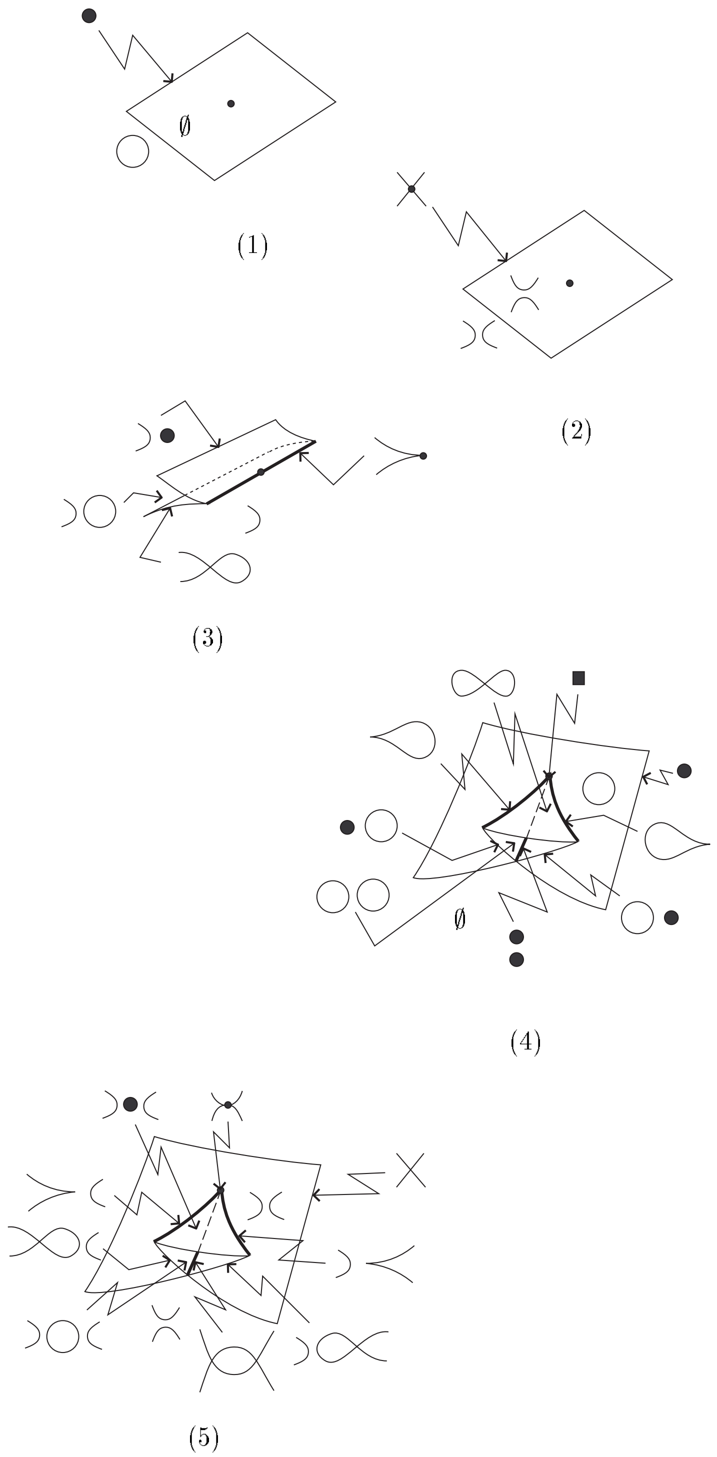

Figure 7. Local degenerations of fibers

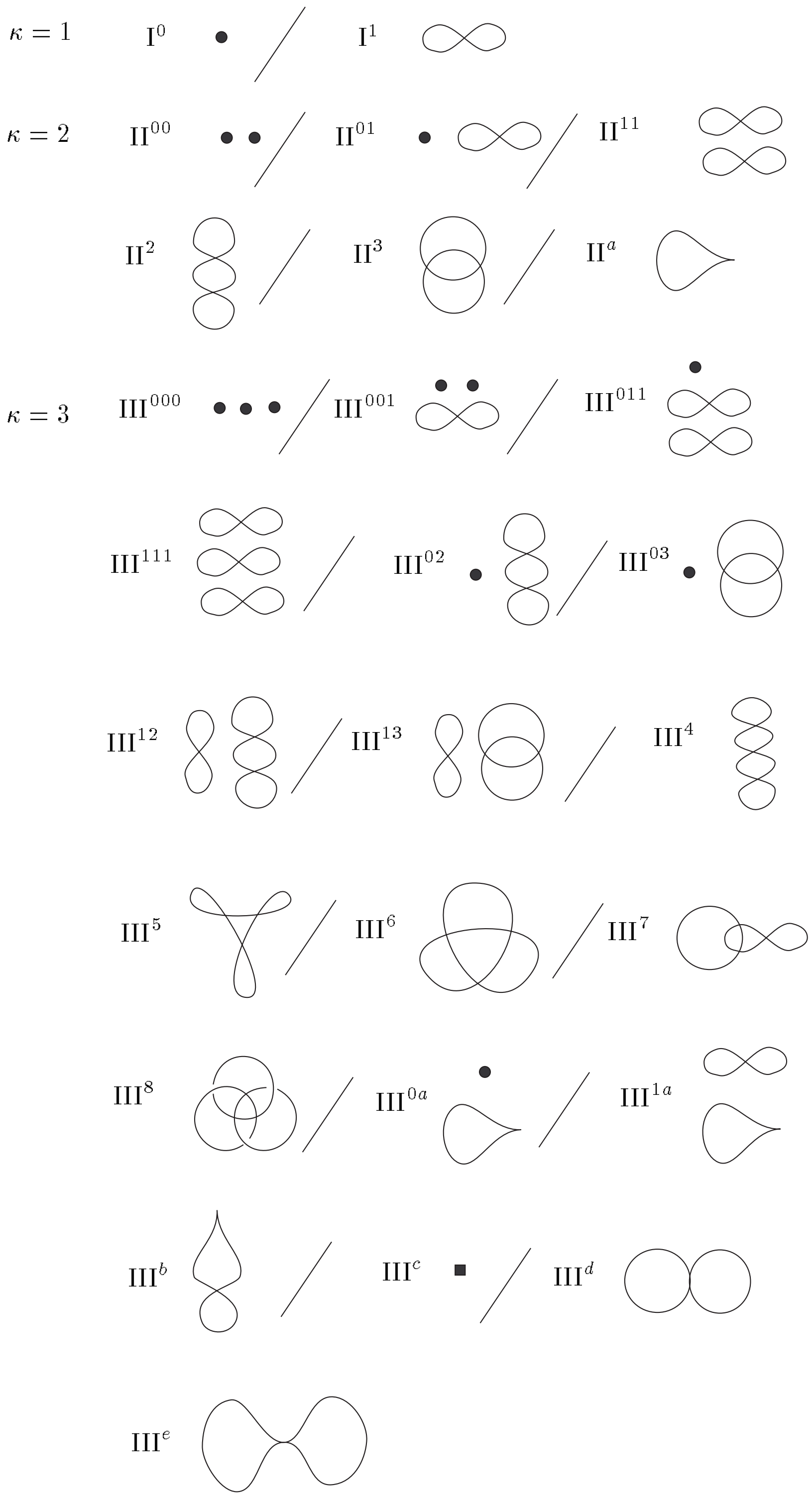

FIGURE 8. List of singular fibers of proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds

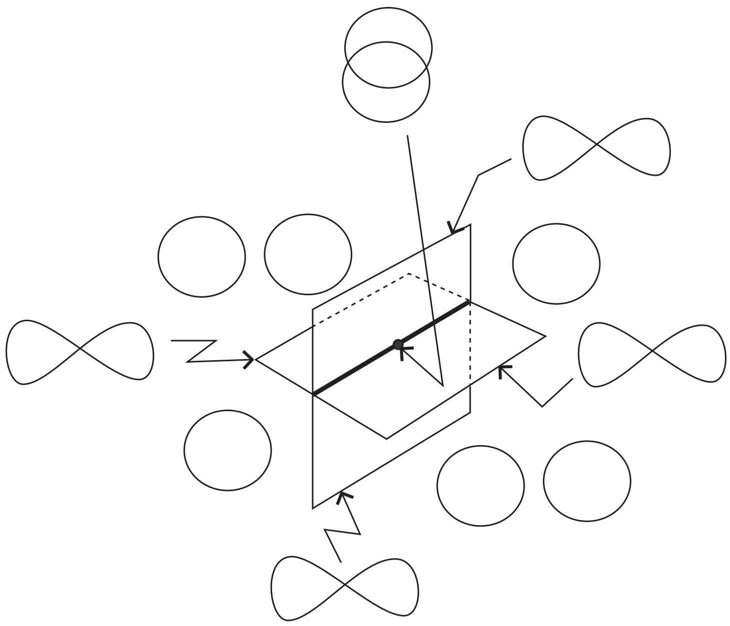

FIGURE 9. Degeneration of fibers around the fiber of type II^3

$\mathrm{I}^*, \mathrm{II}^*$ and III^* mean the names of the corresponding singular fibers, and "/" is used only for separating the figures. Note that we have named the fibers so that each connected fiber has its own digit or letter, and a disconnected fiber has the name consisting of the digits or letters of its connected components. Hence, the number of digits or letters in the superscript coincides with the number of connected components.

It is not difficult to describe the behavior of the map on a neighborhood of each singular fiber in Fig. 8. This can also be regarded as a degeneration of fibers around the singular fiber, or a deformation of the singular fiber. In Fig. 9–12 are depicted the nearby fibers for four of the 27 singular fibers (Fig. 7 (1) and (4) can also be regarded as the deformations of the singular fibers of types I^0 and III^c respectively).[1] Since we are assuming that the source 4-manifold is orientable, the singular fiber as in Fig. 2 (3) never appears in the degenerations.

Remark 4.6. Each singular fiber described in Fig. 8 can be realized as a component of a singular fiber of some C^∞ stable map of a *closed* orientable 4-manifold into $\mathbf{R}^3$. This can be seen as follows. Given a singular fiber, we can first realize it semi-locally; i.e., we can construct a proper C^∞ stable map of an open 4-manifold M_0 into $\mathbf{R}^3$ such that its image coincides with the open unit disk in $\mathbf{R}^3$ and that it has the given singular fiber over the center. Such a map can be constructed, for example, by using a 2-parameter deformation of smooth functions on an orientable surface: in this case, the open 4-manifold M_0 is diffeomorphic to the product of an open orientable surface and an open 2-disk. Then we can extend the map to a smooth map of a closed orientable 4-manifold M containing M_0 into $\mathbf{R}^3$. Perturbing the extended map slightly, we obtain a desired stable map. In fact, we can choose an arbitrary closed orientable 4-manifold as the source manifold M of the desired map.

[1]The degenerations of fibers around all the singular fibers are described in detail in [17].

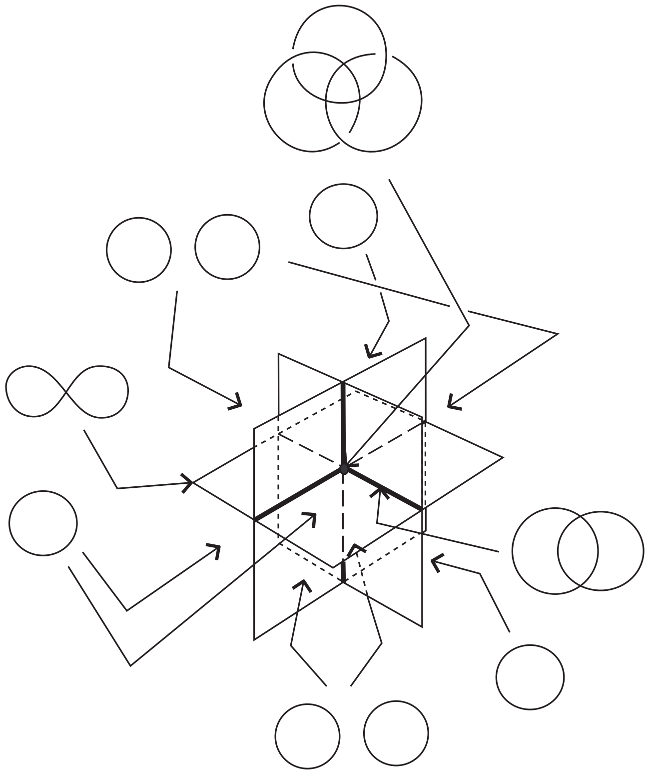

FIGURE 10. Degeneration of fibers around the fiber of type III^8

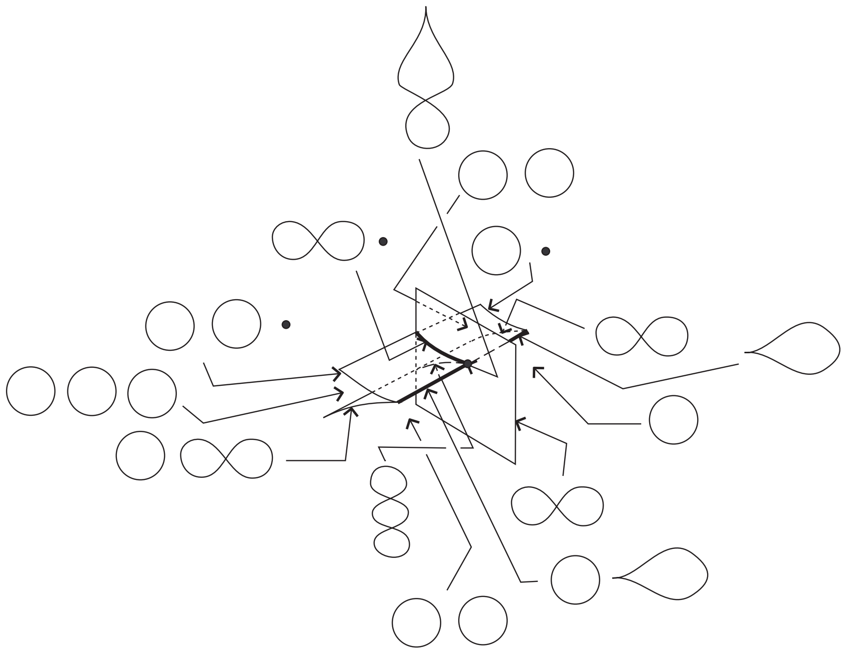

FIGURE 11. Degeneration of fibers around the fiber of type III^b

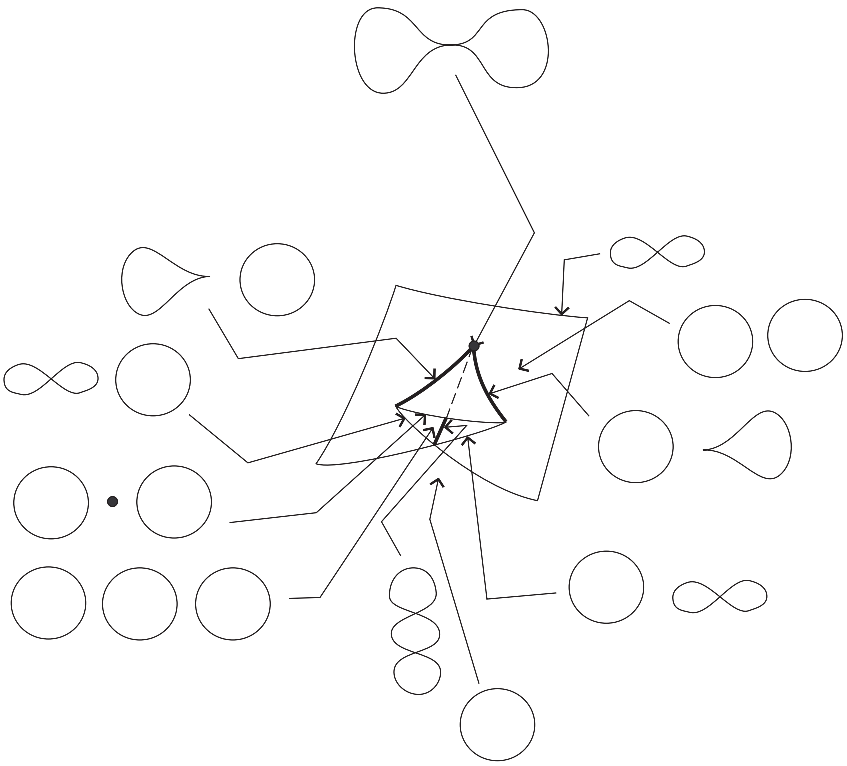

FIGURE 12. Degeneration of fibers around the fiber of type III^e

Proof of Theorem 4.5. Let us take a point $y \in f(S(f))$. We will first show that the union of the components of $f^{-1}(y)$ containing singular points is diffeomorphic to one of the fibers listed in Fig. 8 in the sense of Definition 2.1 (2).

If y corresponds to Fig. 5 (1), then $f^{-1}(y)$ contains exactly one singular point, which is a fold point. Thus, by an argument similar to that in the proof of Theorem 3.1, we see that the component of $f^{-1}(y)$ containing the singular point is diffeomorphic to one of the three figures of Fig. 2. If a fiber as in Fig. 2 (3) appears, then the 4-manifold M must contain a punctured Möbius band times D^2, and hence is nonorientable. Since we have assumed that M is orientable, this does not occur. Hence, we see that the singular fiber $f^{-1}(y)$ is diffeomorphic to the disjoint union of I^0 (or I^1) and a finite number of nonsingular circles.

If y corresponds to Fig. 5 (2), then $f^{-1}(y)$ contains exactly two singular points, say q_1 and q_2, which are fold points. Since they have neighborhoods as in Lemma 4.3 (1) or (2) in $f^{-1}(y)$, and since f is a submersion outside of the singular points, we see that there are only a finite number of possibilities for the diffeomorphism type of the union of the components of $f^{-1}(y)$ containing q_1 and q_2: for example, if both q_1 and q_2 are indefinite fold points, then it is obtained from two copies of the figure as in Fig. 6 (2) by connecting their end points by four arcs. Then we can use Lemma 4.4 to obtain the nearby fibers of each possible singular fiber: for example, for the singular fiber of type II^3, see Fig. 9. Excluding the possibilities such that a singular fiber as in Fig. 2 (3) appears as a nearby fiber, we get the fibers $\mathrm{II}^{00}, \mathrm{II}^{01}, \mathrm{II}^{11}, \mathrm{II}^2$ and II^3.

By similar combinatorial arguments, we obtain the following singular fibers:

(1) if y corresponds to Fig. 5 (3), then we obtain II^a,

(2) if y corresponds to Fig. 5 (4), then we obtain III^{000}, III^{001}, III^{011}, III^{111}, III^{02}, III^{03}, III^{12}, III^{13}, III^{4}, III^{5}, III^{6}, III^{7} and III^{8},
(3) if y corresponds to Fig. 5 (5), then we obtain III^{0a}, III^{1a} and III^{b},
(4) if y corresponds to Fig. 5 (6), then we obtain III^{c}, III^{d} and III^{e}.

Thus we have proved that every singular fiber is diffeomorphic to one of the fibers listed in the theorem.

In order to complete the proof, we have only to show that if two singular fibers are diffeomorphic to each other, then they are C^∞ equivalent in the sense of Definition 2.1 (2), except for the two types of fibers I^0 and III^c.

Let $f_i : M_i \to N_i$, $i = 0, 1$, be proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds. Let us take $y_i \in f_i(S(f_i)) \subset N_i$. Suppose that the singular fibers over y_0 and y_1 are diffeomorphic to each other.

If the singular fibers over y_0 and y_1 are of type I^0, then let $q_i \in S(f_i) \cap (f_i)^{-1}(y_i)$ be the unique singular point on the fibers. Since q_i are definite fold points, there exist neighborhoods V_i of q_i in M_i, U_i of y_i in N_i and diffeomorphisms $\widetilde{\varphi}_0 : (V_0, q_0) \to (V_1, q_1)$ and $\varphi : (U_0, y_0) \to (U_1, y_1)$ which make the following diagram commutative:

$$\begin{array}{ccc} (V_0, q_0) & \xrightarrow{\widetilde{\varphi}_0} & (V_1, q_1) \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ (U_0, y_0) & \xrightarrow{\varphi} & (U_1, y_1). \end{array}$$

Furthermore, by taking the neighborhoods sufficiently small, we may assume that $(U_i, U_i \cap f_i(S(f_i)))$ is as described in Fig. 5 (1), that V_i is a connected component of $(f_i)^{-1}(U_i)$, $U_i \cong \mathrm{Int}\, D^3$, $V_i \cong \mathrm{Int}\, D^4$, and $(f_i)^{-1}(y_i) \cap V_i = \{q_i\}$. Then the maps

$$f_i|_{(f_i)^{-1}(U_i) \smallsetminus V_i} : (f_i)^{-1}(U_i) \smallsetminus V_i \to U_i, \quad i = 0, 1$$

are proper submersions and their fibers are disjoint unions of the same number of copies of the circle. Hence, by Ehresmann's fibration theorem, the diffeomorphism $\widetilde{\varphi}_0 : (V_0, q_0) \to (V_1, q_1)$ extends to a diffeomorphism

$$\widetilde{\varphi} : ((f_0)^{-1}(U_0), (f_0)^{-1}(y_0)) \to ((f_1)^{-1}(U_1), (f_1)^{-1}(y_1))$$

so that the diagram (2.1) in §2 commutes. Hence, the fibers over y_0 and y_1 are equivalent.

The same argument works when the fibers over y_0 and y_1 are of type III^c.

When the fibers over y_0 and y_1 are of type I^1, we can imitate the above argument for the case of I^0; however, we cannot take V_i to be a connected component of $(f_i)^{-1}(U_i)$, since the relevant singular points are indefinite fold points. So, we first take V_i sufficiently small, and then imitate the proof of Theorem 3.1. More precisely, we modify the diffeomorphisms $\widetilde{\varphi}_0 : V_0 \to V_1$ and $\varphi : U_0 \to U_1$, if necessary, by using self-diffeomorphisms of V_0 and U_0 corresponding to those defined by $(x, y, z, w) \mapsto (x, y, w, z)$ and $(X, Y, Z) \to (X, Y, -Z)$ respectively with respect to the coordinates as in Proposition 4.1 (i) so that the diffeomorphism

$$\widetilde{\varphi}_0 : (f_0)^{-1}(y_0) \cap V_0 \to (f_1)^{-1}(y_1) \cap V_1$$

extends to one between the whole fibers $(f_0)^{-1}(y_0)$ and $(f_1)^{-1}(y_1)$. Then we use the relative version of Ehresmann's fibration theorem to extend the diffeomorphism $\widetilde{\varphi}_0 : V_0 \to V_1$ to a fiber preserving diffeomorphism between $(f_0)^{-1}(U_0)$ and $(f_1)^{-1}(U_1)$. Hence, the fibers over y_0 and y_1 are equivalent.

The same argument works when the fiber over y_i contains exactly one singular point: namely, for the cases of II^a, III^d and III^e.

Now suppose that the fibers over y_0 and y_1 are of type II^{00}. Then there exist neighborhoods U_i of y_i such that the sets $U_i \cap f_i(S(f_i))$ are as in Fig. 5 (2). In particular, there exists a diffeomorphism $\varphi : (U_0, y_0) \to (U_1, y_1)$ between the neighborhoods U_i of y_i such that $\varphi(U_0 \cap f_0(S(f_0))) = U_1 \cap f_1(S(f_1))$. Note that we can describe the degeneration of the fibers of f_i over U_i using Lemma 4.4 (for the case of II^3, see Fig. 9). Then we see that the diffeomorphism φ can be chosen so that it preserves the diffeomorphism types of the fibers: i.e., we may assume that $(f_0)^{-1}(y)$ is diffeomorphic to $(f_1)^{-1}(\varphi(y))$ for all $y \in U_0$. Put $(f_i)^{-1}(y_i) \cap S(f_i) = \{q_i, q_i'\}$, where q_i and q_i' are definite fold points. Then the multi-germs $\varphi \circ f_0 : ((f_0)^{-1}(U_0), \{q_0, q_0'\}) \to (U_1, y_1)$ and $f_1 : ((f_1)^{-1}(U_1), \{q_1, q_1'\}) \to (U_1, y_1)$ have the same discriminant set germ $(f_1(S(f_1)), y_1)$ and they satisfy the assumption of [8, (0.6) Theorem]. Hence they are right equivalent; i.e., there exists a diffeomorphism $\widetilde{\varphi}_0 : (V_0, \{q_0, q_0'\}) \to (V_1, \{q_1, q_1'\})$ between sufficiently small neighborhoods V_0 and V_1 of $\{q_0, q_0'\}$ and $\{q_1, q_1'\}$ respectively such that $f_1 \circ \widetilde{\varphi}_0 = \varphi \circ f_0 : (V_0, \{q_0, q_0'\}) \to (U_1, y_1)$ (see also [52]). Then the rest of the proof is the same as that in the case of I^0.

When the fibers over y_0 and y_1 are of type II^{01}, put $(f_i)^{-1}(y_i) \cap S(f_i) = \{q_i, q_i'\}$, where q_i is a definite fold point and q_i' is an indefinite fold point. Then we can imitate the above argument to obtain a diffeomorphism φ between neighborhoods U_i of y_i and a diffeomorphism $\widetilde{\varphi}_0$ between neighborhoods V_i of $\{q_i, q_i'\}$ such that $f_1 \circ \widetilde{\varphi}_0 = \varphi \circ f_0$ on V_0. If we choose the diffeomorphism φ so that it preserves the diffeomorphism types of the fibers, then we see easily that the diffeomorphism $\widetilde{\varphi}_0$ between the local fibers $(f_0)^{-1}(y_0) \cap V_0$ and $(f_1)^{-1}(y_1) \cap V_1$ necessarily extends to one between the whole fibers $(f_0)^{-1}(y_0)$ and $(f_1)^{-1}(y_1)$; in other words, we do not need to modify $\widetilde{\varphi}_0$ or φ as in the proof of Theorem 3.1. Then the rest of the proof is the same as that in the case of I^1.

A similar argument works also in the cases of II^{11}, III^{000}, III^{001}, III^{011}, III^{111}, III^{0a} and III^{1a}.

When the fibers over y_0 and y_1 are of type II^2, we can use almost the same argument. The only difference is that we have to choose the diffeomorphism $\widetilde{\varphi}_0 : V_0 \to V_1$ so that the diffeomorphism $\widetilde{\varphi}_0 : (f_0)^{-1}(y_0) \cap V_0 \to (f_1)^{-1}(y_1) \cap V_1$ between the local fibers extends to a diffeomorphism between the whole fibers $(f_0)^{-1}(y_0)$ and $(f_1)^{-1}(y_1)$. For this, we can use the self-diffeomorphisms of each of the neighborhoods of the indefinite fold points corresponding to those defined by $(x, y, z, w) \mapsto (x, y, \pm z, \pm w)$ with respect to the coordinates as in Proposition 4.1 (i). More precisely, we modify $\widetilde{\varphi}_0$ using these diffeomorphisms as we did in the case of I^1. Note that here, φ is chosen so that it preserves the diffeomorphism types of the fibers, and is fixed. Therefore, we cannot use the self-diffeomorphisms corresponding to those defined by $(x, y, z, w) \mapsto (x, y, \pm w, \pm z)$.

We can use similar arguments also in the cases of II^3, III^{02}, III^{03}, III^{12}, III^{13}, III^4, III^5, III^6, III^7, III^8 and III^b.

In the above argument, we note the following. When the fibers over y_0 and y_1 are of type III^{02}, III^{03}, III^{12}, III^{13}, III^4 or III^7, put $(f_i)^{-1}(y_i) \cap S(f_i) = \{q_i, q_i', q_i''\}$. We name them so that

$$\varphi(f_0(V_{0j} \cap S(f_0))) = f_1(V_{1j} \cap S(f_1))), \quad j = 1, 2, 3,$$

where V_i is the disjoint union of V_{i1}, V_{i2} and V_{i3} which are neighborhoods of q_i, q_i' and q_i'' respectively. Then we see easily that the correspondence $q_0 \mapsto q_1$, $q_0' \mapsto q_1'$, $q_0'' \mapsto q_1''$ coincides with that given by $\widetilde{\varphi}_0$ and extends to a diffeomorphism between the whole fibers $(f_0)^{-1}(y_0)$ and $(f_1)^{-1}(y_1)$, since φ preserves the diffeomorphism types of the fibers. (For the cases of II^2, II^3, III^5, III^6 and III^8, we do not need such an argument by virtue of their symmetries. For the case of III^b, we do not need it either because the two singular points contained in a fiber are of different types.) Therefore, we can apply the argument above.

This completes the proof of Theorem 4.5. $\square$

Remark 4.7. Let $f : M \to N$ be a proper C^∞ stable map of an orientable 4-manifold M into a 3-manifold N and $\mathfrak{F}$ the type of one of the singular fibers appearing in Fig. 8. We define $\mathfrak{F}(f)$ to be the set of points $y \in N$ such that the fiber $f^{-1}(y)$ over y is equivalent to the disjoint union of $\mathfrak{F}$ and some copies of a fiber of the trivial circle bundle. As the above proof shows, each $\mathfrak{F}(f)$ is a submanifold of N, provided that it is nonempty, and its codimension is denoted by $\kappa(\mathfrak{F})$, which is called the *codimension* of the singular fiber of type $\mathfrak{F}$ (or the codimension of the disjoint union of $\mathfrak{F}$ and some copies of a fiber of the trivial circle bundle). See Fig. 8 for the codimension of each singular fiber. Note that the target manifold N is naturally stratified into these submanifolds.

Remark 4.8. As the proof of Theorem 4.5 shows, two singular fibers of proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds are diffeomorphic if and only if they are C^∞ equivalent, except for the singular fibers of types I^0 and III^c.

Furthermore, we also have the following.

Corollary 4.9. *Two fibers of proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds are C^∞ equivalent if and only if they are C^0 equivalent.*

Proof. We have only to prove the statement for arbitrary two fibers in the list given in Theorem 4.5. Suppose that two fibers are C^0 equivalent. Then the degenerations of the fibers around the singular fibers are also topologically equivalent, and their nearby fibers must be homeomorphic. It is not difficult to check that this implies that the two fibers are of the same C^∞ type. $\square$

Remark 4.10. Recall that Damon [7] (see also [6]) has shown that for nice dimensions, two C^∞ stable map germs are topologically right-left equivalent if and only if they are smoothly right-left equivalent. The above corollary shows that this is also true for C^∞ stable map germs along fibers for the dimension pair $(4, 3)$, which is in the nice range, as long as the source manifold is orientable. (In fact, this is also true for the dimension pairs $(2, 1)$ and $(3, 2)$ without the orientability hypothesis. See §3 and Corollary 4.16 below.) Note that even for nice dimensions, this statement for map germs along fibers is not true in general. For example, we can construct two proper Morse functions of 8-dimensional manifolds such that one of them has the standard 7-dimensional sphere as its regular fibers, and that the other has a homotopy 7-sphere not diffeomorphic to the standard 7-sphere [28] as its regular fibers. Then the map germs along (nonsingular) fibers are topologically right-left equivalent, but not smoothly right-left equivalent.

Remark 4.11. Let us denote by $\mathbf{0}$ the smooth right-left equivalence class of a regular fiber. Furthermore, for a fiber of type $\mathfrak{F}$ and a positive integer n, we denote by $\mathfrak{F}_n$ the smooth right-left equivalence class of the fiber consisting of a fiber of type $\mathfrak{F}$ and some copies of a fiber of the trivial circle bundle such that the total number of connected components is equal to n. If we classify the singular fibers of proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds up to homeomorphism in the sense of Definition 2.1 (2), then we get a smaller list than that given in Theorem 4.5. In fact, we have the following, where "$\approx$" means a homeomorphism:

(1) $\mathrm{I}^0_n \approx \mathrm{III}^c_n$ for $n \geq 1$,
(2) $\mathrm{I}^0_n \approx \mathrm{III}^c_n \approx \mathrm{III}^{0a}_n$ for $n \geq 2$,
(3) $\mathrm{I}^1_n \approx \mathrm{III}^b_n \approx \mathrm{III}^d_n \approx \mathrm{III}^e_n$ for $n \geq 1$,
(4) $\mathrm{I}^1_n \approx \mathrm{III}^b_n \approx \mathrm{III}^d_n \approx \mathrm{III}^e_n \approx \mathrm{III}^{1a}_n$ for $n \geq 2$,
(5) $\mathrm{III}^6_n \approx \mathrm{III}^8_n$ for $n \geq 1$,
(6) $\mathrm{II}^a_n \approx \mathbf{0}_n$ for $n \geq 1$.

Furthermore, it is not difficult to see that the above fibers exhaust all the repetitions of the homeomorphism types in the list of smooth right-left equivalence classes of fibers.

Remark 4.12. Suppose that a smooth map $f : M \to N$ between smooth manifolds is given. For two points $q, q' \in M$, we define $q \sim_f q'$ if $f(q) = f(q')$ and q and q' belong to the same connected component of an f-fiber. We define $W_f = M/\sim_f$ to be the quotient space and $q_f : M \to W_f$ the quotient map. Then it is easy to see that there exists a unique continuous map $\overline{f} : W_f \to N$ such that the diagram

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "q_f"'] & & N \\ & W_f \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

is commutative. The space W_f or the above commutative diagram is called the *Stein factorization* of f (see [25]). It is known that if f is a topologically stable map, then W_f is a polyhedron and all the maps appearing in the above diagram are triangulable (for details, see [17]).

Kushner, Levine and Porto [23, 25] have determined the local structures of Stein factorizations of proper C^∞ stable maps of 3-manifolds into surfaces by using their classification of singular fibers. Similarly, by using our classification of singular fibers, we can determine the local structures of Stein factorizations of proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds. For details, see [17].

Remark 4.13. For proper C^∞ stable maps of possibly nonorientable 4-manifolds into 3-manifolds, a similar classification of singular fibers is obtained in [53].

4.2. **Stable maps of surfaces and 3-manifolds.** In this subsection, let us mention similar classifications of singular fibers of proper C^∞ stable Morse functions on surfaces and those of proper C^∞ stable maps of 3-manifolds into surfaces. Let us begin by the following remark.

Remark 4.14. We can obtain a classification of singular fibers of proper C^∞ stable maps of orientable 3-manifolds into surfaces similar to Theorem 4.5. The list we get is nothing but the singular fibers with $\kappa = 1$ and 2 in Fig. 8. The list itself was already obtained by Kushner, Levine and Porto [23, 25], although they did not describe explicitly the equivalence relation for their classification.

In fact, we can easily get the following list of C^∞ right-left equivalence classes of singular fibers for proper C^∞ stable maps of (not necessarily orientable) 3-manifolds into surfaces. Details are left to the reader.

Theorem 4.15. *Let $f : M \to N$ be a proper C^∞ stable map of a 3-manifold M into a surface N. Then, every singular fiber of f is equivalent to the disjoint union of one of the fibers as in Fig.* 13 *and a finite number of copies of a fiber of the trivial circle bundle.*

Note that the above list itself is mentioned in the introduction of [25]. As a corollary to Theorems 3.1 and 4.15, we get the following, which we can prove by an argument similar to that in the proof of Corollary 4.9. Details are left to the reader.

Corollary 4.16. *Let us consider two fibers of proper C^∞ stable Morse functions on surfaces, or two fibers of proper C^∞ stable maps of 3-manifolds into surfaces. Then, the following conditions are equivalent to each other.*

(1) *They are diffeomorphic.*
(2) *They are C^0 equivalent.*

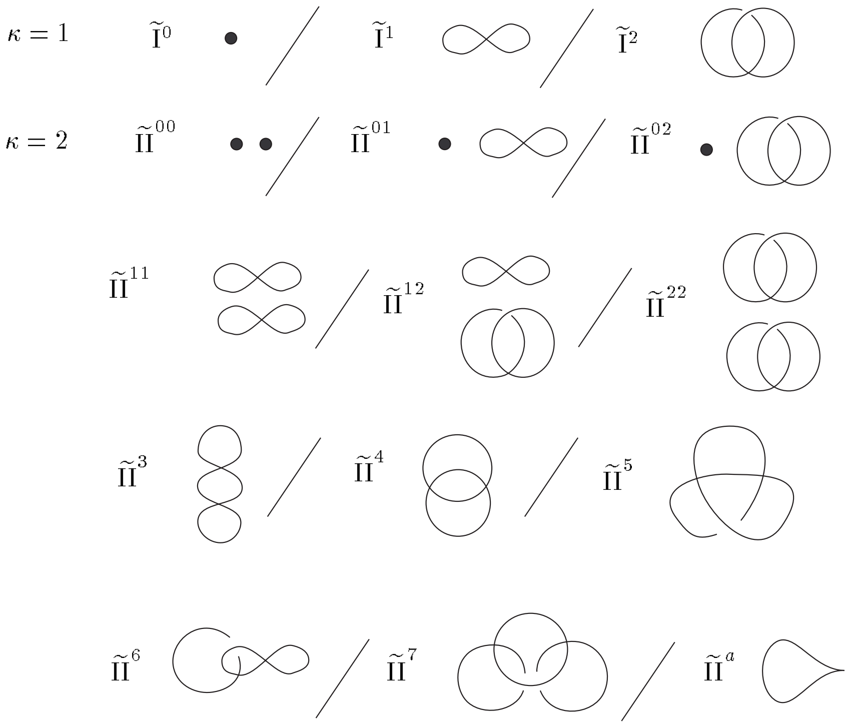

FIGURE 13. List of singular fibers of proper C^∞ stable maps of 3-manifolds into surfaces

(3) *They are C^∞ equivalent.*

We warn the reader that the fibers as depicted in Fig. 2 (2) and (3) (or the fibers $\widetilde{\mathrm{I}}^1$ and $\widetilde{\mathrm{I}}^2$) are homeomorphic to each other, although they are not C^0 equivalent nor diffeomorphic to each other. Compare these results with Remark 4.8 and Corollary 4.9.

As an important consequence of the above mentioned result, we show the following.

Corollary 4.17. *Let $f_0 : M_0 \to N_0$ and $f_1 : M_1 \to N_1$ be two proper C^∞ stable maps of surfaces into 1-dimensional manifolds. Then, the maps f_0 and f_1 are C^0 right-left equivalent if and only if they are C^∞ right-left equivalent.*

Proof. Suppose that f_0 and f_1 are C^0 right-left equivalent so that we have homeomorphisms $\widetilde{\varphi} : M_0 \to M_1$ and $\varphi : N_0 \to N_1$ satisfying $f_1 \circ \widetilde{\varphi} = \varphi \circ f_0$. Since $f_0(S(f_0))$ and $f_1(S(f_1))$ are discrete sets and φ sends $f_0(S(f_0))$ homeomorphically onto $f_1(S(f_1))$, we see that there exists a diffeomorphism $\psi : N_0 \to N_1$ which approximates φ such that $\psi|_{f_0(S(f_0))} = \varphi|_{f_0(S(f_0))}$.

Then by Corollary 4.16 together with the proof of Theorem 3.1, we see that for each point $y \in f_0(S(f_0))$, there exist a small neighborhood U_y of y in N_0 and a diffeomorphism $\widetilde{\psi}_y : (f_0)^{-1}(U_y) \to (f_1)^{-1}(U_{y'})$ such that the diagram

$$\begin{array}{ccc} ((f_0)^{-1}(U_y), (f_0)^{-1}(y)) & \xrightarrow{\widetilde{\psi}_y} & ((f_1)^{-1}(U_{y'}), (f_1)^{-1}(y')) \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ (U_y, y) & \xrightarrow{\psi} & (U_{y'}, y') \end{array}$$

is commutative, where $y' = \psi(y)$ and $U_{y'} = \psi(U_y)$ is a neighborhood of y' in N_1. Here, we choose the diffeomorphism $\widetilde{\psi}_y$ so that it approximates $\widetilde{\varphi}|_{(f_0)^{-1}(U_y)}$.

Since the collection of homeomorphisms $\widetilde{\varphi}|_{(f_0)^{-1}(U_y)}$, $y \in f_0(S(f_0))$, extends to a homeomorphism $\widetilde{\varphi}$ such that $f_1 \circ \widetilde{\varphi} = \varphi \circ f_0$, the collection of diffeomorphisms $\widetilde{\psi}_y$, $y \in f_0(S(f_0))$, also extends to a homeomorphism $\widetilde{\psi}$ such that $f_1 \circ \widetilde{\psi} = \psi \circ f_0$.

Now, it is well-known that two C^∞ S^1-bundles are C^0 equivalent if and only if they are C^∞ equivalent. This is true also for C^∞ bundles with fiber a union of finite copies of S^1. Hence the homeomorphism $\widetilde{\psi}$ above can be chosen to be a diffeomorphism. Hence, the C^∞ maps f_0 and f_1 are C^∞ right-left equivalent to each other. This completes the proof. □

Problem 4.18. Let $f_0 : M_0 \to N_0$ and $f_1 : M_1 \to N_1$ be two proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds (or two proper C^∞ stable maps of 3-manifolds into surfaces). If f_0 and f_1 are C^0 right-left equivalent, then are they C^∞ right-left equivalent?

For the above problem and Corollary 4.17, refer to [6, §4], for example. Note that there have been known a lot of examples of 4-manifold pairs which are mutually homeomorphic, but are not diffeomorphic. If the answer to the above problem is affirmative, then such 4-manifolds would not admit C^∞ stable maps that are C^0 right-left equivalent.

5. Relations among the numbers of singular fibers

Let $f : M \to N$ be a C^∞ stable map of a closed orientable 4-manifold into a 3-manifold. In this section, we consider a natural stratification of N induced by the equivalence classes of fibers of f, and obtain some relations among the numbers of singular fibers of codimension three.

Let $f : M \to N$ be a C^∞ stable map of a closed orientable 4-manifold M into a 3-manifold N and $\mathfrak{F}$ the equivalence class of one of the singular fibers appearing in Fig. 8. We define $\mathfrak{F}(f)$ to be the set of points $y \in N$ such that the fiber $f^{-1}(y)$ over y is equivalent to the union of $\mathfrak{F}$ and some copies of a fiber of the trivial circle bundle. Furthermore, we define $\mathfrak{F}_{\mathrm{o}}(f)$ (resp. $\mathfrak{F}_{\mathrm{e}}(f)$) to be the subset of $\mathfrak{F}(f)$ consisting of the points $y \in N$ such that $b_0(f^{-1}(y))$ is odd (resp. even), where b_0 denotes the number of connected components. We denote the closures of $\mathfrak{F}(f)$, $\mathfrak{F}_{\mathrm{o}}(f)$, and $\mathfrak{F}_{\mathrm{e}}(f)$ in N by $\overline{\mathfrak{F}(f)}$, $\overline{\mathfrak{F}_{\mathrm{o}}(f)}$, and $\overline{\mathfrak{F}_{\mathrm{e}}(f)}$, respectively. It is easy to see that each of $\overline{\mathfrak{F}(f)}$, $\overline{\mathfrak{F}_{\mathrm{o}}(f)}$, or $\overline{\mathfrak{F}_{\mathrm{e}}(f)}$ is a $(3-\kappa)$-dimensional subcomplex of N, where κ is the codimension of $\mathfrak{F}$. In particular, if the codimension κ is equal to two, then $\overline{\mathfrak{F}_{\mathrm{o}}(f)}$ and $\overline{\mathfrak{F}_{\mathrm{e}}(f)}$ are finite graphs embedded in N. Their vertices correspond to points over which lies a singular fiber with $\kappa = 3$. For a singular fiber $\mathfrak{F}'$ of $\kappa = 3$, the degree of the vertex corresponding to $\mathfrak{F}'_{\mathrm{o}}(f)$ (or $\mathfrak{F}'_{\mathrm{e}}(f)$) in the graph $\overline{\mathfrak{F}_{\mathrm{o}}(f)}$ is given in Table 1, which can be obtained by using the description of nearby fibers as in Fig. 10–12. Note that the degrees in the graph $\overline{\mathfrak{F}_{\mathrm{e}}(f)}$ can be obtained by interchanging $\mathfrak{F}'_{\mathrm{o}}(f)$ with $\mathfrak{F}'_{\mathrm{e}}(f)$ in the table.

In the following, for a finite set X, we denote by $|X|$ the number of its elements. Since the sum of the degrees over all vertices is always an even number for any finite graph, we obtain the following.

Proposition 5.1. *Let $f : M \to N$ be a C^∞ stable map of a closed orientable 4-manifold into a 3-manifold. Then the following numbers are always even.*

(1) $|\mathrm{III}^{000}(f)| + |\mathrm{III}^{001}(f)| + |\mathrm{III}^{0a}_{\mathrm{e}}(f)| + |\mathrm{III}^{c}_{\mathrm{e}}(f)|$.
(2) $|\mathrm{III}^{000}(f)| + |\mathrm{III}^{001}(f)| + |\mathrm{III}^{0a}_{\mathrm{o}}(f)| + |\mathrm{III}^{c}_{\mathrm{o}}(f)|$.
(3) $|\mathrm{III}^{0a}_{\mathrm{o}}(f)| + |\mathrm{III}^{1a}_{\mathrm{e}}(f)| + |\mathrm{III}^{b}_{\mathrm{e}}(f)|$.
(4) $|\mathrm{III}^{0a}_{\mathrm{e}}(f)| + |\mathrm{III}^{1a}_{\mathrm{o}}(f)| + |\mathrm{III}^{b}_{\mathrm{o}}(f)|$.

	$\overline{\mathrm{II}^{00}_{\mathrm{o}}(f)}$	$\overline{\mathrm{II}^{01}_{\mathrm{o}}(f)}$	$\overline{\mathrm{II}^{11}_{\mathrm{o}}(f)}$	$\overline{\mathrm{II}^{2}_{\mathrm{o}}(f)}$	$\overline{\mathrm{II}^{3}_{\mathrm{o}}(f)}$	$\overline{\mathrm{II}^{a}_{\mathrm{o}}(f)}$
$\mathrm{III}^{000}_{\mathrm{o}}(f)$	3	0	0	0	0	0
$\mathrm{III}^{000}_{\mathrm{e}}(f)$	3	0	0	0	0	0
$\mathrm{III}^{001}_{\mathrm{o}}(f)$	1	2	0	0	0	0
$\mathrm{III}^{001}_{\mathrm{e}}(f)$	1	2	0	0	0	0
$\mathrm{III}^{011}_{\mathrm{o}}(f)$	0	2	1	0	0	0
$\mathrm{III}^{011}_{\mathrm{e}}(f)$	0	2	1	0	0	0
$\mathrm{III}^{111}_{\mathrm{o}}(f)$	0	0	3	0	0	0
$\mathrm{III}^{111}_{\mathrm{e}}(f)$	0	0	3	0	0	0
$\mathrm{III}^{02}_{\mathrm{o}}(f)$	0	2	0	1	0	0
$\mathrm{III}^{02}_{\mathrm{e}}(f)$	0	2	0	1	0	0
$\mathrm{III}^{03}_{\mathrm{o}}(f)$	0	4	0	0	1	0
$\mathrm{III}^{03}_{\mathrm{e}}(f)$	0	0	0	0	1	0
$\mathrm{III}^{12}_{\mathrm{o}}(f)$	0	0	2	1	0	0
$\mathrm{III}^{12}_{\mathrm{e}}(f)$	0	0	2	1	0	0
$\mathrm{III}^{13}_{\mathrm{o}}(f)$	0	0	4	0	1	0
$\mathrm{III}^{13}_{\mathrm{e}}(f)$	0	0	0	0	1	0
$\mathrm{III}^{4}_{\mathrm{o}}(f)$	0	0	0	3	0	0
$\mathrm{III}^{4}_{\mathrm{e}}(f)$	0	0	1	2	0	0
$\mathrm{III}^{5}_{\mathrm{o}}(f)$	0	0	0	3	0	0
$\mathrm{III}^{5}_{\mathrm{e}}(f)$	0	0	0	3	0	0
$\mathrm{III}^{6}_{\mathrm{o}}(f)$	0	0	0	3	3	0
$\mathrm{III}^{6}_{\mathrm{e}}(f)$	0	0	0	0	0	0
$\mathrm{III}^{7}_{\mathrm{o}}(f)$	0	0	0	4	1	0
$\mathrm{III}^{7}_{\mathrm{e}}(f)$	0	0	0	0	1	0
$\mathrm{III}^{8}_{\mathrm{o}}(f)$	0	0	0	0	6	0
$\mathrm{III}^{8}_{\mathrm{e}}(f)$	0	0	0	0	0	0
$\mathrm{III}^{0a}_{\mathrm{o}}(f)$	0	1	0	0	0	1
$\mathrm{III}^{0a}_{\mathrm{e}}(f)$	1	0	0	0	0	1
$\mathrm{III}^{1a}_{\mathrm{o}}(f)$	0	0	1	0	0	1
$\mathrm{III}^{1a}_{\mathrm{e}}(f)$	0	1	0	0	0	1
$\mathrm{III}^{b}_{\mathrm{o}}(f)$	0	0	0	1	0	1
$\mathrm{III}^{b}_{\mathrm{e}}(f)$	0	1	0	0	0	1
$\mathrm{III}^{c}_{\mathrm{o}}(f)$	0	0	0	0	0	2
$\mathrm{III}^{c}_{\mathrm{e}}(f)$	1	0	0	0	0	0
$\mathrm{III}^{d}_{\mathrm{o}}(f)$	0	0	0	0	1	2
$\mathrm{III}^{d}_{\mathrm{e}}(f)$	0	0	0	0	0	0
$\mathrm{III}^{e}_{\mathrm{o}}(f)$	0	0	0	1	0	0
$\mathrm{III}^{e}_{\mathrm{e}}(f)$	0	0	0	0	0	2

TABLE 1. Degrees of each vertex in the graphs

(5) $|\mathrm{III}^{011}(f)| + |\mathrm{III}^{111}(f)| + |\mathrm{III}^{4}_{\mathrm{e}}(f)| + |\mathrm{III}^{1a}_{\mathrm{o}}(f)|$.
(6) $|\mathrm{III}^{011}(f)| + |\mathrm{III}^{111}(f)| + |\mathrm{III}^{4}_{\mathrm{o}}(f)| + |\mathrm{III}^{1a}_{\mathrm{e}}(f)|$.
(7) $|\mathrm{III}^{02}(f)| + |\mathrm{III}^{12}(f)| + |\mathrm{III}^{4}_{\mathrm{o}}(f)| + |\mathrm{III}^{5}(f)| + |\mathrm{III}^{6}_{\mathrm{o}}(f)| + |\mathrm{III}^{b}_{\mathrm{o}}(f)| + |\mathrm{III}^{e}_{\mathrm{o}}(f)|$.
(8) $|\mathrm{III}^{02}(f)| + |\mathrm{III}^{12}(f)| + |\mathrm{III}^{4}_{\mathrm{e}}(f)| + |\mathrm{III}^{5}(f)| + |\mathrm{III}^{6}_{\mathrm{e}}(f)| + |\mathrm{III}^{b}_{\mathrm{e}}(f)| + |\mathrm{III}^{e}_{\mathrm{e}}(f)|$.
(9) $|\mathrm{III}^{03}(f)| + |\mathrm{III}^{13}(f)| + |\mathrm{III}^{6}_{\mathrm{o}}(f)| + |\mathrm{III}^{7}(f)| + |\mathrm{III}^{d}_{\mathrm{o}}(f)|$.
(10) $|\mathrm{III}^{03}(f)| + |\mathrm{III}^{13}(f)| + |\mathrm{III}^{6}_{\mathrm{e}}(f)| + |\mathrm{III}^{7}(f)| + |\mathrm{III}^{d}_{\mathrm{e}}(f)|$.
(11) $|\mathrm{III}^{0a}(f)| + |\mathrm{III}^{1a}(f)| + |\mathrm{III}^{b}(f)|$.

In fact, items (1)–(10) of the above proposition correspond to the graphs $\overline{\mathrm{II}^{00}_{\mathrm{o}}(f)}$, $\overline{\mathrm{II}^{00}_{\mathrm{e}}(f)}$, $\overline{\mathrm{II}^{01}_{\mathrm{o}}(f)}$, $\overline{\mathrm{II}^{01}_{\mathrm{e}}(f)}$, $\overline{\mathrm{II}^{11}_{\mathrm{o}}(f)}$, $\overline{\mathrm{II}^{11}_{\mathrm{e}}(f)}$, $\overline{\mathrm{II}^{2}_{\mathrm{o}}(f)}$, $\overline{\mathrm{II}^{2}_{\mathrm{e}}(f)}$, $\overline{\mathrm{II}^{3}_{\mathrm{o}}(f)}$, and $\overline{\mathrm{II}^{3}_{\mathrm{e}}(f)}$ respectively. Item (11) corresponds to both $\overline{\mathrm{II}^{a}_{\mathrm{o}}(f)}$ and $\overline{\mathrm{II}^{a}_{\mathrm{e}}(f)}$.

Eliminating the terms of the forms $|\mathfrak{F}_{\mathrm{o}}(f)|$ and $|\mathfrak{F}_{\mathrm{e}}(f)|$, we obtain the following.

Corollary 5.2. *Let $f : M \to N$ be a C^∞ stable map of a closed orientable 4-manifold into a 3-manifold. Then the following numbers are always even.*

(1) $|\mathrm{III}^{0a}(f)| + |\mathrm{III}^{c}(f)|$.
(2) $|\mathrm{III}^{0a}(f)| + |\mathrm{III}^{1a}(f)| + |\mathrm{III}^{b}(f)|$.
(3) $|\mathrm{III}^{4}(f)| + |\mathrm{III}^{1a}(f)|$.
(4) $|\mathrm{III}^{4}(f)| + |\mathrm{III}^{6}(f)| + |\mathrm{III}^{b}(f)| + |\mathrm{III}^{e}(f)|$.
(5) $|\mathrm{III}^{6}(f)| + |\mathrm{III}^{d}(f)|$.

Remark 5.3. It is easy to see that the five numbers appearing in Corollary 5.2 are all even if and only if the following five hold.

(1) $|\mathrm{III}^{0a}(f)| \equiv |\mathrm{III}^{c}(f)| \pmod 2$.
(2) $|\mathrm{III}^{1a}(f)| \equiv |\mathrm{III}^{4}(f)| \pmod 2$.
(3) $|\mathrm{III}^{6}(f)| \equiv |\mathrm{III}^{d}(f)| \pmod 2$.
(4) $|\mathrm{III}^{b}(f)| \equiv |\mathrm{III}^{4}(f)| + |\mathrm{III}^{c}(f)| \pmod 2$.
(5) $|\mathrm{III}^{c}(f)| + |\mathrm{III}^{d}(f)| + |\mathrm{III}^{e}(f)| \equiv 0 \pmod 2$.

Note that the left hand side of congruence (5) is nothing but the total number of swallowtails. Note also that item (11) of Proposition 5.1 represents the number of cuspidal intersections as in Fig. 5 (5).

Remark 5.4. Adding items (2), (3), (6), (8) and (10) of Proposition 5.1, we obtain

$$\begin{aligned}&|\mathrm{III}^{000}(f)| + |\mathrm{III}^{001}(f)| + |\mathrm{III}^{011}(f)| + |\mathrm{III}^{111}(f)| + |\mathrm{III}^{02}(f)| + |\mathrm{III}^{03}(f)| \\ &+|\mathrm{III}^{12}(f)| + |\mathrm{III}^{13}(f)| + |\mathrm{III}^{4}(f)| + |\mathrm{III}^{5}(f)| + |\mathrm{III}^{7}(f)| \\ &+|\mathrm{III}^{c}_{\mathrm{o}}(f)| + |\mathrm{III}^{d}_{\mathrm{e}}(f)| + |\mathrm{III}^{e}_{\mathrm{e}}(f)| \equiv 0 \pmod 2.\end{aligned}$$

This and congruence (1) of Remark 5.3 have also been obtained in [17] by using methods different from ours.

Remark 5.5. By using the same method, we can obtain similar co-existence results for singular fibers of proper C^∞ stable maps of closed 3-manifolds into surfaces. More precisely, using the notation introduced in Theorem 4.15, we have the following.

(1) $|\widetilde{\mathrm{II}}^{01}(f)| + |\widetilde{\mathrm{II}}^{a}_{\mathrm{e}}(f)| \equiv 0 \pmod 2$.
(2) $|\widetilde{\mathrm{II}}^{01}(f)| + |\widetilde{\mathrm{II}}^{a}_{\mathrm{o}}(f)| \equiv 0 \pmod 2$.
(3) $|\widetilde{\mathrm{II}}^{02}(f)| + |\widetilde{\mathrm{II}}^{12}(f)| + |\widetilde{\mathrm{II}}^{6}(f)| \equiv 0 \pmod 2$.

Details are left to the reader (compare this with Table 3 of §10).

We end this section by posing a problem.

Problem 5.6. Let $\mathcal{S}$ be the $\mathbf{Z}_2$-vector space consisting of 38-tuples of elements of $\mathbf{Z}_2$ such that the congruences in Proposition 5.1 hold, where each of the 38 components corresponds to $|\mathrm{III}^{000}_{\mathrm{o}}(f)|, |\mathrm{III}^{000}_{\mathrm{e}}(f)|$, etc. Then, for an arbitrary element

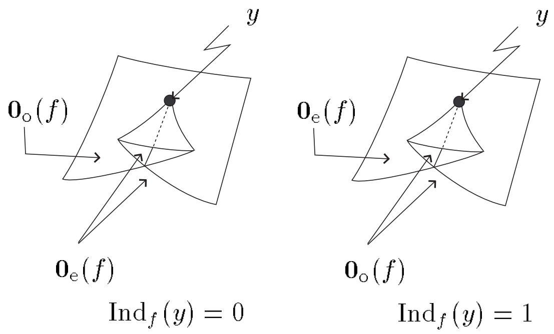

FIGURE 14. Index of a swallowtail

of $\mathcal{S}$, does there exist a C^∞ stable map of some closed orientable 4-manifold into some 3-manifold which realizes it as the parities of the numbers of corresponding singular fibers? In other words, do the congruences in Proposition 5.1 exhaust all the possible relations among the parities of the numbers of singular fibers of the form $\mathfrak{F}_\mathrm{o}(f)$ or $\mathfrak{F}_\mathrm{e}(f)$?

6. Parity of the Euler characteristic of the source 4-manifold

In this section, using the co-existence results for singular fibers obtained in the previous section, we study the relationship between the number of singular fibers of a certain type and the Euler characteristic of the source 4-manifold. In the following, χ will denote the Euler characteristic.

Let $f : M \to N$ be a C^∞ stable map of a closed orientable 4-manifold into a 3-manifold. Set

$$\begin{aligned} \mathbf{0}_\mathrm{o}(f) &= \{y \in N \smallsetminus f(S(f)) : b_0(f^{-1}(y)) \equiv 1 \pmod 2\}, \\ \mathbf{0}_\mathrm{e}(f) &= \{y \in N \smallsetminus f(S(f)) : b_0(f^{-1}(y)) \equiv 0 \pmod 2\}. \end{aligned}$$

It is easy to see that they are disjoint open sets of N. Furthermore, since M is compact, the closure $\overline{\mathbf{0}_\mathrm{o}(f)}$ of $\mathbf{0}_\mathrm{o}(f)$ is compact. Let y and y' be points in N belonging to adjacent regions of $N \smallsetminus f(S(f))$. Since M is orientable, the difference between the numbers of components of the fibers over y and y' is always equal to one. Hence, we have

$$\overline{\mathbf{0}_\mathrm{o}(f)} \cap \overline{\mathbf{0}_\mathrm{e}(f)} = \partial \mathbf{0}_\mathrm{o}(f) = \partial \mathbf{0}_\mathrm{e}(f) = f(S(f)),$$

where for a subset X of a topological space, ∂X denotes $\overline{X} \smallsetminus \mathrm{Int}\, X$. In other words, $(N, f(S(f)))$ is two colorable in the sense of [31] (see also [30]).

Note that the map $f|_{S(f)} : S(f) \to N$ is a topologically stable singular surface in the sense of [31]. Then, for each cross cap $y \in f(S(f))$, which corresponds to a swallowtail point of f, we can define the index $\mathrm{Ind}_f(y) \in \{0, 1\}$ by using the coloring $(\mathbf{0}_\mathrm{o}(f), \mathbf{0}_\mathrm{e}(f))$ of $(N, f(S(f)))$. More precisely, it is defined as in Fig. 14 (for details, see [31]).

Then by Szűcs' formula [47] (see also [31]), we have

$$T(f(S(f))) + \sum_y \mathrm{Ind}_f(y) \equiv \chi(S(f)) \pmod 2, \tag{6.1}$$

where y runs through the cross caps of $f(S(f))$ corresponding to Fig. 5 (6), and $T(f(S(f)))$ denotes the number of triple points of $f(S(f))$ corresponding to Fig. 5 (4). On the other hand, by using the degenerations of the fibers around the singular fibers corresponding to swallowtails as in Fig. 12, we obtain the following:

$$\mathrm{Ind}_f(y) = \begin{cases} 0, & \text{if } y \in \mathrm{III}^c_{\mathrm{o}}(f) \cup \mathrm{III}^d_{\mathrm{o}}(f) \cup \mathrm{III}^e_{\mathrm{e}}(f), \\ 1, & \text{if } y \in \mathrm{III}^c_{\mathrm{e}}(f) \cup \mathrm{III}^d_{\mathrm{e}}(f) \cup \mathrm{III}^e_{\mathrm{o}}(f). \end{cases}$$

Hence, applying (6.1), we have

$$\begin{aligned} &|\mathrm{III}^{000}(f)| + |\mathrm{III}^{001}(f)| + |\mathrm{III}^{011}(f)| + |\mathrm{III}^{111}(f)| + |\mathrm{III}^{02}(f)| + |\mathrm{III}^{03}(f)| \\ &\quad + |\mathrm{III}^{12}(f)| + |\mathrm{III}^{13}(f)| + |\mathrm{III}^{4}(f)| + |\mathrm{III}^{5}(f)| + |\mathrm{III}^{6}(f)| + |\mathrm{III}^{7}(f)| \\ &\quad + |\mathrm{III}^{8}(f)| + |\mathrm{III}^c_{\mathrm{e}}(f)| + |\mathrm{III}^d_{\mathrm{e}}(f)| + |\mathrm{III}^e_{\mathrm{o}}(f)| \equiv \chi(S(f)) \pmod 2. \end{aligned}$$

On the other hand, adding items (1), (3), (5), (7), (9) and (11) in Proposition 5.1, we obtain

$$\begin{aligned} &|\mathrm{III}^{000}(f)| + |\mathrm{III}^{001}(f)| + |\mathrm{III}^{011}(f)| + |\mathrm{III}^{111}(f)| + |\mathrm{III}^{02}(f)| + |\mathrm{III}^{03}(f)| \\ &\quad + |\mathrm{III}^{12}(f)| + |\mathrm{III}^{13}(f)| + |\mathrm{III}^{4}(f)| + |\mathrm{III}^{5}(f)| + |\mathrm{III}^{7}(f)| + |\mathrm{III}^c_{\mathrm{e}}(f)| \\ &\quad + |\mathrm{III}^d_{\mathrm{o}}(f)| + |\mathrm{III}^e_{\mathrm{o}}(f)| \equiv 0 \pmod 2. \end{aligned}$$

Adding the above two congruences, we obtain

$$|\mathrm{III}^{6}(f)| + |\mathrm{III}^{8}(f)| + |\mathrm{III}^{d}(f)| \equiv \chi(S(f)) \pmod 2.$$

Since $|\mathrm{III}^{6}(f)| \equiv |\mathrm{III}^{d}(f)| \pmod 2$ by Corollary 5.2 (5), we get

$$|\mathrm{III}^{8}(f)| \equiv \chi(S(f)) \pmod 2.$$

Since we always have

$$\chi(S(f)) \equiv \chi(M) \pmod 2$$

by [11, 39], we finally obtain the following theorem, which can be regarded as a 4-dimensional version of Corollary 3.4.

Theorem 6.1. *Let $f : M \to N$ be a C^∞ stable map of a closed orientable 4-manifold into a 3-manifold. Then we have*

$$\chi(M) \equiv |\mathrm{III}^{8}(f)| \pmod 2.$$

Remark 6.2. The above theorem holds also for C^∞ stable maps of closed (not necessarily orientable) 4-manifolds into 3-manifolds such that every fiber has an orientable neighborhood.

Remark 6.3. The results in the previous and the present sections can be generalized to C^∞ stable maps of possibly nonorientable closed 4-manifolds into 3-manifolds. For details, see [53] (see also Remark 4.13).

Remark 6.4. A result corresponding to Remark 3.5 does not hold for singular fibers of types III^* for C^∞ stable maps of 4-manifolds into 3-manifolds. This is because we can increase the number of fibers of a given type of codimension three as much as we want. For details, see Remark 4.6.

We end this section by posing a problem.

Problem 6.5. Is it possible to obtain an integral formula giving the signature of the source oriented 4-manifold in terms of the algebraic numbers of some singular fibers?

In order to appropriately define "algebraic numbers" of singular fibers, we should probably determine those singular fibers which are "orientable", and define their signs.

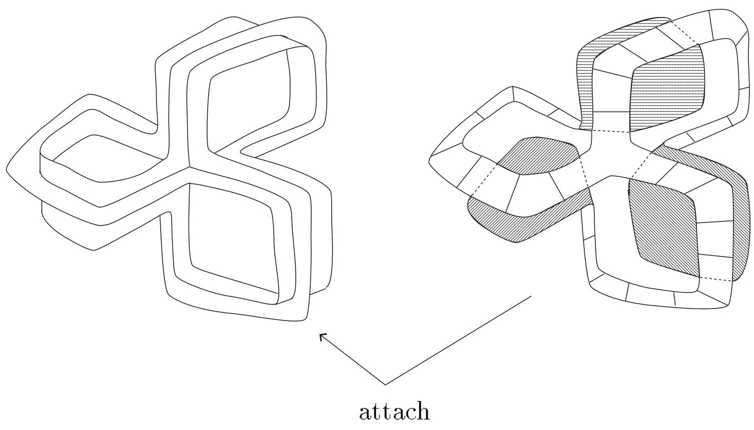

FIGURE 15. Boy surface

7. Some explicit examples of stable maps of 4-manifolds

In this section, we give explicit examples of C^∞ stable maps of 4-manifolds into $\mathbf{R}^3$. Note that there have already been known some explicit examples of such stable maps that have only definite fold points as their singularities (see [43, 37, 38, 41, 42]). Such maps have singular fibers of types I^0, II^{00}, and III^{000}, and have no other singular fibers. Furthermore, the source 4-manifolds of such maps always have even Euler characteristics. Here we construct more complicated maps having a singular fiber of type III^8 such that the source 4-manifold has odd Euler characteristic.

Let us first construct a C^∞ stable map $f : \mathbf{C}P^2 \sharp 2\overline{\mathbf{C}P^2} \to \mathbf{R}^3$ which satisfies the following properties.

(1) The map f has only fold points as its singularities.
(2) The singular set $S(f)$ is the union of three 2-sphere components consisting of definite fold points and a projective plane component consisting of indefinite fold points.
(3) The discriminant set $f(S(f))$ is a disjoint union of three embedded 2-spheres and the Boy surface in $\mathbf{R}^3$ (see Fig. 15).
(4) The fibers of f can be completely described (details will be given in Fig. 16).

Recall that the Boy surface P, which is the image of an immersion $\mathbf{R}P^2 \looparrowright \mathbf{R}^3$, is constructed by attaching a 2-disk as in the right hand side of Fig. 15 to the image of an immersion of the Möbius band as in the left hand side of Fig. 15, from the front side.

Note that $\mathbf{R}^3 \smallsetminus P$ consists exactly of two regions. Let S_0 be a 2-sphere embedded in the unbounded region of $\mathbf{R}^3 \smallsetminus P$ such that the bounded region of $\mathbf{R}^3 \smallsetminus S_0$ contains P. Furthermore, let S_1 and S_2 be two disjoint concentric 2-spheres embedded in the bounded region of $\mathbf{R}^3 \smallsetminus P$ such that S_2 is contained in the bounded region of $\mathbf{R}^3 \smallsetminus S_1$. Note that $\mathbf{R}^3 \smallsetminus (P \cup S_0 \cup S_1 \cup S_2)$ consists exactly of five regions and that $P \cup S_0 \cup S_1 \cup S_2$ naturally induces a stratification of $\mathbf{R}^3$: we have five strata of dimension three, seven strata of dimension two, three strata of dimension one, and one stratum of dimension zero. Let us denote by A^i_j the strata of dimension i. We enumerate them as follows (see Fig. 16):

(1) the closure of A^2_j contains $A^0_1 \cup A^1_j$, $j = 1, 2, 3$, and the closure of A^2_4 contains $A^0_1 \cup A^1_1 \cup A^1_2 \cup A^1_3$,
(2) $A^2_5 = S_0, A^2_6 = S_1, A^2_7 = S_2$,
(3) A^3_1 is the unbounded region of $\mathbf{R}^3 \smallsetminus S_0$,

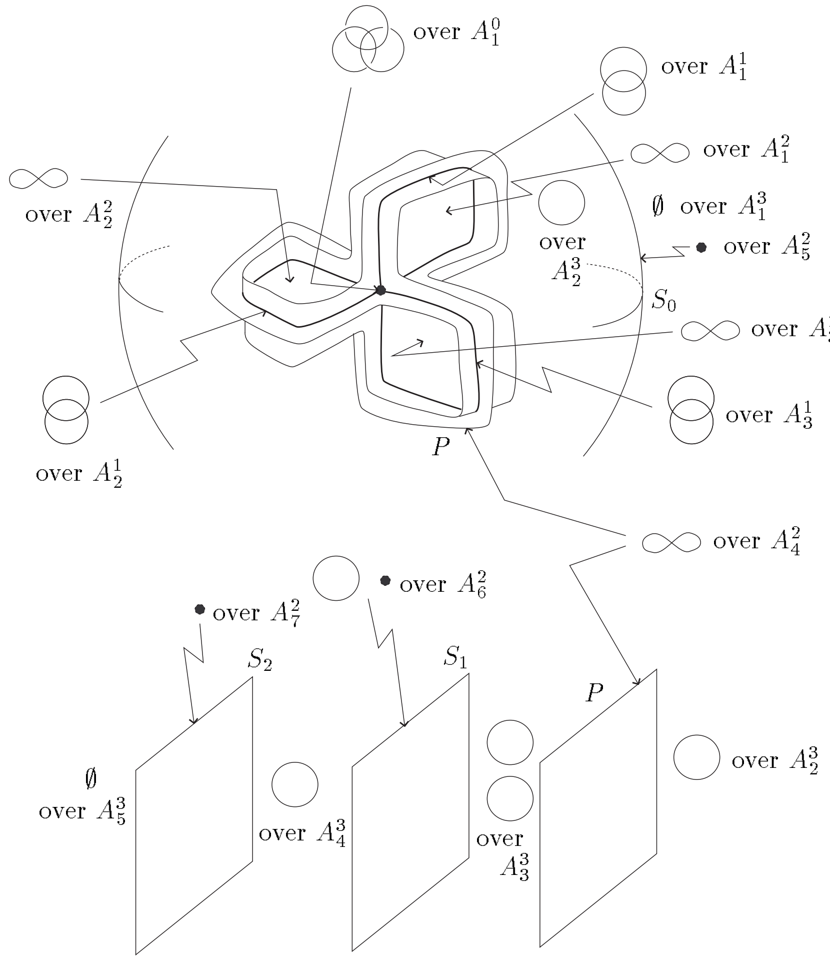

FIGURE 16. Fibers over the points in $\mathbf{R}^3$

(4) A_2^3 is the region between S_0 and the Boy surface,
(5) A_3^3 is the region between the Boy surface and S_1,
(6) A_4^3 is the region between S_1 and S_2, and
(7) A_5^3 is the bounded region of $\mathbf{R}^3 \smallsetminus S_2$.

We shall construct a fold map $f : \mathbf{C}P^2 \sharp 2\overline{\mathbf{C}P^2} \to \mathbf{R}^3$ such that $f(S_0(f)) = S_0 \cup S_1 \cup S_2$ and $f(S_1(f)) = P$, where a *fold map* is a smooth map with only fold points as its singularities. In particular, $S_0(f)$ is diffeomorphic to the disjoint union of three 2-spheres and $S_1(f)$ is diffeomorphic to $\mathbf{R}P^2$.

Over the points on each stratum we put fibers as depicted in Fig. 16, where the lower figure depicts a part of the 2-disk (contained in P) as in the right hand side of Fig. 15 together with parts of S_1 and S_2, which sit inside the bounded region of $\mathbf{R}^3 \smallsetminus P$. It is easy to see that the regular parts of the fibers can be oriented consistently. Hence, if such a smooth map is constructed, then the source 4-manifold will be orientable.

Let $N(A_1^0)$ be a small closed disk neighborhood of the zero dimensional stratum A_1^0 such that its boundary two sphere is transverse to the other strata. Let $N(A_j^1) \cong D^2 \times [0,1]$ denote the closure of $\widetilde{N}(A_j^1) \smallsetminus N(A_1^0)$, where $\widetilde{N}(A_j^1)$ is a small tubular neighborhood of the 1-dimensional stratum A_j^1 such that its boundary is transverse

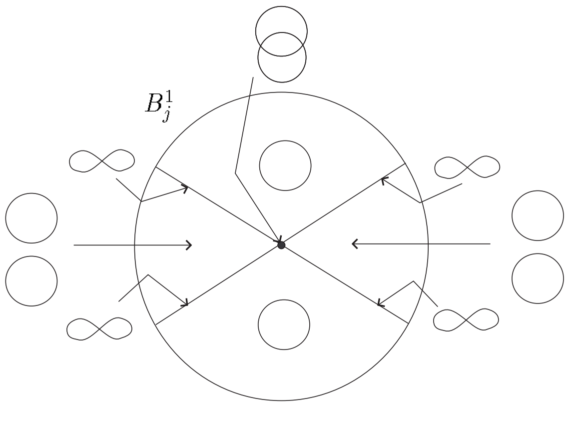

FIGURE 17. Fibers over the points in B^1_j for g^1_j

to the strata of higher dimensions ($j = 1, 2, 3$). We may assume that $N(A^1_j) \cong D^2 \times [0,1]$ is attached to $N(A^0_1)$ along $D^2 \times \{0,1\}$ and that $N(A^0_1) \cup N(A^1_1) \cup N(A^1_2) \cup N(A^1_3)$ is a regular neighborhood of $A^0_1 \cup A^1_1 \cup A^1_2 \cup A^1_3$ in $\mathbf{R}^3$. Similarly, we construct $N(A^2_j)$, $j = 1, 2, \ldots, 7$, and $N(A^3_j)$, $j = 1, 2, \ldots, 5$, so that the family of closed sets $\{N(A^i_j)\}_{0 \leq i \leq 3}$ covers $\mathbf{R}^3$ and that distinct members intersect only along their boundaries. Furthermore, we put $\widehat{A}^i_j = A^i_j \cap N(A^i_j)$. We may assume that the natural projection $N(A^i_j) \to \widehat{A}^i_j$ is a smooth $(3-i)$-disk bundle.

Let us now construct a closed orientable 4-manifold M and a C^∞ stable map $f : M \to \mathbf{R}^3$ such that $f(S(f))$ and the fibers are as depicted in Fig. 16. Our strategy is to first construct compact 4-manifolds M^i_j and smooth maps $f^i_j : M^i_j \to N(A^i_j)$, and then glue them together.

As we have noted in Remark 4.6, we can construct a compact orientable 4-manifold M^0_1 and a smooth map $f^0_1 : M^0_1 \to \mathbf{R}^3$ which has only fold points as its singularities such that $f^0_1(M^0_1) = N(A^0_1)$ and that the fibers are consistent with Fig. 16 (see also Fig. 10). In our case, M^0_1 is diffeomorphic to $T^2_{(3)} \times D^2$, where for a surface F, we denote by $F_{(\ell)}$ the surface obtained from F by taking off ℓ open disks whose closures do not intersect each other, and T^2 denotes the 2-dimensional torus.

Let B^1_j be a 2-disk fiber of the bundle $N(A^1_j) \to \widehat{A}^1_j$, $j = 1, 2, 3$. Then we can construct a compact orientable 3-manifold N^1_j and a smooth map $g^1_j : N^1_j \to B^1_j$ which has only fold points as its singularities such that its fibers are as depicted in Fig. 17 (for details, see [23, 25, 36], for example). Then we can construct a smooth map $f^1_j : M^1_j = N^1_j \times [0,1] \to N(A^1_j)$ by putting $f^1_j = g^1_j \times \mathrm{id}_{[0,1]}$, where we identify $N(A^1_j)$ with $B^1_j \times [0,1]$. Note that M^1_j is diffeomorphic to $T^2_{(2)} \times [-1,1] \times [0,1]$.

Similarly, for each of the four strata A^2_j diffeomorphic to an open disk, $j = 1, 2, 3, 4$, by using a Morse function $S^2_{(3)} \to [-1,1]$ as in Fig. 3 (2), we can construct a smooth map $f^2_j : M^2_j \to N(A^2_j) \cong [-1,1] \times D^2$ which has only fold points as its singularities such that its fibers are as depicted in Fig. 16. Note that M^2_j is diffeomorphic to $S^2_{(3)} \times D^2$. For the other three strata A^2_j diffeomorphic to a 2-sphere, $j = 5, 6, 7$, we do not construct f^2_j for the moment.

Now let us piece together the smooth maps constructed above. First, we attach $f^0_1 : M^0_1 \to N(A^0_1)$ and $f^1_j : N^1_j \times [0,1] \to N(A^1_j)$, $j = 1, 2, 3$, by using appropriate embeddings $\varphi^1_j : N^1_j \times \{0,1\} \to \partial M^0_1$. This is possible by the classification of singular fibers of C^∞ stable maps of 3-manifolds into surfaces (see Remark 4.14

and Theorem 4.15), since f_1^0 and f_j^1 have the same singular fiber of $\kappa = 2$ on the attaching part. Note that then the natural map

$$(f_1^0 \cup f_j^1)^{-1}((N(A_1^0) \cup N(A_j^1)) \cap N(A_j^2)) \to \partial \widehat{A}_j^2 \tag{7.1}$$

is the projection of a smooth $S^2_{(3)}$-bundle over a circle, $j = 1, 2, 3$.

Note that we have a nontrivial diffeomorphism $\varphi : N_j^1 \to N_j^1$ such that $g_j^1 \circ \varphi = g_j^1$. (This corresponds to the rotation through the angle π around the center of the square representing $T^2_{(2)}$ in [36, Fig. 1].) Thus, we may assume that the $S^2_{(3)}$-bundle (7.1) is trivial by changing the embedding φ_j^1 by $\varphi_j^1 \circ \widetilde{\varphi}$ if necessary, where $\widetilde{\varphi} : N_j^1 \times \{0,1\} \to N_j^1 \times \{0,1\}$ is the identity on $N_j^1 \times \{0\}$ and is φ on $N_j^1 \times \{1\}$. Let us denote the resulting map $f_1^0 \cup f_1^1 \cup f_2^1 \cup f_3^1$ by $\widetilde{f}^1$. Then, we can check that the natural map

$$(\widetilde{f}^1)^{-1}((N(A_1^0) \cup N(A_1^1) \cup N(A_2^1) \cup N(A_3^1)) \cap N(A_4^2)) \to \partial \widehat{A}_4^2 \tag{7.2}$$

is also the projection of a trivial $S^2_{(3)}$-bundle over a circle.

Since the $S^2_{(3)}$-bundles (7.1) and (7.2) are trivial, we can now attach $f_j^2 : M_j^2 \cong S^2_{(3)} \times D^2 \to N(A_j^2)$, $j = 1, 2, 3, 4$, to $\widetilde{f}^1$. Let us denote the resulting map $\widetilde{f}^1 \cup f_1^2 \cup f_2^2 \cup f_3^2 \cup f_4^2$ by

$$\begin{aligned}\widetilde{f}^2 : \widetilde{M}^2 \to N(A_1^0) &\cup N(A_1^1) \cup N(A_2^1) \cup N(A_3^1) \\ &\cup N(A_1^2) \cup N(A_2^2) \cup N(A_3^2) \cup N(A_4^2) \subset \mathbf{R}^3.\end{aligned}$$

Note that the image X of the above map is nothing but the regular neighborhood of the Boy surface P. Let $\partial X = \partial_0 X \cup \partial_1 X$ be the connected components of ∂X, where

$$\partial_0 X = X \cap N(A_2^3) \quad \text{and} \quad \partial_1 X = X \cap N(A_3^3),$$

both of which are diffeomorphic to the 2-sphere. Note that

$$\widetilde{f}^2|_{(\widetilde{f}^2)^{-1}(\partial_0 X)} : (\widetilde{f}^2)^{-1}(\partial_0 X) \to \partial_0 X \tag{7.3}$$

is the projection of a smooth orientable S^1-bundle over a 2-sphere, and that

$$\widetilde{f}^2|_{(\widetilde{f}^2)^{-1}(\partial_1 X)} : (\widetilde{f}^2)^{-1}(\partial_1 X) \to \partial_1 X \tag{7.4}$$

is the projection of a smooth orientable $(S^1 \cup S^1)$-bundle over a 2-sphere. Note also that the latter is a disjoint union of two orientable S^1-bundles, since $\partial_1 X$ is simply connected.

Let M_5^2 be the total space of the D^2-bundle associated with the S^1-bundle (7.3), and M_6^2, M_7^2 the total spaces of the D^2-bundles associated with the two S^1-bundles (7.4). Then, by extending the maps (7.3) and (7.4), we can construct smooth maps

$$f_5^2 : M_5^2 \to N(A_5^2) \cup N(A_2^3), \tag{7.5}$$

$$f_6^2 : M_6^2 \to N(A_6^2) \cup N(A_3^3), \tag{7.6}$$

$$f_7^2 : M_7^2 \to N(A_7^2) \cup N(A_4^3) \cup N(A_6^2) \cup N(A_3^3) \tag{7.7}$$

with only definite fold points as their singularities such that their singular sets correspond to the zero sections of the D^2-bundles, $f_5^2(S_0(f_5^2)) = A_5^2$, $f_6^2(S_0(f_6^2)) = A_6^2$, and $f_7^2(S_0(f_7^2)) = A_7^2$. Then, their fibers are as depicted in Fig. 16. By our construction, we can glue (7.5), (7.6), (7.7) and $\widetilde{f}^2$ to get a smooth map

$$f : M \to \mathbf{R}^3$$

of a smooth closed 4-manifold M into $\mathbf{R}^3$.

Note that f has only fold points as its singularities and that its fibers are exactly as depicted in Fig. 16. Then by Proposition 4.1, f is a C^∞ stable map.

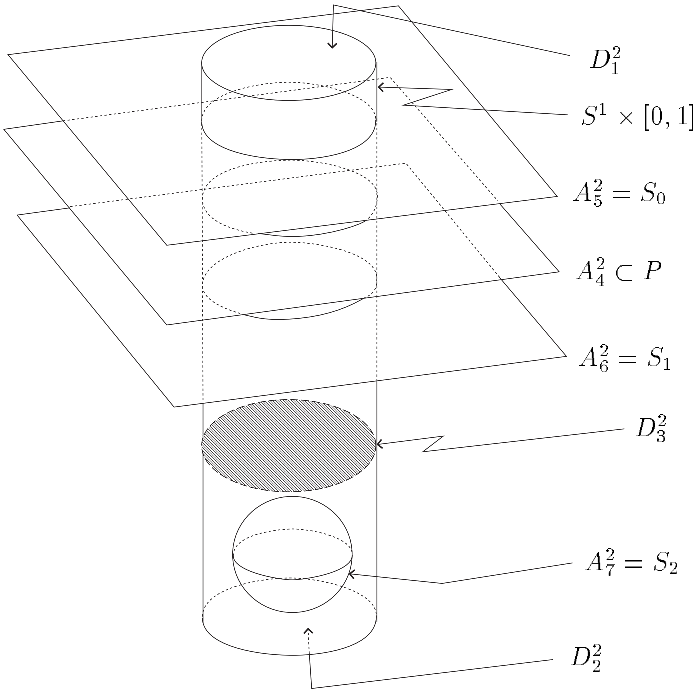

FIGURE 18. Embedded 2-sphere $Y = D_1^2 \cup (S^1 \times [0,1]) \cup D_2^2$ in $\mathbf{R}^3$

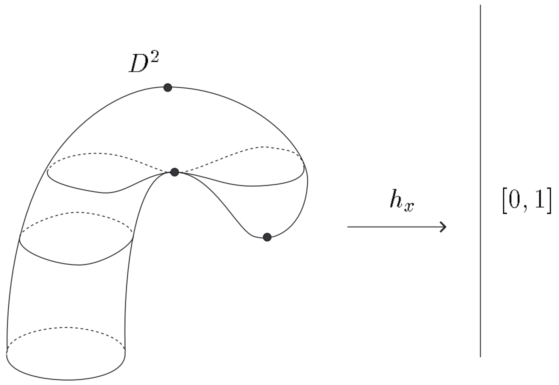

FIGURE 19. Morse function $h_x : D^2 \to [0,1]$

In order to prove that M is diffeomorphic to $\mathbf{C}P^2 \sharp 2\overline{\mathbf{C}P^2}$, let us consider a C^∞ stable map $g : M' \to \mathbf{R}^3$ constructed as follows. Let $Y = D_1^2 \cup (S^1 \times [0,1]) \cup D_2^2$ be a 2-sphere embedded in $\mathbf{R}^3$ which intersects A_5^2, A_4^2 and A_6^2 transversely as shown in Fig. 18, where D_1^2 and D_2^2 are copies of 2-disks. We take Y so that the 3-disk $\widetilde{Y}$ bounded by Y contains $A_7^2 = S_2$ in its interior. Note that the natural map

$$f^{-1}(S^1 \times [0,1]) \xrightarrow{\quad f \quad} S^1 \times [0,1] \xrightarrow{\quad \pi_1 \quad} S^1$$

is a trivial D^2-bundle, where π_1 is the projection to the first factor. Note also that the map $h_x = f|_{f^{-1}(\{x\} \times [0,1])} : D^2 \to [0,1]$ is a Morse function as described in Fig. 19 for all $x \in S^1$ and is independent of the choice of x.

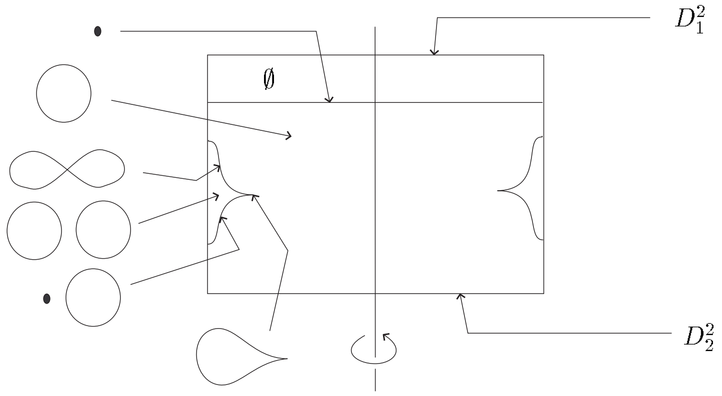

FIGURE 20. Fibers of $g_{\widetilde{Y}}$

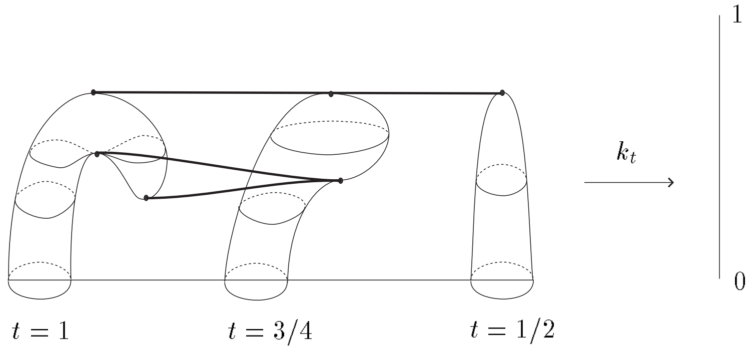

FIGURE 21. Deformation of functions on the 2-disk

Let us replace the map $f|_{f^{-1}(\widetilde{Y})}$ by the smooth map $g_{\widetilde{Y}}$ whose fibers are as described in Fig. 20 (in fact, the real figure is obtained by rotating the rectangle around the vertical line in the center).

Let us explain the reason why such a replacement is possible. We identify $\widetilde{Y}$ with $D^2 \times [0,1]$ so that $D^2 \times \{\varepsilon\}$ corresponds to $D^2_{2-\varepsilon}$ for $\varepsilon = 0, 1$. Let Δ be a small concentric 2-disk in the interior of D^2. By using a generic deformation of functions $k_t : D^2 \to [0,1]$, $t \in [1/2, 1]$, as shown in Fig. 21, we can construct the smooth map

$$g_1 : S^1 \times [1/2, 1] \times D^2 \to \overline{\widetilde{Y} \smallsetminus (\Delta \times [0,1])} \cong S^1 \times [1/2, 1] \times [0,1]$$

by putting $g_1(x, t, q) = (x, t, k_t(q))$. Note that g_1 has only fold points and cusp points as its singularities and is consistent with $f|_{f^{-1}(\mathbf{R}^3 \smallsetminus \mathrm{Int}\, \widetilde{Y})}$ along $(S^1 \times \{1\} \times D^2) \cup (S^1 \times [1/2, 1] \times \partial D^2)$.

Then, using the Morse function $k_{1/2}$, we define the smooth map $g_2 : \Delta \times D^2 \to \Delta \times [0,1]$ by $g_2(x, q) = (x, k_{1/2}(q))$. Obviously, this is consistent with $g_1|_{g_1^{-1}(\partial\Delta \times [0,1])}$ along $\partial\Delta \times D^2 = S^1 \times \{1/2\} \times D^2$, although we do not know if it is consistent with $f|_{f^{-1}(\mathbf{R}^3 \smallsetminus \mathrm{Int}\, \widetilde{Y})}$ along

$$g_2^{-1}(\Delta \times \{0\}) = \Delta \times \partial D^2 = f^{-1}(\Delta \times \{0\}). \tag{7.8}$$

However, we have a plenty of diffeomorphisms $D^2 \to D^2$ that preserve the Morse function $k_{1/2}$. For example, all the diffeomorphisms in the rotation group $SO(2)$ satisfy this property. Hence, changing the identification $g_2^{-1}(\partial\Delta \times [0,1]) \cong \partial\Delta \times D^2$

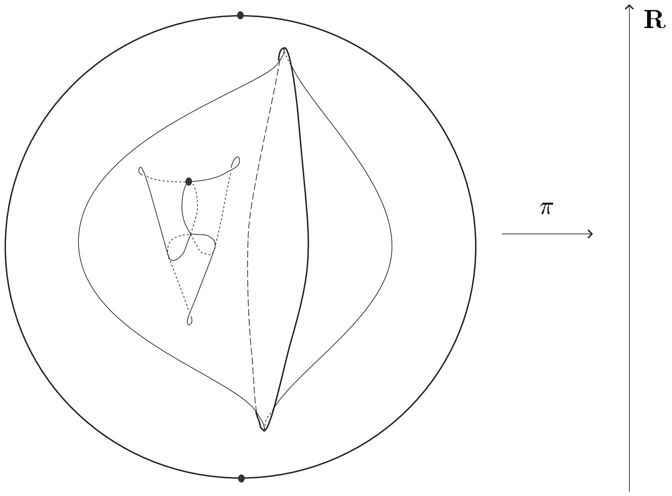

FIGURE 22. Image of $S(g)$ by g

if necessary, we can arrange so that g_2 is consistent with $f|_{f^{-1}(\mathbf{R}^3 \smallsetminus \mathrm{Int}\,\widetilde{Y})}$ along (7.8). Therefore, we obtain a C^∞ stable map $g : M' \to \mathbf{R}^3$ by gluing $f|_{f^{-1}(\mathbf{R}^3 \smallsetminus \mathrm{Int}\,\widetilde{Y})}$, g_1 and g_2, where $g_{\widetilde{Y}} = g_1 \cup g_2$ (see Fig. 20 again). Note that the singular set $S(g)$ is the union of a 2-sphere component consisting of definite fold points and a projective plane component containing the set of cusp points.

Lemma 7.1. *The smooth closed* 4*-manifold* M' *is diffeomorphic to* $\mathbf{C}P^2$ *or* $\overline{\mathbf{C}P^2}$.

Proof. Let $\pi : \mathbf{R}^3 \to \mathbf{R}$ be a projection. By locating $g(S(g))$ as in Fig. 22 by an isotopy of $\mathbf{R}^3$, we may assume that $\pi \circ g : M' \to \mathbf{R}$ is a Morse function with exactly three critical points (for such a construction of Morse functions, refer to [11] for more details). We see easily that their indices are equal to $0, 2$ and 4. Thus, M' has a handlebody decomposition $h^0 \cup h^2 \cup h^4$, where h^i denotes an i-handle. Let k be the knot in $\partial h^0 = S^3$ along which the 2-handle h^2 is attached to h^0. Since the resulting handlebody $h^0 \cup h^2$ has boundary diffeomorphic to S^3, the knot k must be trivial and the framing must be equal to ± 1 by [14, 15]. Hence, M' is diffeomorphic to $\mathbf{C}P^2$ or $\overline{\mathbf{C}P^2}$ (for details, see [20]). $\square$

Remark 7.2. In this way we have completed the construction of a C^∞ stable map $g : \mathbf{C}P^2 \to \mathbf{R}^3$ with the following properties.

(1) The map g has only fold and cusp points as its singularities.
(2) The set $C(g)$ of its cusp points constitutes a circle, and the singular set $S(g)$ is the union of a 2-sphere component consisting of definite fold points and a projective plane component which contains $C(g)$.
(3) The discriminant set $g(S(g))$ is as described in Fig. 22.
(4) The fibers of g can be completely described.

Presumably, the C^∞ stable map $g : M' = \mathbf{C}P^2 \to \mathbf{R}^3$ thus constructed coincides with Kobayashi's example presented in [21, 22].

By choosing an appropriate orientation for M', we may assume that it is orientation preservingly diffeomorphic to $\overline{\mathbf{C}P^2}$. By our construction, it is easy to see that $g^{-1}(\widetilde{Y})$ is diffeomorphic to D^4. Hence $f^{-1}(\mathbf{R}^3 \smallsetminus \mathrm{Int}\,\widetilde{Y})$ is diffeomorphic to $\overline{\mathbf{C}P^2} - \mathrm{Int}\,D^4$.

Let us determine the diffeomorphism type of $f^{-1}(\widetilde{Y})$. Take a properly embedded 2-disk D_3^2 in $\widetilde{Y}$ as in Fig. 18, and let $\widetilde{Y}_1$ and $\widetilde{Y}_2$ be the 3-disks such that $\widetilde{Y} = \widetilde{Y}_1 \cup \widetilde{Y}_2$, $\widetilde{Y_1} \cap \widetilde{Y}_2 = D_3^2$, and $\widetilde{Y}_2 \supset S_2$. Then it is easy to see that $f^{-1}(\widetilde{Y}_1)$ and $f^{-1}(\widetilde{Y}_2)$ are diffeomorphic to $D^2 \times D^2$ and to the total space E of a D^2-bundle over S^2, respectively. More precisely, $f^{-1}(\widetilde{Y})$ is obtained from E by attaching a 2-handle along the boundary of a D^2-fiber of the fibration $E \to S^2$. Hence, $f^{-1}(\widetilde{Y})$ is diffeomorphic either to $\mathbf{C}P^2\sharp\overline{\mathbf{C}P^2} \setminus \operatorname{Int} D^4$ or to $S^2 \times S^2 \setminus \operatorname{Int} D^4$.

Therefore, the source 4-manifold $M = f^{-1}(\mathbf{R}^3)$ of f is diffeomorphic either to $\overline{\mathbf{C}P^2}\sharp(\mathbf{C}P^2\sharp\overline{\mathbf{C}P^2})$ or to $\overline{\mathbf{C}P^2}\sharp(S^2 \times S^2)$. In both cases, M is diffeomorphic to $\mathbf{C}P^2\sharp 2\overline{\mathbf{C}P^2}$ (for details, see [20], for example). This completes the construction of the desired C^∞ stable map $f : \mathbf{C}P^2\sharp 2\overline{\mathbf{C}P^2} \to \mathbf{R}^3$ as promised at the beginning of this section.

It is an easy task to check that all the results obtained in §§5 and 6 are valid for the above constructed C^∞ stable maps.

Remark 7.3. The author has shown that $\mathbf{C}P^2$ does not admit a fold map into $\mathbf{R}^3$ (see [36, 39, 45, 1, 40, 34]). This implies that the normal bundle of the definite fold component of f in M corresponding to S_2 is nontrivial, for if it were trivial, then we could construct a smooth map $g'' : M' \to \mathbf{R}^3$ with only fold points as its singularities. In fact, we can show that the normal Euler numbers of the definite fold components of f in M corresponding to S_0, S_1 and S_2 are equal to $1, -2$ and -2 respectively.

Using the example constructed above, we can show the following.

Proposition 7.4. *For every $n \geq 1$, there exists a smooth map*

$$f_n : n\mathbf{C}P^2\sharp(n+1)\overline{\mathbf{C}P^2} \to \mathbf{R}^3$$

with only fold points as its singularities.

Proof. Recall that there exists a smooth map $\ell : \mathbf{C}P^2\sharp\overline{\mathbf{C}P^2} \to \mathbf{R}^3$ with only definite fold points as its singularities (for example, see [37]). Note also that such a map can be constructed explicitly. Then, we can construct the desired map f_n from $f = f_1 : \mathbf{C}P^2\sharp 2\overline{\mathbf{C}P^2} \to \mathbf{R}^3$ and $n-1$ copies of ℓ by the connected sum construction (for details, see [37]). □

Remark 7.5. Sakuma [44] had conjectured that no closed orientable 4-manifold with odd Euler characteristic can admit a fold map into $\mathbf{R}^3$. The above proposition gives explicit counter-examples to his conjecture. Note that a more precise result has been obtained in [40] about fold maps of 4-manifolds into $\mathbf{R}^3$.

8. Generalities

In this and the following sections, we formalize the idea used in §§5 and 6 in a more general setting and develop a general theory.

First, let us prepare the following notation. For a pair of nonnegative integers (n, p), we denote by $\mathcal{T}_{\mathrm{pr}}(n, p)$ (or by $\mathcal{S}^\infty_{\mathrm{pr}}(n, p)$) the set of all proper Thom maps (resp. proper C^∞ stable maps) between manifolds of dimensions n and p (for Thom maps, see Example 2.3 of §2). Furthermore, we denote by $\mathcal{S}^0_{\mathrm{pr}}(n, p)$ the set of all C^0 stable maps which are elements of $\mathcal{T}_{\mathrm{pr}}(n, p)$. Note that we have $\mathcal{S}^\infty_{\mathrm{pr}}(n, p) \subset \mathcal{T}_{\mathrm{pr}}(n, p)$. However, the author does not know if a proper C^0 stable map is a Thom map or not, so that we adopt the above convention. Note also that $\mathcal{S}^0_{\mathrm{pr}}(n, p) = \mathcal{S}^\infty_{\mathrm{pr}}(n, p)$ for nice dimension pairs (n, p) in the sense of Mather [27] by [9, 51] (see also Remark 4.2).

In the following, we call $k = p - n$ the *codimension* of a map in these sets. For a fixed k, we put

$$\begin{aligned} \widetilde{\mathcal{T}}_{\mathrm{pr}}(k) &= \bigcup_{p-n=k} \mathcal{T}_{\mathrm{pr}}(n,p), \\ \widetilde{\mathcal{S}}^{\infty}_{\mathrm{pr}}(k) &= \bigcup_{p-n=k} \mathcal{S}^{\infty}_{\mathrm{pr}}(n,p), \\ \widetilde{\mathcal{S}}^{0}_{\mathrm{pr}}(k) &= \bigcup_{p-n=k} \mathcal{S}^{0}_{\mathrm{pr}}(n,p). \end{aligned}$$

In the following, for a Thom map $f : M \to N$ in $\mathcal{T}_{\mathrm{pr}}(n,p)$, $\mathcal{M}$ and $\mathcal{N}$ will denote Whitney stratifications of M and N respectively such that f satisfies the Thom regularity condition [12, Chapter I, §3] with respect to them. For a C^0 equivalence class $\mathfrak{F}$ of fibers, we denote by $\mathfrak{F}(f)$ the set of points in N over which lies a fiber of type $\mathfrak{F}$.

Lemma 8.1. *The subspace $\mathfrak{F}(f)$ of N is a union of strata of $\mathcal{N}$ and is a C^0 submanifold of N of constant codimension if it is nonempty. Furthermore, this codimension does not depend on a particular choice of $f \in \mathcal{T}_{\mathrm{pr}}(n,p)$.*

Proof. The first assertion has already been shown in Example 2.3. In order to show the second assertion, let us take a top dimensional stratum Σ contained in $\mathfrak{F}(f)$. Note that for each point $y \in \Sigma$, there exists a neighborhood U_y of y in N such that $U_y \cap \mathfrak{F}(f) = U_y \cap \Sigma$, since Σ is top dimensional. On the other hand, by the definition of C^0 equivalence, for each point y' of $\mathfrak{F}(f)$, there exists a neighborhood $U_{y'}$ of y' in N such that $(U_{y'}, U_{y'} \cap \mathfrak{F}(f))$ is homeomorphic to $(U_y, U_y \cap \mathfrak{F}(f))$. Hence the assertion about $\mathfrak{F}(f)$ follows. Using a similar argument, we can prove the final assertion. This completes the proof. $\square$

Note that by virtue of the above lemma, the codimension of a C^0 type $\mathfrak{F}$ of fibers makes sense, and we denote it by $\kappa(\mathfrak{F})$.

Let us introduce the following notion which will play an important role throughout the rest of the paper.

Definition 8.2. Suppose that an equivalence relation $\rho = \rho_{n,p}$ among the fibers of proper Thom maps between smooth manifolds of dimensions n and p is given. We say that the relation ρ is *admissible* if the following conditions are satisfied.

(1) If two fibers are C^0 equivalent, then they are also equivalent with respect to ρ.
(2) For any two proper Thom maps $f_i : M_i \to N_i$ in $\mathcal{T}_{\mathrm{pr}}(n,p)$ and for any points $y_i \in N_i$, $i = 0, 1$, such that the fibers over y_i are equivalent to each other with respect to ρ, there exist neighborhoods U_i of y_i in N_i, $i = 0, 1$, and a homeomorphism $\varphi : U_0 \to U_1$ such that $\varphi(y_0) = y_1$ and $\varphi(U_0 \cap \widetilde{\mathfrak{F}}(f_0)) = U_1 \cap \widetilde{\mathfrak{F}}(f_1)$ for every equivalence class $\widetilde{\mathfrak{F}}$ of fibers with respect to ρ, where $\widetilde{\mathfrak{F}}(f_i)$ is the set of points in N_i over which lies a fiber of f_i of type $\widetilde{\mathfrak{F}}$.

For example, the C^0 equivalence is clearly admissible in the above sense. We denote the C^0 equivalence relation among the fibers of elements of $\mathcal{T}_{\mathrm{pr}}(n,p)$ by $\rho^0_{n,p}$.

In the following argument, we fix an admissible equivalence relation $\rho = \rho_{n,p}$ as in Definition 8.2.

Lemma 8.3. *For every equivalence class $\widetilde{\mathfrak{F}}$ with respect to an admissible equivalence relation ρ, and for every proper Thom map $f : M \to N$ in $\mathcal{T}_{\mathrm{pr}}(n,p)$, the subspace $\widetilde{\mathfrak{F}}(f)$ of N is a union of strata of $\mathcal{N}$ and is a C^0 submanifold of N of constant* *codimension if it is nonempty. Furthermore, this codimension does not depend on a particular choice of $f \in \mathcal{T}_{\mathrm{pr}}(n,p)$.*

Proof. By Definition 8.2 (1) and Lemma 8.1, $\widetilde{\mathfrak{F}}(f)$ is a union of strata. Hence, the rest of the assertion follows from an argument similar to that in the proof of Lemma 8.1 together with Definition 8.2 (2). $\square$

By virtue of the above lemma, the codimension of $\widetilde{\mathfrak{F}}$ makes sense, and we denote it by $\kappa(\widetilde{\mathfrak{F}})$.

For an equivalence class $\widetilde{\mathfrak{F}}$ of fibers with respect to ρ with $\kappa = \kappa(\widetilde{\mathfrak{F}})$, let $\partial\widetilde{\mathfrak{F}}$ be the set of equivalence classes $\widetilde{\mathfrak{G}}$ of fibers with respect to ρ of codimension $\kappa+1$ such that $\widetilde{\mathfrak{G}}(f) \subset \overline{\widetilde{\mathfrak{F}}(f)} \setminus \widetilde{\mathfrak{F}}(f)$ for every $f \in \mathcal{T}_{\mathrm{pr}}(n,p)$. For $\widetilde{\mathfrak{G}} \in \partial\widetilde{\mathfrak{F}}$, we take a proper Thom map $f \in \mathcal{T}_{\mathrm{pr}}(n,p)$ with $\widetilde{\mathfrak{G}}(f) \neq \emptyset$. Then we take a top dimensional stratum $\Sigma \subset \widetilde{\mathfrak{G}}(f)$, and let B_Σ be a small disk which intersects Σ transversely exactly at its center and whose dimension coincides with the codimension of Σ. Then $B_\Sigma \cap \overline{\widetilde{\mathfrak{F}}(f)}$ consists of a finite number of arcs which have $B_\Sigma \cap \Sigma$ as a common end point. Let $n_{\widetilde{\mathfrak{F}}}(\widetilde{\mathfrak{G}}) \in \mathbf{Z}_2$ denote the number of such arcs modulo two, which clearly does not depend on the choice of B_Σ, Σ or f by Definition 8.2 (2). Then, by considering the homological boundary of $\overline{\widetilde{\mathfrak{F}}(f)}$, we have the following.

Proposition 8.4. *For every equivalence class $\widetilde{\mathfrak{F}}$ of fibers with respect to an admissible equivalence relation ρ, and for every $f : M \to N$ in $\mathcal{T}_{\mathrm{pr}}(n,p)$, the $\mathbf{Z}_2$-chain*

$$\sum_{\widetilde{\mathfrak{G}} \in \partial\widetilde{\mathfrak{F}}} n_{\widetilde{\mathfrak{F}}}(\widetilde{\mathfrak{G}})\overline{\widetilde{\mathfrak{G}}(f)} \tag{8.1}$$

(of closed support) is a cycle in N and represents the zero homology class in the homology $H^c_{p-\kappa-1}(N;\mathbf{Z}_2)$ of closed support, where κ denotes the codimension of $\widetilde{\mathfrak{F}}$.

Proof. By the definition of $n_{\widetilde{\mathfrak{F}}}(\widetilde{\mathfrak{G}})$, we see that the $\mathbf{Z}_2$-chain (8.1) coincides the boundary of the $\mathbf{Z}_2$-chain $\overline{\widetilde{\mathfrak{F}}(f)}$ in N. Hence the result follows. $\square$

Remark 8.5. In the above proposition, if $\widetilde{\mathfrak{F}}$ does not contain the empty fiber and the source manifold M is compact, then the $\mathbf{Z}_2$-chain (8.1) has compact support and represents the zero homology class in the usual homology $H_{p-\kappa-1}(N;\mathbf{Z}_2)$.

We warn the reader that the sum appearing in the right hand side of (8.1) may contain infinitely many terms if the source manifold M of f is not compact.

Note that all the results obtained in §5 are special cases of the above proposition. Some applications of Proposition 8.4 to other specific situations will be given in §16.

9. Universal complex of singular fibers

In this section, based on the idea given in the previous section, we define a complex of singular fibers for a specific map, and then we define its universal versions for various classes of maps. We will see later that this is a generalization of Vassiliev's universal complex of multi-singularities [50]. Here we develop a rather detailed theory of such universal complexes in order to better understand what is the essential point behind our results obtained in §§5 and 6, and to obtain further related results.

9.1. **Complex of singular fibers for a specific map.** Let $f : M \to N$ be a proper smooth map of a smooth n-dimensional manifold M into a smooth p-dimensional manifold N such that f is a Thom map in the sense of Example 2.3, as in the previous section: in other words, $f \in \mathcal{T}_{\rm pr}(n,p)$.

In the following, we fix an equivalence relation $\rho = \rho_{n,p}$ for the set of fibers of such maps which is admissible in the sense of Definition 8.2. Let us construct a complex of fibers for f with coefficients in $\mathbf{Z}_2$ with respect to the admissible equivalence relation ρ as follows.

For $\kappa \geq 0$, let $C^\kappa(f,\rho)$ be the $\mathbf{Z}_2$-vector space consisting of all formal linear combinations,

$$\sum_{\kappa(\widetilde{\mathfrak{F}})=\kappa} m_{\widetilde{\mathfrak{F}}}\widetilde{\mathfrak{F}} \quad (m_{\widetilde{\mathfrak{F}}} \in \mathbf{Z}_2),$$

which may possibly contain infinitely many terms if M is noncompact, of the equivalence classes $\widetilde{\mathfrak{F}}$ of fibers of f with codimension κ with respect to the equivalence relation ρ. If there are no such fibers, then we simply put $C^\kappa(f,\rho) = 0$. Furthermore, for $\kappa < 0$, we also put $C^\kappa(f,\rho) = 0$. For two equivalence classes of fibers $\widetilde{\mathfrak{F}}$ and $\widetilde{\mathfrak{G}}$ of f with $\kappa(\widetilde{\mathfrak{F}}) = \kappa(\widetilde{\mathfrak{G}}) - 1$, we define the *incidence coefficient* $[\widetilde{\mathfrak{F}} : \widetilde{\mathfrak{G}}]_f \in \mathbf{Z}_2$ by putting $[\widetilde{\mathfrak{F}} : \widetilde{\mathfrak{G}}]_f = n_{\widetilde{\mathfrak{F}}}(\widetilde{\mathfrak{G}}) \in \mathbf{Z}_2$ if $\widetilde{\mathfrak{G}}(f) \subset \overline{\widetilde{\mathfrak{F}}(f)} \setminus \widetilde{\mathfrak{F}}(f)$, and $[\widetilde{\mathfrak{F}} : \widetilde{\mathfrak{G}}]_f = 0$ otherwise. Define the $\mathbf{Z}_2$-linear map

$$\delta_\kappa(f) : C^\kappa(f,\rho) \to C^{\kappa+1}(f,\rho)$$

by

$$\delta_\kappa(f)(\widetilde{\mathfrak{F}}) = \sum_{\kappa(\widetilde{\mathfrak{G}})=\kappa+1} [\widetilde{\mathfrak{F}} : \widetilde{\mathfrak{G}}]_f \widetilde{\mathfrak{G}}, \tag{9.1}$$

for $\widetilde{\mathfrak{F}}$ with $\kappa(\widetilde{\mathfrak{F}}) = \kappa$. We warn the reader that the sum appearing in the right hand side of (9.1) may possibly contain infinitely many terms if M is noncompact. Nevertheless, for a given equivalence class $\widetilde{\mathfrak{G}}$ of fibers of f with codimension $\kappa+1$, the number of equivalence classes $\widetilde{\mathfrak{F}}$ of fibers of f with codimension κ such that $[\widetilde{\mathfrak{F}} : \widetilde{\mathfrak{G}}]_f \neq 0$ is finite by virtue of the local finiteness of the Whitney regular stratifications and the definition of an admissible equivalence relation. Hence, the linear map $\delta_\kappa(f)$ is well-defined.

The following lemma can be proved by an argument similar to that in [50, 8.3.4 Lemma] or [33, Lemma 1.5]. The details are left to the reader.

Lemma 9.1. $\delta_{\kappa+1}(f) \circ \delta_\kappa(f) = 0$.

Therefore, $\mathcal{C}(f,\rho) = (C^\kappa(f,\rho), \delta_\kappa(f))_\kappa$ constitutes a complex and its cohomology groups $H^\kappa(f,\rho)$ are well-defined.

9.2. **Universal complex of singular fibers for Thom maps between manifolds of fixed dimensions.** The above construction can be generalized to get a "universal" complex of singular fibers for proper Thom maps between manifolds of dimensions n and p as follows.

Let ρ be an admissible equivalence relation as in Definition 8.2 for the fibers of elements of $\mathcal{T}_{\rm pr}(n,p)$. For $\kappa \in \mathbf{Z}$, let $C^\kappa(\mathcal{T}_{\rm pr}(n,p),\rho)$ be the $\mathbf{Z}_2$-vector space consisting of all formal linear combinations,

$$\sum_{\kappa(\widetilde{\mathfrak{F}})=\kappa} m_{\widetilde{\mathfrak{F}}}\widetilde{\mathfrak{F}} \quad (m_{\widetilde{\mathfrak{F}}} \in \mathbf{Z}_2),$$

which may possibly contain infinitely many terms, of the equivalence classes $\widetilde{\mathfrak{F}}$ of fibers of proper Thom maps between manifolds of dimensions n and p with $\kappa(\widetilde{\mathfrak{F}}) = \kappa$ with respect to the equivalence relation $\rho = \rho_{n,p}$. If there is no such equivalence class (for example, if $\kappa > p$ or $\kappa < 0$), then we simply put $C^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p),\rho) = 0$. For two equivalence classes $\widetilde{\mathfrak{F}}$ and $\widetilde{\mathfrak{G}}$ of fibers of elements of $\mathcal{T}_{\mathrm{pr}}(n,p)$ with $\kappa(\widetilde{\mathfrak{F}}) = \kappa(\widetilde{\mathfrak{G}}) - 1$, we define the *incidence coefficient* $[\widetilde{\mathfrak{F}} : \widetilde{\mathfrak{G}}] \in \mathbf{Z}_2$ by putting $[\widetilde{\mathfrak{F}} : \widetilde{\mathfrak{G}}] = n_{\widetilde{\mathfrak{F}}}(\widetilde{\mathfrak{G}}) \in \mathbf{Z}_2$ if $\widetilde{\mathfrak{G}}(f) \subset \overline{\widetilde{\mathfrak{F}}(f)} \setminus \widetilde{\mathfrak{F}}(f)$ for every $f \in \mathcal{T}_{\mathrm{pr}}(n,p)$, and $[\widetilde{\mathfrak{F}} : \widetilde{\mathfrak{G}}] = 0$ otherwise. Then the $\mathbf{Z}_2$-linear map $\delta_\kappa : C^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p),\rho) \to C^{\kappa+1}(\mathcal{T}_{\mathrm{pr}}(n,p),\rho)$ is defined by

$$\delta_\kappa(\widetilde{\mathfrak{F}}) = \sum_{\kappa(\widetilde{\mathfrak{G}})=\kappa+1} [\widetilde{\mathfrak{F}} : \widetilde{\mathfrak{G}}]\widetilde{\mathfrak{G}}, \tag{9.2}$$

for $\widetilde{\mathfrak{F}}$ with $\kappa(\widetilde{\mathfrak{F}}) = \kappa$. (See (9.1) and the subsequent remark). Note that the incidence coefficient, and hence the map δ_κ, is well-defined by virtue of Definition 8.2 (2). Furthermore, we can prove that $\delta_{\kappa+1} \circ \delta_\kappa = 0$ as in Lemma 9.1. We call the resulting complex $\mathcal{C}(\mathcal{T}_{\mathrm{pr}}(n,p),\rho) = (C^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p),\rho), \delta_\kappa)_\kappa$ the *universal complex of singular fibers for proper Thom maps between manifolds of dimensions n and p with respect to the admissible equivalence relation $\rho = \rho_{n,p}$*, and we denote its cohomology group of dimension κ by $H^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p),\rho)$.

For $f \in \mathcal{T}_{\mathrm{pr}}(n,p)$, let $C^\kappa(f^c,\rho)$ be the linear subspace of $C^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p),\rho)$ spanned by all the equivalence classes $\widetilde{\mathfrak{F}}$ of fibers of elements of $\mathcal{T}_{\mathrm{pr}}(n,p)$ of codimension κ with respect to ρ such that no fiber of f belongs to $\widetilde{\mathfrak{F}}$.

Lemma 9.2. *For $f \in \mathcal{T}_{\mathrm{pr}}(n,p)$, the following holds.*

1. *We have $\delta_\kappa(C^\kappa(f^c,\rho)) \subset C^{\kappa+1}(f^c,\rho)$ for every $\kappa \in \mathbf{Z}$. Hence, $\mathcal{C}(f^c,\rho) = (C^\kappa(f^c,\rho), \delta_\kappa|_{C^\kappa(f^c,\rho)})_\kappa$ constitutes a subcomplex of $\mathcal{C}(\mathcal{T}_{\mathrm{pr}}(n,p),\rho)$.*
2. *The quotient complex*
$$\mathcal{C}(\mathcal{T}_{\mathrm{pr}}(n,p),\rho)/\mathcal{C}(f^c,\rho) = (C^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p),\rho)/C^\kappa(f^c,\rho), \overline{\delta}_\kappa)_\kappa$$
is naturally isomorphic to $\mathcal{C}(f,\rho)$, where
$$\overline{\delta}_\kappa : C^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p),\rho)/C^\kappa(f^c,\rho) \to C^{\kappa+1}(\mathcal{T}_{\mathrm{pr}}(n,p),\rho)/C^{\kappa+1}(f^c,\rho)$$
is the well-defined $\mathbf{Z}_2$-linear map induced by δ_κ.

Proof. Let $\widetilde{\mathfrak{F}} \in C^\kappa(f^c,\rho)$ be an equivalence class of fibers of codimension κ which contains no fiber of f. For an equivalence class $\widetilde{\mathfrak{G}} \in C^{\kappa+1}(\mathcal{T}_{\mathrm{pr}}(n,p),\rho)$ of fibers of codimension $\kappa+1$, if $[\widetilde{\mathfrak{F}} : \widetilde{\mathfrak{G}}] \neq 0$, then $\widetilde{\mathfrak{G}}(f) \subset \overline{\widetilde{\mathfrak{F}}(f)} \setminus \widetilde{\mathfrak{F}}(f)$. Since $\widetilde{\mathfrak{F}}$ does not contain any fiber of f, we have $\widetilde{\mathfrak{F}}(f) = \emptyset$, and hence $\widetilde{\mathfrak{G}}(f) = \emptyset$. Thus, we have $\widetilde{\mathfrak{G}} \in C^{\kappa+1}(f^c,\rho)$ and item (1) follows.

Let $\pi_\kappa : C^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p),\rho) \to C^\kappa(f,\rho)$ be the natural projection: i.e., π_κ is the linear map defined by

$$\pi_\kappa(\widetilde{\mathfrak{F}}) = \begin{cases} \widetilde{\mathfrak{F}}, & \text{if } \widetilde{\mathfrak{F}} \in C^\kappa(f,\rho), \\ 0, & \text{otherwise}, \end{cases}$$

for an equivalence class $\widetilde{\mathfrak{F}} \in C^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p),\rho)$ of fibers. Then, it is easy to see that the system of $\mathbf{Z}_2$-linear maps $\{\pi_\kappa\}_\kappa$ defines a surjective cochain map and the kernel of π_κ coincides with $C^\kappa(f^c,\rho)$. Hence, item (2) follows. This completes the proof. □

In view of the above lemma, the complex $\mathcal{C}(\mathcal{T}_{\mathrm{pr}}(n,p),\rho)$ is universal in the sense that the complex $\mathcal{C}(f,\rho)$ for a specific Thom map f is obtained as a quotient complex.

Remark 9.3. We will see in §10 that the universal complex of singular fibers with respect to the C^0 equivalence as defined above corresponds to increasing the generators of each cochain group of Vassiliev's universal complex of multi-singularities

[50] according to the topological structures of the fibers (see Definition 10.9 and Remark 10.10).

9.3. **Universal complex of singular fibers for Thom maps with fixed codimension.** As we have noticed in Remark 4.14, a singular fiber of a codimension k map into a p-dimensional manifold can naturally be identified with a singular fiber of a codimension k map into a $(p+1)$-dimensional manifold. This is formalized as follows.

Definition 9.4. Let $f : M \to N$ be a proper Thom map between manifolds of dimensions n and p with $k = p - n$. For a positive integer ℓ, we call the map

$$f \times \mathrm{id}_{\mathbf{R}^\ell} : M \times \mathbf{R}^\ell \to N \times \mathbf{R}^\ell$$

the *ℓ-th suspension* of f. (When $\ell = 1$, we sometimes call it the *suspension* of f and denote it by Σf.) Furthermore, to the fiber of f over a point $y \in N$, we can associate the fiber of $f \times \mathrm{id}_{\mathbf{R}^\ell}$ over $y \times \{0\}$. We say that the latter fiber is obtained from the original fiber by the *ℓ-th suspension*. Note that the ℓ-th suspension of a proper Thom map is again a proper Thom map.

By considering the suspension as above, we can define a cochain map

$$\mathcal{C}(\mathcal{T}_{\mathrm{pr}}(n+\ell, p+\ell), \rho_{n+\ell,p+\ell}) \to \mathcal{C}(\mathcal{T}_{\mathrm{pr}}(n,p), \rho_{n,p})$$

as long as the equivalence relations for the dimension pairs are consistent with each other in a certain sense, which is specified as follows.

Definition 9.5. Let us fix an integer k. Suppose that for each dimension pair (n,p) with $p - n = k$ and $\min(n,p) \geq 0$, we are given an admissible equivalence relation $\rho_{n,p}$ for the fibers of proper Thom maps between manifolds of dimensions n and p. Such a system of equivalence relations

$$\mathcal{R}_k = \{\rho_{n,p} \,:\, p - n = k, \, \min(n,p) \geq 0\},$$

which is often written simply as $\{\rho_{n,p}\}_{p-n=k}$ or $\{\rho_{p-k,p}\}_p$, is said to be *stable* if the following condition is satisfied: if two fibers of proper Thom maps between manifolds of dimensions n and p are equivalent with respect to $\rho_{n,p}$, then their ℓ-th suspensions are also equivalent with respect to $\rho_{n+\ell,p+\ell}$ for all $\ell > 0$. Note that the ℓ-th suspensions are fibers of proper Thom maps between manifolds of dimensions $n + \ell$ and $p + \ell$.

For example, the set of C^0 equivalence relations $\{\rho^0_{p-k,p}\}_p$ gives a stable system of admissible equivalence relations for the fibers of proper Thom maps of codimension k, and we denote it by $\mathcal{R}^0_k$.

Suppose that a stable system of admissible equivalence relations $\mathcal{R}_k$ as in Definition 9.5 is given for the fibers of proper Thom maps of codimension k. Then, for every pair (n,p) with $p - n = k$ and a positive integer ℓ, the suspension induces a natural map

$$s_\kappa : C^\kappa(\mathcal{T}_{\mathrm{pr}}(n+\ell, p+\ell), \rho_{n+\ell,p+\ell}) \to C^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p), \rho_{n,p}) \tag{9.3}$$

for $\kappa \in \mathbf{Z}$. More precisely, when $0 \leq \kappa \leq p$, for an equivalence class $\widetilde{\mathfrak{F}} \in C^\kappa(\mathcal{T}_{\mathrm{pr}}(n+\ell, p+\ell), \rho_{n+\ell,p+\ell})$ of fibers with respect to $\rho_{n+\ell,p+\ell}$, we define $s_\kappa(\widetilde{\mathfrak{F}}) \in C^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p), \rho_{n,p})$ to be the (possibly infinite) sum of all those equivalence classes of fibers of codimension κ with respect to $\rho_{n,p}$ whose ℓ-th suspensions are contained in $\widetilde{\mathfrak{F}}$. For $\kappa > p$ or $\kappa < 0$, we simply put $s_\kappa = 0$. Note that s_κ is a well-defined $\mathbf{Z}_2$-linear map by virtue of Definition 9.5.

Lemma 9.6. *The $\mathbf{Z}_2$-linear map s_κ of (9.3) is a monomorphism for every $\kappa \leq p$.*

Proof. For $\kappa < 0$, the assertion is clear. Suppose $0 \leq \kappa \leq p$. For an equivalence class $\widetilde{\mathfrak{F}} \in C^{\kappa}(\mathcal{T}_{\mathrm{pr}}(n+\ell, p+\ell), \rho_{n+\ell,p+\ell})$ of fibers, there exists a proper Thom map $f : M \to N$ between manifolds of dimensions $n+\ell$ and $p+\ell$ such that its fiber over a point $y \in N$ is a representative of $\widetilde{\mathfrak{F}}$. By the proof of Lemma 8.3, we may assume that the stratum containing y is of codimension κ. Let B be a small open disk of dimension p embedded in N centered at y which is transverse to all the strata. Then $f|_{f^{-1}(B)} : f^{-1}(B) \to B$ is a proper Thom map and the ℓ-th suspension of its fiber over y is C^0 equivalent to the fiber of f over y by Thom's second isotopy lemma. Moreover, the codimension of the equivalence class containing the fiber of $f|_{f^{-1}(B)}$ over y is equal to κ. Hence, $s_{\kappa}(\widetilde{\mathfrak{F}})$ never vanishes. Since $\{\rho_{p-k,p}\}_p$ is stable, this shows that s_{κ} is a monomorphism. $\square$

Remark 9.7. We warn the reader that the equivalence class with respect to $\rho_{n+\ell,p+\ell}$ of the ℓ-th suspension of a fiber whose equivalence class with respect to $\rho_{n,p}$ is of codimension κ may not be of codimension κ. The codimension can decrease by suspension.

Remark 9.8. We see easily that for a κ with $0 \leq \kappa \leq p$, the $\mathbf{Z}_2$-linear map s_{κ} of (9.3) is an isomorphism if and only if the following two hold.

(1) If an equivalence class of fibers with respect to $\rho_{n,p}$ has codimension κ, then the equivalence class of their ℓ-th suspensions with respect to $\rho_{n+\ell,p+\ell}$ has also codimension κ.
(2) Two fibers whose equivalence classes with respect to $\rho_{n,p}$ have codimension κ are equivalent with respect to $\rho_{n,p}$ if and only if their ℓ-th suspensions are equivalent with respect to $\rho_{n+\ell,p+\ell}$.

In particular, the $\mathbf{Z}_2$-linear maps s_{κ} are isomorphisms for all κ with $0 \leq \kappa \leq p$ if and only if the following holds: two fibers are equivalent with respect to $\rho_{n,p}$ if and only if their ℓ-th suspensions are equivalent with respect to $\rho_{n+\ell,p+\ell}$.

By virtue of Definition 9.5, we can prove the following.

Lemma 9.9. *The system of $\mathbf{Z}_2$-linear maps $\{s_{\kappa}\}_{\kappa}$ defines a cochain map*

$$\mathcal{C}(\mathcal{T}_{\mathrm{pr}}(n+\ell, p+\ell), \rho_{n+\ell,p+\ell}) \to \mathcal{C}(\mathcal{T}_{\mathrm{pr}}(n,p), \rho_{n,p}).$$

In other words, we have $\delta_{\kappa} \circ s_{\kappa} = s_{\kappa+1} \circ \delta_{\kappa}$ for all $\kappa \in \mathbf{Z}$.

Proof. We may assume that $0 \leq \kappa \leq p-1$. Let $\widetilde{\mathfrak{F}}$ be an equivalence class of fibers in $C^{\kappa}(\mathcal{T}_{\mathrm{pr}}(n+\ell, p+\ell), \rho_{n+\ell,p+\ell})$, and $\widetilde{\mathfrak{G}}$ an equivalence class in $C^{\kappa+1}(\mathcal{T}_{\mathrm{pr}}(n,p), \rho_{n,p})$. Let us consider the coefficients of $\widetilde{\mathfrak{G}}$ in $\delta_{\kappa} \circ s_{\kappa}(\widetilde{\mathfrak{F}})$ and in $s_{\kappa+1} \circ \delta_{\kappa}(\widetilde{\mathfrak{F}})$.

Case 1. The equivalence class of the ℓ-th suspension of $\widetilde{\mathfrak{G}}$ has codimension strictly smaller than $\kappa+1$.

The relevant coefficient in $s_{\kappa+1} \circ \delta_{\kappa}(\widetilde{\mathfrak{F}})$ is clearly zero by the definition of $s_{\kappa+1}$. On the other hand, if the relevant coefficient in $\delta_{\kappa} \circ s_{\kappa}(\widetilde{\mathfrak{F}})$ is not zero, then there is a codimension κ equivalence class $\widetilde{\mathfrak{H}}$ whose coboundary contains $\widetilde{\mathfrak{G}}$ and whose ℓ-th suspension is contained in $\widetilde{\mathfrak{F}}$. By our assumption, the ℓ-th suspension of $\widetilde{\mathfrak{G}}$ has codimension strictly smaller than $\kappa+1$, and hence either the ℓ-th suspension of $\widetilde{\mathfrak{H}}$ has codimension strictly smaller than κ, or the ℓ-th suspension of $\widetilde{\mathfrak{G}}$ is equivalent to the ℓ-th suspension of $\widetilde{\mathfrak{H}}$.

The first case does not occur, since the ℓ-th suspension of $\widetilde{\mathfrak{H}}$ is contained in $\widetilde{\mathfrak{F}}$, which is of codimension κ.

If the second case occurs, then the equivalence class of the ℓ-th suspension of $\widetilde{\mathfrak{G}}$ has codimension κ. Since by Lemma 8.3, the equivalence class determines a topological submanifold of codimension κ, there must be a unique codimension κ

equivalence class $\widetilde{\mathfrak{H}}'$ $(\neq \widetilde{\mathfrak{H}})$ whose coboundary contains $\widetilde{\mathfrak{G}}$ and whose ℓ-th suspension is contained in $\widetilde{\mathfrak{F}}$. Hence, we see that the coefficient of $\widetilde{\mathfrak{G}}$ in $\delta_\kappa \circ s_\kappa(\widetilde{\mathfrak{F}})$ is equal to zero.

Hence, the relevant coefficients coincide with each other in this case.

Case 2. The equivalence class of the ℓ-th suspension of $\widetilde{\mathfrak{G}}$ has codimension $\kappa+1$.

The coefficient of $\widetilde{\mathfrak{G}}$ in $\delta_\kappa \circ s_\kappa(\widetilde{\mathfrak{F}})$ is equal to the number of codimension κ equivalence classes whose coboundaries contain $\widetilde{\mathfrak{G}}$ and whose ℓ-th suspensions are contained in $\widetilde{\mathfrak{F}}$. On the other hand, the coefficient of $\widetilde{\mathfrak{G}}$ in $s_{\kappa+1} \circ \delta_\kappa(\widetilde{\mathfrak{F}})$ is not zero if and only if the ℓ-th suspension of $\widetilde{\mathfrak{G}}$ is contained in the coboundary of $\widetilde{\mathfrak{F}}$. Hence, the relevant coefficients coincide with each other in this case as well. This completes the proof. $\square$

It follows easily from the definition of s_κ that the composition of

$$s_\kappa : C^\kappa(\mathcal{T}_{\mathrm{pr}}(n+\ell+\ell', p+\ell+\ell'), \rho_{n+\ell+\ell', p+\ell+\ell'}) \to C^\kappa(\mathcal{T}_{\mathrm{pr}}(n+\ell, p+\ell), \rho_{n+\ell, p+\ell})$$

and

$$s_\kappa : C^\kappa(\mathcal{T}_{\mathrm{pr}}(n+\ell, p+\ell), \rho_{n+\ell, p+\ell}) \to C^\kappa(\mathcal{T}_{\mathrm{pr}}(n, p), \rho_{n,p})$$

coincides with

$$s_\kappa : C^\kappa(\mathcal{T}_{\mathrm{pr}}(n+\ell+\ell', p+\ell+\ell'), \rho_{n+\ell+\ell', p+\ell+\ell'}) \to C^\kappa(\mathcal{T}_{\mathrm{pr}}(n, p), \rho_{n,p}).$$

By this observation together with Lemma 9.9, for a fixed integer k, the projective limit

$$\mathcal{C}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}_k) = \lim_{\overleftarrow{p}} \mathcal{C}(\mathcal{T}_{\mathrm{pr}}(p-k, p), \rho_{p-k,p}) \tag{9.4}$$

is well-defined as a cochain complex. We call $\mathcal{C}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}_k)$ the *universal complex of singular fibers for codimension k proper Thom maps with respect to the stable system of admissible equivalence relations $\mathcal{R}_k$*. We write its cohomology group of dimension κ by $H^\kappa(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}_k)$.

Remark 9.10. Recall that the projective limit

$$C^\kappa(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}_k) = \lim_{\overleftarrow{p}} C^\kappa(\mathcal{T}_{\mathrm{pr}}(p-k, p), \rho_{p-k,p})$$

is identified with the subspace of the product

$$\Pi_p C^\kappa(\mathcal{T}_{\mathrm{pr}}(p-k, p), \rho_{p-k,p})$$

consisting of all elements $(c_p)_p$ with $s_\kappa(c_{p+\ell}) = c_p$ for all p and ℓ.

As a direct consequence of Lemmas 9.6 and 9.9, we have the following.

Lemma 9.11. *The natural map*

$$\Phi^\kappa_{n,p} : C^\kappa(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}_k) \to C^\kappa(\mathcal{T}_{\mathrm{pr}}(n, p), \rho_{n,p}) \tag{9.5}$$

induced by the projection is a monomorphism if $\kappa \leq p$. Furthermore, the system of $\mathbf{Z}_2$-linear maps $\{\Phi^\kappa_{n,p}\}_\kappa$ defines a cochain map

$$\mathcal{C}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}_k) \to \mathcal{C}(\mathcal{T}_{\mathrm{pr}}(n, p), \rho_{n,p}).$$

The $\mathbf{Z}_2$-linear map $\Phi^\kappa_{n,p}$ defined above can be identified with the map (9.6) which will be defined in §9.4.

9.4. **Another description of the universal complex of singular fibers for Thom maps with fixed codimension.** The complex $\mathcal{C}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}_k)$ can also be constructed by using another method, as explained below.

Definition 9.12. Let $f_i : M_i \to N_i$, $i = 0, 1$, be proper Thom maps with the same codimension $k = \dim N_i - \dim M_i$. We say that the fibers over $y_i \in N_i$, $i = 0, 1$, are *stably* C^0 (or C^∞) *equivalent* if the fibers of $f_i \times \mathrm{id}_{\mathbf{R}^{\ell_i}} : M_i \times \mathbf{R}^{\ell_i} \to N_i \times \mathbf{R}^{\ell_i}$ over $y_i \times \{0\}$ are C^0 (resp. C^∞) equivalent to each other for some nonnegative integers ℓ_i, $i = 0, 1$, with $\dim N_0 + \ell_0 = \dim N_1 + \ell_1$.

Definition 9.13. Suppose that an equivalence relation $\widehat{\mathcal{R}}_k$ among the fibers of proper Thom maps of codimension k is given. We say that the relation $\widehat{\mathcal{R}}_k$ is *stably admissible* if the following conditions are satisfied.

1. (1) If two fibers are stably C^0 equivalent, then they are also equivalent with respect to $\widehat{\mathcal{R}}_k$.
2. (2) For every positive integer ℓ, two fibers are equivalent with respect to $\widehat{\mathcal{R}}_k$ if and only if their ℓ-th suspensions are equivalent with respect to $\widehat{\mathcal{R}}_k$.
3. (3) For any proper Thom maps $f_i : M_i \to N_i$, $i = 0, 1$, of codimension k and for any points $y_i \in N_i$ whose corresponding fibers are equivalent with respect to $\widehat{\mathcal{R}}_k$, there exist neighborhoods U_i of $y_i \times \{0\}$ in $N_i \times \mathbf{R}^{\ell_i}$ for some nonnegative integers ℓ_i, $i = 0, 1$, with $\dim N_0 + \ell_0 = \dim N_1 + \ell_1$, and a homeomorphism $\varphi : U_0 \to U_1$ such that $\varphi(y_0 \times \{0\}) = y_1 \times \{0\}$ and $\varphi(U_0 \cap \widehat{\mathfrak{F}}(f_0 \times \mathrm{id}_{\mathbf{R}^{\ell_0}})) = U_1 \cap \widehat{\mathfrak{F}}(f_1 \times \mathrm{id}_{\mathbf{R}^{\ell_1}})$ for every equivalence class $\widehat{\mathfrak{F}}$ of fibers with respect to $\widehat{\mathcal{R}}_k$, where $\widehat{\mathfrak{F}}(f_i \times \mathrm{id}_{\mathbf{R}^{\ell_i}})$ is the set of points in $N_i \times \mathbf{R}^{\ell_i}$ over which lies a fiber of $f_i \times \mathrm{id}_{\mathbf{R}^{\ell_i}}$ of type $\widehat{\mathfrak{F}}$.

For example, the stable C^0 equivalence is a stably admissible equivalence relation, and we denote it by $\widehat{\mathcal{R}}^0_k$.

The following lemma can be proved by an argument similar to that in the proof of Lemma 8.3.

Lemma 9.14. *For every equivalence class $\widehat{\mathfrak{F}}$ with respect to a stably admissible equivalence relation $\widehat{\mathcal{R}}_k$, and for every proper Thom map $f : M \to N$ in $\widetilde{\mathcal{T}}_{\mathrm{pr}}(k)$, the subspace $\widehat{\mathfrak{F}}(f)$ of N is a union of strata of $\mathcal{N}$. Furthermore, we have the following.*

1. (1) *For every $y \in \widehat{\mathfrak{F}}(f)$, there exists a nonnegative integer ℓ such that $\widehat{\mathfrak{F}}(f) \times \mathbf{R}^\ell$ is a C^0 submanifold of $N \times \mathbf{R}^\ell$ at $y \times \{0\}$.*
2. (2) *The codimension of $\widehat{\mathfrak{F}}(f) \times \mathbf{R}^\ell$ in $N \times \mathbf{R}^\ell$ at $y \times \{0\}$ does not depend on the choice of y or f.*

By virtue of the above lemma, the codimension of $\widehat{\mathfrak{F}}$ makes sense, and we denote it by $\kappa(\widehat{\mathfrak{F}})$.

Let $\widehat{\mathcal{R}}_k$ be a stably admissible equivalence relation among the fibers of proper Thom maps of codimension k. Then, we can naturally construct the cochain complex $\mathcal{C}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \widehat{\mathcal{R}}_k) = (C^\kappa(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \widehat{\mathcal{R}}_k), \delta_\kappa)_\kappa$ as follows: $C^\kappa(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \widehat{\mathcal{R}}_k)$ is the $\mathbf{Z}_2$-vector space consisting of all formal linear combinations, which may possibly contain infinitely many terms, of the equivalence classes $\widehat{\mathfrak{F}}$ of fibers of proper Thom maps of codimension k with $\kappa(\widehat{\mathfrak{F}}) = \kappa$ and

$$\delta_\kappa : C^\kappa(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \widehat{\mathcal{R}}_k) \to C^{\kappa+1}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \widehat{\mathcal{R}}_k)$$

is defined in a way similar to $\delta_\kappa(f)$ (see (9.1) and the subsequent remark). (Here, we simply put $C^\kappa(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \widehat{\mathcal{R}}_k) = 0$ for $\kappa < 0$.) Note that the incidence coefficient is well-defined by virtue of Definition 9.13 (2) and (3). We write the cohomology group of dimension κ of the cochain complex $\mathcal{C}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \widehat{\mathcal{R}}_k)$ by $H^\kappa(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \widehat{\mathcal{R}}_k)$.

Let us now discuss the relationship between the complex thus obtained and that of §9.3. Suppose that a stable system $\mathcal{R}_k = \{\rho_{p-k,p}\}_p$ of admissible equivalence relations for the fibers of proper Thom maps of codimension k in the sense of Definition 9.5 is given. Then, we can naturally define a new equivalence relation $\widehat{\mathcal{R}}_k$ for the fibers of proper Thom maps of codimension k as follows: two such fibers are equivalent if some of their suspensions are equivalent in the original sense. Then we can easily check that this new equivalence relation $\widehat{\mathcal{R}}_k$ is stably admissible in the sense of Definition 9.13. For example, if we consider the system of C^0 equivalence relations $\mathcal{R}^0_k = \{\rho^0_{p-k,p}\}_p$, then it defines a stable system of admissible equivalence relations, and the new equivalence relation is nothing but the stable C^0 equivalence $\widehat{\mathcal{R}}^0_k$.

Then, we get the following.

Proposition 9.15. *The complex $\mathcal{C}(\widetilde{\mathcal{T}}_{\rm pr}(k), \widehat{\mathcal{R}}_k)$ with respect to the new equivalence relation $\widehat{\mathcal{R}}_k$ is naturally isomorphic to the universal complex $\mathcal{C}(\widetilde{\mathcal{T}}_{\rm pr}(k), \mathcal{R}_k)$, defined by* (9.4)*, of singular fibers for codimension k proper Thom maps with respect to the original stable system of admissible equivalence relations $\mathcal{R}_k = \{\rho_{p-k,p}\}_p$.*

Proof. For every pair (n, p) with $p-n = k$ and for every κ, we can naturally define the $\mathbf{Z}_2$-linear map

$$\Phi^{\kappa}_{n,p} : C^{\kappa}(\widetilde{\mathcal{T}}_{\rm pr}(k), \widehat{\mathcal{R}}_k) \to C^{\kappa}(\mathcal{T}_{\rm pr}(n,p), \rho_{n,p}) \tag{9.6}$$

by associating to each equivalence class $\widehat{\mathfrak{F}}$ of codimension κ with respect to $\widehat{\mathcal{R}}_k$ the sum of all those equivalence classes of codimension κ with respect to $\rho_{n,p}$ which are contained in $\widehat{\mathfrak{F}}$. It is not difficult to show that $\Phi_{n,p} = \{\Phi^{\kappa}_{n,p}\}_{\kappa}$ defines a cochain map

$$\mathcal{C}(\widetilde{\mathcal{T}}_{\rm pr}(k), \widehat{\mathcal{R}}_k) \to \mathcal{C}(\mathcal{T}_{\rm pr}(n,p), \rho_{n,p})$$

(see the proof of Lemma 9.9) and that $s_{\kappa} \circ \Phi^{\kappa}_{n+\ell,p+\ell} = \Phi^{\kappa}_{n,p}$ for every positive integer ℓ, where s_{κ} is the $\mathbf{Z}_2$-linear map (9.3) induced by the suspension. Hence, $\{\Phi_{p-k,p}\}_p$ induces a cochain map

$$\Phi : \mathcal{C}(\widetilde{\mathcal{T}}_{\rm pr}(k), \widehat{\mathcal{R}}_k) \to \varprojlim_{p} \mathcal{C}(\mathcal{T}_{\rm pr}(p-k,p), \rho_{p-k,p}) = \mathcal{C}(\widetilde{\mathcal{T}}_{\rm pr}(k), \mathcal{R}_k)$$

by the universality of the projective limit. Furthermore, it is not difficult to show that Φ is injective. Finally, Φ is surjective by virtue of the definitions of $\widehat{\mathcal{R}}_k$ and the projective limit. Hence, we have the desired conclusion. This completes the proof. $\square$

Conversely, suppose that a stably admissible equivalence relation $\widehat{\mathcal{R}}_k$ among the fibers of proper Thom maps of codimension k is given. Then, for every pair (n, p) with $p - n = k$, we can define the equivalence relation $\rho_{n,p}$ among the fibers of elements of $\mathcal{T}_{\rm pr}(n,p)$ as follows: two such fibers are equivalent with respect to $\rho_{n,p}$ if they are equivalent with respect to $\widehat{\mathcal{R}}_k$ and in Definition 9.13 (3), ℓ_i can be chosen to be zero, i.e., if there exist neighborhoods U_i of y_i in N_i, $i = 0, 1$, and a homeomorphism $\varphi : U_0 \to U_1$ such that $\varphi(y_0) = y_1$ and $\varphi(U_0 \cap \widehat{\mathfrak{F}}(f_0)) = U_1 \cap \widehat{\mathfrak{F}}(f_1)$ for every equivalence class $\widehat{\mathfrak{F}}$ of fibers with respect to $\widehat{\mathcal{R}}_k$, where $f_i : M_i \to N_i$ are elements of $\mathcal{T}_{\rm pr}(n,p)$ whose fibers over $y_i \in N_i$ are the given ones, and $\widehat{\mathfrak{F}}(f_i)$ is the set of points in N_i over which lies a fiber of f_i of type $\widehat{\mathfrak{F}}$.

Lemma 9.16. (1) *The relation $\rho_{n,p}$ defined as above is an admissible equivalence relation in the sense of Definition* 8.2.

(2) *The system of equivalence relations $\mathcal{R}_k = \{\rho_{p-k,p}\}_p$ is stable in the sense of Definition* 9.5.

Proof. (1) We can show that the C^0 equivalence implies the equivalence with respect to $\rho_{n,p}$, since C^0 equivalence implies the equivalence with respect to $\widehat{\mathcal{R}}_k$.

Suppose that $f_i : M_i \to N_i$, $i = 0, 1$, are elements of $\mathcal{T}_{\mathrm{pr}}(n, p)$ whose fibers over $y_i \in N_i$ are equivalent to each other with respect to $\rho_{n,p}$. Then, there exist neighborhoods U_i of y_i in N_i, $i = 0, 1$, and a homeomorphism $\varphi : U_0 \to U_1$ such that $\varphi(y_0) = y_1$ and $\varphi(U_0 \cap \widehat{\mathfrak{F}}(f_0)) = U_1 \cap \widehat{\mathfrak{F}}(f_1)$ for every equivalence class $\widehat{\mathfrak{F}}$ of fibers with respect to $\widehat{\mathcal{R}}_k$. Then, the fiber of f_0 over an arbitrary point $y \in U_0$ is equivalent, with respect to $\rho_{n,p}$, to that of f_1 over $\varphi(y)$, by the very definition of the equivalence relation $\rho_{n,p}$. Hence, we have $\varphi(U_0 \cap \widetilde{\mathfrak{F}}(f_0)) = U_1 \cap \widetilde{\mathfrak{F}}(f_1)$ for every equivalence class $\widetilde{\mathfrak{F}}$ with respect to $\rho_{n,p}$. Hence, (1) holds.

(2) This follows immediately from the definition of the equivalence relation $\rho_{n,p}$. This completes the proof. $\square$

We see easily that the stably admissible equivalence relation among the fibers of proper Thom maps of codimension k constructed from $\mathcal{R}_k$ coincides with the original equivalence relation $\widehat{\mathcal{R}}_k$. Therefore, by Proposition 9.15, the complex $\mathcal{C}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \widehat{\mathcal{R}}_k)$ is naturally isomorphic to the universal complex $\mathcal{C}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}_k)$, defined by (9.4), of singular fibers for codimension k proper Thom maps with respect to the stable system of admissible equivalence relations $\mathcal{R}_k = \{\rho_{p-k,p}\}_p$.

For the stable C^0 equivalence, we have the following problem, the answer to which the author does not know.

Problem 9.17. Let $f_i : M_i \to N_i$, $i = 0, 1$, be proper Thom maps such that $n = \dim M_0 = \dim M_1$ and $p = \dim N_0 = \dim N_1$. For points $y_i \in N_i$, if the fibers of f_i over y_i are stably C^0 (or C^∞) equivalent, then are they C^0 (resp. C^∞) equivalent? In other words, is the natural cochain map

$$\mathcal{C}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}_k^0) \to \mathcal{C}(\mathcal{T}_{\mathrm{pr}}(n, p), \rho_{n,p}^0)$$

of the universal complex with respect to the stable C^0 equivalence to that with respect to the C^0 equivalence an epimorphism?

Note that if f_i are codimension -1 proper C^0 stable maps of manifolds of dimension less than or equal to 4, then the answer to the above problem is shown to be affirmative by using an argument similar to that in the proof of Corollary 4.9. (In the 4-dimensional case, we should assume the orientability of the source manifold, while for the other dimensions, it is not necessary. See Corollary 4.16 and the subsequent remark.)

9.5. **Changing the equivalence relation.** Suppose that we are given two admissible equivalence relations $\rho = \rho_{n,p}$ and $\overline{\rho} = \overline{\rho}_{n,p}$ for the fibers of elements of $\mathcal{T}_{\mathrm{pr}}(n, p)$. If every equivalence class with respect to $\rho_{n,p}$ is a union of equivalence classes with respect to $\overline{\rho}_{n,p}$, then we say that $\rho_{n,p}$ is *weaker* than $\overline{\rho}_{n,p}$ and write $\rho_{n,p} \leq \overline{\rho}_{n,p}$. In this case, we can naturally define the $\mathbf{Z}_2$-linear map

$$\varepsilon_{\rho,\overline{\rho}} : \mathcal{C}(\mathcal{T}_{\mathrm{pr}}(n, p), \rho) \to \mathcal{C}(\mathcal{T}_{\mathrm{pr}}(n, p), \overline{\rho}) \tag{9.7}$$

by associating to a class $\widetilde{\mathfrak{F}}$ of codimension κ with respect to ρ the sum of all the equivalence classes with respect to $\overline{\rho}$ of codimension κ contained in $\widetilde{\mathfrak{F}}$. This clearly defines a cochain map (for example, see the proof of Lemma 9.9). Note that the associated map

$$C^\kappa(\mathcal{T}_{\mathrm{pr}}(n, p), \rho) \to C^\kappa(\mathcal{T}_{\mathrm{pr}}(n, p), \overline{\rho})$$

is a monomorphism for every κ.

Suppose that we are given two stable systems of admissible equivalence relations $\mathcal{R}_k = \{\rho_{p-k,p}\}_p$ and $\overline{\mathcal{R}}_k = \{\overline{\rho}_{p-k,p}\}_p$ for the fibers of codimension k proper Thom maps. If $\rho_{p-k,p} \leq \overline{\rho}_{p-k,p}$ for every p, then we say that $\mathcal{R}_k$ is *weaker* than $\overline{\mathcal{R}}_k$ and write $\mathcal{R}_k \leq \overline{\mathcal{R}}_k$. In this case, the system of cochain maps $\{\varepsilon_{\rho_{p-k,p}, \overline{\rho}_{p-k,p}}\}_p$ induces the cochain map

$$\varepsilon_{\mathcal{R}_k, \overline{\mathcal{R}}_k} : \mathcal{C}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}_k) \to \mathcal{C}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \overline{\mathcal{R}}_k).$$

Note that the associated map

$$C^\kappa(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}_k) \to C^\kappa(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \overline{\mathcal{R}}_k)$$

is a monomorphism for every κ.

In particular, if we consider the C^0 equivalence $\rho^0_{n,p}$ among the fibers of elements of $\mathcal{T}_{\mathrm{pr}}(n,p)$, then we have $\rho_{n,p} \leq \rho^0_{n,p}$ for any admissible equivalence relation $\rho_{n,p}$. Hence, we have the cochain map

$$\varepsilon_{\rho_{n,p}, \rho^0_{n,p}} : \mathcal{C}(\mathcal{T}_{\mathrm{pr}}(n,p), \rho_{n,p}) \to \mathcal{C}(\mathcal{T}_{\mathrm{pr}}(n,p), \rho^0_{n,p}).$$

In other words, since this cochain map is always a monomorphism, we may regard $\mathcal{C}(\mathcal{T}_{\mathrm{pr}}(n,p), \rho_{n,p})$ as a subcomplex of $\mathcal{C}(\mathcal{T}_{\mathrm{pr}}(n,p), \rho^0_{n,p})$.

Furthermore, if we consider the stable system of admissible equivalence relations $\mathcal{R}^0_k = \{\rho^0_{p-k,p}\}_p$ induced by the C^0 equivalence, then $\mathcal{R}_k \leq \mathcal{R}^0_k$ for any stable system $\mathcal{R}_k$ of admissible equivalence relations. Hence, we have the cochain map

$$\varepsilon_{\mathcal{R}_k, \mathcal{R}^0_k} : \mathcal{C}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}_k) \to \mathcal{C}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}^0_k).$$

We can show that this is always a monomorphism, and hence $\mathcal{C}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}_k)$ can be regarded as a subcomplex of $\mathcal{C}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}^0_k)$.

9.6. **Changing the class of maps.** So far, we have worked with the whole set of proper Thom maps of a fixed codimension. By restricting the class of Thom maps that we consider, we can also obtain the universal complex of singular fibers for such a class of maps.

First, let us consider maps between manifolds of fixed dimensions.

Definition 9.18. A C^0 equivalence class $\mathfrak{F}$ of fibers of elements of $\mathcal{T}_{\mathrm{pr}}(n,p)$ is said to be *under* another C^0 equivalence class $\mathfrak{G}$ of fibers if for some (and hence every) representative $f : M, f^{-1}(y) \to N, y$ of $\mathfrak{F}$, there is a point y' arbitrarily close to y over which lies a fiber of type $\mathfrak{G}$. In this case, we also say that $\mathfrak{G}$ is *over* $\mathfrak{F}$.

Let $\Gamma = \Gamma_{n,p}$ be an ascending set of C^0 equivalence classes of fibers of elements of $\mathcal{T}_{\mathrm{pr}}(n,p)$, where "*ascending*" means that for an arbitrary equivalence class in the set, every class over it also belongs to the set. We say that a proper Thom map $f : M \to N$ between smooth manifolds of dimensions n and p is a Γ-*map* if its fibers all lie in Γ. We use the same notation $\Gamma = \Gamma_{n,p}$ for the set of all Γ-maps, when there is no confusion.

If for an arbitrary equivalence class in the set Γ, every class under it also belongs to the set, then we say that it is an *descending* set. For example, the set of all C^0 equivalence classes of fibers of a fixed Thom map $f \in \mathcal{T}_{\mathrm{pr}}(n,p)$ is ascending, while the set f^c of all C^0 equivalence classes of fibers of elements of $\mathcal{T}_{\mathrm{pr}}(n,p)$ which do not appear for f is a descending set. Note that $\mathcal{C}(f^c, \rho^0)$ is a subcomplex of $\mathcal{C}(\mathcal{T}_{\mathrm{pr}}(n,p), \rho^0)$ (see Lemma 9.2), essentially because the set f^c is descending.

Let $\Gamma = \Gamma_{n,p}$ be as above and let $\rho^\Gamma = \rho^\Gamma_{n,p}$ be an equivalence relation among the fibers of Γ-maps which is admissible in the same sense as in Definition 8.2. Then, we can naturally define the universal complex $\mathcal{C}(\Gamma_{n,p}, \rho^\Gamma)$ of singular fibers for Γ-maps with respect to the admissible equivalence relation ρ^Γ. We write the corresponding cohomology group of dimension κ by $H^\kappa(\Gamma_{n,p}, \rho^\Gamma)$.

Suppose that the equivalence relation ρ^Γ is the restriction to Γ of an admissible equivalence relation $\rho = \rho_{n,p}$ among the fibers of elements of $\mathcal{T}_{\mathrm{pr}}(n,p)$. Let $C^\kappa(\Gamma^c, \rho)$ be the linear subspace of $C^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p), \rho)$ spanned by all the equivalence classes $\widetilde{\mathfrak{F}}$ of fibers of elements of $\mathcal{T}_{\mathrm{pr}}(n,p)$ of codimension κ with respect to ρ such that no fiber of a Γ-map belongs to $\widetilde{\mathfrak{F}}$. Then, by an argument similar to that in the proof of Lemma 9.2, we can prove the following. Details are left to the reader.

Lemma 9.19. *For an ascending set $\Gamma = \Gamma_{n,p}$ of C^0 equivalence classes of fibers of elements of $\mathcal{T}_{\mathrm{pr}}(n,p)$, the following holds.*

(1) *We have $\delta_\kappa(C^\kappa(\Gamma^c,\rho)) \subset C^{\kappa+1}(\Gamma^c,\rho)$ for every $\kappa \in \mathbf{Z}$. Hence, $\mathcal{C}(\Gamma^c,\rho) = (C^\kappa(\Gamma^c,\rho), \delta_\kappa|_{C^\kappa(\Gamma^c,\rho)})_\kappa$ constitutes a subcomplex of $\mathcal{C}(\mathcal{T}_{\mathrm{pr}}(n,p),\rho)$.*

(2) *The quotient complex*

$$\mathcal{C}(\mathcal{T}_{\mathrm{pr}}(n,p),\rho)/\mathcal{C}(\Gamma^c,\rho) = (C^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p),\rho)/C^\kappa(\Gamma^c,\rho), \overline{\delta}_\kappa)_\kappa$$

is naturally isomorphic to $\mathcal{C}(\Gamma,\rho^\Gamma)$, where

$$\overline{\delta}_\kappa : C^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p),\rho)/C^\kappa(\Gamma^c,\rho) \to C^{\kappa+1}(\mathcal{T}_{\mathrm{pr}}(n,p),\rho)/C^{\kappa+1}(\Gamma^c,\rho)$$

is the well-defined $\mathbf{Z}_2$-linear map induced by δ_κ.

More generally, if Γ and Γ' are two ascending sets of singular fibers of elements of $\mathcal{T}_{\mathrm{pr}}(n,p)$ such that $\Gamma \subset \Gamma'$, and if the admissible equivalence relation ρ^Γ for Γ is the restriction to Γ of an admissible equivalence relation $\rho^{\Gamma'}$ for Γ', then we naturally have the cochain map

$$\pi_{\Gamma',\Gamma} : \mathcal{C}(\Gamma',\rho^{\Gamma'}) \to \mathcal{C}(\Gamma,\rho^\Gamma)$$

induced by the projection. Note that the corresponding $\mathbf{Z}_2$-linear map on every dimension is an epimorphism.

Furthermore, if $\rho^{\Gamma'}$ and $\overline{\rho}^{\Gamma'}$ are two admissible equivalence relations for Γ' with $\rho^{\Gamma'} \leq \overline{\rho}^{\Gamma'}$ in a sense similar to §9.5, then we can naturally define the cochain map

$$\varepsilon_{\rho^{\Gamma'},\overline{\rho}^{\Gamma'}} : \mathcal{C}(\Gamma',\rho^{\Gamma'}) \to \mathcal{C}(\Gamma',\overline{\rho}^{\Gamma'}).$$

Note that the corresponding $\mathbf{Z}_2$-linear map on every dimension is a monomorphism. If ρ^Γ and $\overline{\rho}^\Gamma$ are the restrictions to Γ of $\rho^{\Gamma'}$ and $\overline{\rho}^{\Gamma'}$ respectively, then we have the commutative diagram of cochain complexes:

$$\begin{array}{ccc} \mathcal{C}(\Gamma',\rho^{\Gamma'}) & \xrightarrow{\varepsilon_{\rho^{\Gamma'},\overline{\rho}^{\Gamma'}}} & \mathcal{C}(\Gamma',\overline{\rho}^{\Gamma'}) \\ {\scriptstyle \pi_{\Gamma',\Gamma}}\downarrow & & \downarrow{\scriptstyle \pi_{\Gamma',\Gamma}} \\ \mathcal{C}(\Gamma,\rho^\Gamma) & \xrightarrow{\varepsilon_{\rho^\Gamma,\overline{\rho}^\Gamma}} & \mathcal{C}(\Gamma,\overline{\rho}^\Gamma). \end{array} \tag{9.8}$$

Let us denote by $\mathcal{C}(\Gamma' \smallsetminus \Gamma,\rho^{\Gamma'})$ and $\mathcal{C}(\Gamma' \smallsetminus \Gamma,\overline{\rho}^{\Gamma'})$ the kernels of the $\mathbf{Z}_2$-linear maps

$$\pi_{\Gamma',\Gamma} : \mathcal{C}(\Gamma',\rho^{\Gamma'}) \to \mathcal{C}(\Gamma,\rho^\Gamma)$$

and

$$\pi_{\Gamma',\Gamma} : \mathcal{C}(\Gamma',\overline{\rho}^{\Gamma'}) \to \mathcal{C}(\Gamma,\overline{\rho}^\Gamma)$$

respectively. Note that they are subcomplexes of $\mathcal{C}(\Gamma',\rho^{\Gamma'})$ and $\mathcal{C}(\Gamma',\overline{\rho}^{\Gamma'})$ respectively spanned by those equivalence classes of fibers in Γ' which contain no fiber in Γ. Furthermore, we define $\mathcal{C}(\Gamma,\overline{\rho}^\Gamma/\rho^\Gamma)$ and $\mathcal{C}(\Gamma',\overline{\rho}^{\Gamma'}/\rho^{\Gamma'})$ to be the cokernels of $\varepsilon_{\rho^\Gamma,\overline{\rho}^\Gamma}$ and $\varepsilon_{\rho^{\Gamma'},\overline{\rho}^{\Gamma'}}$ respectively. It is easy to show that $\varepsilon_{\rho^{\Gamma'},\overline{\rho}^{\Gamma'}}$ induces a monomorphism

$$\mathcal{C}(\Gamma' \smallsetminus \Gamma,\rho^{\Gamma'}) \to \mathcal{C}(\Gamma' \smallsetminus \Gamma,\overline{\rho}^{\Gamma'})$$

and we denote its cokernel by $\mathcal{C}(\Gamma' \setminus \Gamma, \overline{\rho}^{\Gamma'}/\rho^{\Gamma'})$. Then we naturally have the following commutative diagram:

(9.9)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{C}(\Gamma' \setminus \Gamma, \rho^{\Gamma'}) & \to & \mathcal{C}(\Gamma' \setminus \Gamma, \overline{\rho}^{\Gamma'}) & \to & \mathcal{C}(\Gamma' \setminus \Gamma, \overline{\rho}^{\Gamma'}/\rho^{\Gamma'}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{C}(\Gamma', \rho^{\Gamma'}) & \xrightarrow{\varepsilon_{\rho^{\Gamma'},\overline{\rho}^{\Gamma'}}} & \mathcal{C}(\Gamma', \overline{\rho}^{\Gamma'}) & \to & \mathcal{C}(\Gamma', \overline{\rho}^{\Gamma'}/\rho^{\Gamma'}) & \to & 0 \\
 & & \pi_{\Gamma',\Gamma}\Big\downarrow & & \Big\downarrow\pi_{\Gamma',\Gamma} & & \downarrow & & \\
0 & \to & \mathcal{C}(\Gamma, \rho^{\Gamma}) & \xrightarrow{\varepsilon_{\rho^{\Gamma},\overline{\rho}^{\Gamma}}} & \mathcal{C}(\Gamma, \overline{\rho}^{\Gamma}) & \to & \mathcal{C}(\Gamma, \overline{\rho}^{\Gamma}/\rho^{\Gamma}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where all the rows and columns are exact. Therefore, we have the corresponding commutative diagram of long exact sequences of cohomologies as well.

Now, let us vary the dimension pair (n, p) keeping the codimension $p - n = k$ fixed. Let

$$\widetilde{\Gamma} = \widetilde{\Gamma}_k = \bigcup_{p-n=k} \Gamma_{n,p}$$

be a set of C^0 equivalence classes of fibers of proper Thom maps of codimension k such that each $\Gamma_{n,p}$ is an ascending set of C^0 equivalence classes of fibers of elements of $\mathcal{T}_{\mathrm{pr}}(n, p)$, and that $\widetilde{\Gamma}$ is closed under suspension in the sense of Definition 9.4. (For example, the set of all C^0 equivalence classes of fibers of elements of $\widetilde{\mathcal{S}}^0_{\mathrm{pr}}(k)$ is such a set.)

We say that a proper Thom map of codimension k is a $\widetilde{\Gamma}_k$-map if its fibers all lie in $\widetilde{\Gamma}_k$. We use the same notation $\widetilde{\Gamma} = \widetilde{\Gamma}_k$ for the set of all $\widetilde{\Gamma}_k$-maps, when there is no confusion.

Let $\mathcal{R}_k^{\widetilde{\Gamma}} = \{\rho_{p-k,p}^{\Gamma_{p-k,p}}\}_p$ be a system of equivalence relations, where each $\rho_{p-k,p}^{\Gamma_{p-k,p}}$ is an admissible equivalence relation among the fibers of $\Gamma_{p-k,p}$-maps. Furthermore, we assume that the system $\mathcal{R}_k^{\widetilde{\Gamma}}$ of admissible equivalence relations is stable in the sense of Definition 9.5.

Then, we can naturally define the universal complex of singular fibers

$$\mathcal{C}(\widetilde{\Gamma}_k, \mathcal{R}_k^{\widetilde{\Gamma}})$$

for $\widetilde{\Gamma}_k$-maps with respect to the stable system of admissible equivalence relations $\mathcal{R}_k^{\widetilde{\Gamma}}$. As in the case of Thom maps, we have two definitions for the universal complexes, which are equivalent to each other as in Proposition 9.15. We write its cohomology group of dimension κ by $H^\kappa(\widetilde{\Gamma}_k, \mathcal{R}_k^{\widetilde{\Gamma}})$.

Note that if the stable system of admissible equivalence relations $\mathcal{R}_k^{\widetilde{\Gamma}}$ is the restriction of a stable system of admissible equivalence relations $\mathcal{R}_k$ among the fibers of proper Thom maps of codimension k, then we see that the complex $\mathcal{C}(\widetilde{\Gamma}_k, \mathcal{R}_k^{\widetilde{\Gamma}})$ is a quotient complex of the universal complex $\mathcal{C}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}_k)$ in view of the construction given in §9.4.

More generally, if $\widetilde{\Gamma}$ and $\widetilde{\Gamma}'$ are two ascending sets of singular fibers of elements of $\widetilde{\mathcal{T}}_{\mathrm{pr}}(k)$ which are closed under suspension such that $\widetilde{\Gamma} \subset \widetilde{\Gamma}'$, and if the stable system of admissible equivalence relations $\mathcal{R}_k^{\widetilde{\Gamma}}$ for $\widetilde{\Gamma}$ is the restriction to $\widetilde{\Gamma}$ of a stable system of admissible equivalence relations $\mathcal{R}_k^{\widetilde{\Gamma}'}$ for $\widetilde{\Gamma}'$, then we naturally have the cochain map

$$\pi_{\widetilde{\Gamma}',\widetilde{\Gamma}} : \mathcal{C}(\widetilde{\Gamma}', \mathcal{R}_k^{\widetilde{\Gamma}'}) \to \mathcal{C}(\widetilde{\Gamma}, \mathcal{R}_k^{\widetilde{\Gamma}})$$

induced by the natural projection. Note that the corresponding $\mathbf{Z}_2$-linear map on every dimension is an epimorphism.

Furthermore, if $\mathcal{R}^{\widetilde{\Gamma}'}$ and $\overline{\mathcal{R}}^{\widetilde{\Gamma}'}$ are two stable systems of admissible equivalence relations for $\widetilde{\Gamma}'$ with $\mathcal{R}^{\widetilde{\Gamma}'} \leq \overline{\mathcal{R}}^{\widetilde{\Gamma}'}$ in a sense similar to §9.5, then we can naturally define the cochain map

$$\varepsilon_{\mathcal{R}^{\widetilde{\Gamma}'}, \overline{\mathcal{R}}^{\widetilde{\Gamma}'}} : \mathcal{C}(\widetilde{\Gamma}', \mathcal{R}^{\widetilde{\Gamma}'}) \to \mathcal{C}(\widetilde{\Gamma}', \overline{\mathcal{R}}^{\widetilde{\Gamma}'}).$$

Note that the corresponding $\mathbf{Z}_2$-linear map on every dimension is a monomorphism.

If $\mathcal{R}^{\widetilde{\Gamma}}$ and $\overline{\mathcal{R}}^{\widetilde{\Gamma}}$ are the restrictions to $\widetilde{\Gamma}$ of $\mathcal{R}^{\widetilde{\Gamma}'}$ and $\overline{\mathcal{R}}^{\widetilde{\Gamma}'}$ respectively, then we have the commutative diagram of cochain maps:

$$\begin{array}{ccc}
\mathcal{C}(\widetilde{\Gamma}', \mathcal{R}^{\widetilde{\Gamma}'}) & \xrightarrow{\varepsilon_{\mathcal{R}^{\widetilde{\Gamma}'}, \overline{\mathcal{R}}^{\widetilde{\Gamma}'}}} & \mathcal{C}(\widetilde{\Gamma}', \overline{\mathcal{R}}^{\widetilde{\Gamma}'}) \\
{\scriptstyle \pi_{\widetilde{\Gamma}', \widetilde{\Gamma}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\widetilde{\Gamma}', \widetilde{\Gamma}}} \\
\mathcal{C}(\widetilde{\Gamma}, \mathcal{R}^{\widetilde{\Gamma}}) & \xrightarrow{\varepsilon_{\mathcal{R}^{\widetilde{\Gamma}}, \overline{\mathcal{R}}^{\widetilde{\Gamma}}}} & \mathcal{C}(\widetilde{\Gamma}, \overline{\mathcal{R}}^{\widetilde{\Gamma}}).
\end{array}$$

Note that we can extend the above commutative diagram as in (9.9) so that we obtain exact rows and columns.

Remark 9.20. Let $\widetilde{\Gamma} = \widetilde{\Gamma}_k = \cup_p \Gamma_{p-k,p}$ be as above and $\mathcal{R}_k^{\widetilde{\Gamma}} = \{\rho_{p-k,p}^{\Gamma_{p-k,p}}\}_p$ be a stable system of admissible equivalence relations for the fibers of $\widetilde{\Gamma}$-maps. Then we have the natural $\mathbf{Z}_2$-linear map

$$\Phi_{p-k,p}^{\kappa} : C^{\kappa}(\widetilde{\Gamma}, \mathcal{R}_k^{\widetilde{\Gamma}}) \to C^{\kappa}(\Gamma_{p-k,p}, \rho_{p-k,p}^{\Gamma_{p-k,p}})$$

induced by the projection for every p, since $C^{\kappa}(\widetilde{\Gamma}, \mathcal{R}_k^{\widetilde{\Gamma}})$ is the projective limit and hence is a $\mathbf{Z}_2$-submodule of the product of all $C^{\kappa}(\Gamma_{p-k,p}, \rho_{p-k,p}^{\Gamma_{p-k,p}})$ (see Remark 9.10). (Note that for $\widetilde{\Gamma} = \widetilde{\mathcal{T}}_{\mathrm{pr}}(k)$, this map has already been defined. See (9.5) and (9.6).) Set $n = p - k$. For $0 \leq \kappa \leq p$, $\Phi_{n,p}^{\kappa}$ is a monomorphism if and only if every equivalence class of fibers in $\widetilde{\Gamma}$ with respect to $\widehat{\mathcal{R}}_k$ of codimension κ contains a suspension of a fiber in $\Gamma_{p-k,p}$ whose equivalence class with respect to $\rho_{p-k,p}^{\Gamma_{p-k,p}}$ has codimension κ, where $\widehat{\mathcal{R}}_k$ is the stably admissible equivalence relation among the fibers in $\widetilde{\Gamma}$ defined just before Proposition 9.15 (compare this assertion with Lemma 9.11). On the other hand, $\Phi_{n,p}^{\kappa}$ is an epimorphism if and only if the following two conditions hold.

(1) If an equivalence class of fibers in $\Gamma_{n,p}$ with respect to $\rho_{n,p}^{\Gamma_{n,p}}$ has codimension κ, then the equivalence class of their ℓ-th suspensions with respect to $\rho_{n+\ell,p+\ell}^{\Gamma_{n+\ell,p+\ell}}$ has also codimension κ for all $\ell \geq 1$.

(2) Two fibers in $\Gamma_{n,p}$ whose equivalence classes with respect to $\rho_{n,p}^{\Gamma_{n,p}}$ have codimension κ are equivalent with respect to $\rho_{n,p}^{\Gamma_{n,p}}$ if and only if their ℓ-th suspensions are equivalent with respect to $\rho_{n+\ell,p+\ell}^{\Gamma_{n+\ell,p+\ell}}$ for some $\ell \geq 0$.

Compare this with Problem 9.17, Lemma 9.6 and Remark 9.8.

When a class of proper Thom maps is given, let us consider the following definitions.

Definition 9.21. (1) Let $\Gamma_{n,p} = \Gamma$ be a subset of $\mathcal{T}_{\mathrm{pr}}(n,p)$. We denote by $\Gamma_{n,p}^* = \Gamma^*$ the set of all C^0 equivalence classes of fibers of elements of $\Gamma_{n,p}$. Then, it is clear that $\Gamma_{n,p}^*$ is an ascending set and the set of all $\Gamma_{n,p}^*$-maps contain the original set $\Gamma_{n,p}$ of maps. For an admissible equivalence relation ρ^{Γ} among the elements of

$\Gamma^*_{n,p}$, we define the *universal complex of singular fibers for* $\Gamma_{n,p}$ *with respect to* ρ^Γ by

$$\mathcal{C}(\Gamma_{n,p}, \rho^\Gamma) = \mathcal{C}(\Gamma^*_{n,p}, \rho^\Gamma).$$

Furthermore, we denote the corresponding cohomology group of dimension κ by $H^\kappa(\Gamma_{n,p}, \rho^\Gamma)$. We call the set of $\Gamma^*_{n,p}$-maps the *completion* of $\Gamma_{n,p}$. When the set of $\Gamma^*_{n,p}$-maps coincides with the original set $\Gamma_{n,p}$, we say that the set $\Gamma_{n,p}$ is *complete*.

(2) Let $\widetilde{\Gamma}_k = \widetilde{\Gamma}$ be a subset of $\widetilde{\mathcal{T}}_{\mathrm{pr}}(k)$. We denote by $\widetilde{\Gamma}^*_k = \widetilde{\Gamma}^*$ the set of all C^0 equivalence classes of fibers of elements of $\widetilde{\Gamma}_k$ and their suspensions. Then, we have

$$\widetilde{\Gamma}^*_k = \bigcup_{p-n=k} \Gamma^*_{n,p},$$

where $\Gamma^*_{n,p}$ is the set of C^0 equivalence classes in $\widetilde{\Gamma}^*_k$ of fibers of maps between manifolds of dimensions n and p, and each $\Gamma^*_{n,p}$ is an ascending set. Furthermore, $\widetilde{\Gamma}^*_k$ is closed under suspension. Then, it is clear that the set of all $\widetilde{\Gamma}^*_k$-maps contain the original set $\widetilde{\Gamma}_k$ of maps. For a stable system of admissible equivalence relations $\mathcal{R}^{\widetilde{\Gamma}}_k$ among the elements of $\widetilde{\Gamma}^*_k$, we define the *universal complex of singular fibers for* $\widetilde{\Gamma}_k$ *with respect to* $\mathcal{R}^{\widetilde{\Gamma}}_k$ by

$$\mathcal{C}(\widetilde{\Gamma}_k, \mathcal{R}^{\widetilde{\Gamma}}_k) = \mathcal{C}(\widetilde{\Gamma}^*_k, \mathcal{R}^{\widetilde{\Gamma}}_k).$$

Furthermore, we denote the corresponding cohomology group of codimension κ by $H^\kappa(\widetilde{\Gamma}_k, \mathcal{R}^{\widetilde{\Gamma}}_k)$. We call the set of $\widetilde{\Gamma}^*_k$-maps the *completion* of $\widetilde{\Gamma}_k$. When the set of $\widetilde{\Gamma}^*_k$-maps coincides with the original set $\widetilde{\Gamma}_k$, we say that the set $\widetilde{\Gamma}_k$ is *complete*.

Example 9.22. For example, the set $\mathcal{S}^0_{\mathrm{pr}}(n,p)$ is not complete, since there exist nonstable Thom maps whose fibers are all C^0 equivalent to a fiber of a C^0 stable map. On the other hand, $\mathcal{T}_{\mathrm{pr}}(n,p)$ is clearly complete.

In the following, if $\Gamma = \Gamma_{n,p} \subset \Gamma'_{n,p} = \Gamma' \subset \mathcal{T}_{\mathrm{pr}}(n,p)$ and ρ^Γ is the restriction of an admissible equivalence relation $\rho^{\Gamma'}$ for the fibers of elements of $\Gamma'_{n,p}$, we sometimes write $\mathcal{C}(\Gamma_{n,p}, \rho^{\Gamma'})$ in place of $\mathcal{C}(\Gamma_{n,p}, \rho^\Gamma)$ when there is no confusion. For the universal complexes for the fibers of codimension k maps, we sometimes use the same convention as well.

Example 9.23. Let $\mathcal{M}_{\mathrm{pr}}(n,p)$ be the set of all proper Morin maps in $\mathcal{T}_{\mathrm{pr}}(n,p)$ which satisfy the normal crossing condition as in [13, Chapter VI, §5], and set

$$\widetilde{\mathcal{M}}_{\mathrm{pr}}(k) = \bigcup_{p-n=k} \mathcal{M}_{\mathrm{pr}}(n,p).$$

(Here, a smooth map is called a *Morin map* if its singularities are all of Morin types [29].) Furthermore, we denote by $\mathcal{M}_{\mathrm{pr}}(n,p)^{\mathrm{ori}}$ the subset of $\mathcal{M}_{\mathrm{pr}}(n,p)$ consisting of those maps whose source manifolds are orientable, and we set

$$\widetilde{\mathcal{M}}_{\mathrm{pr}}(k)^{\mathrm{ori}} = \bigcup_{p-n=k} \mathcal{M}_{\mathrm{pr}}(n,p)^{\mathrm{ori}}.$$

(Note that the sets $\widetilde{\mathcal{M}}_{\mathrm{pr}}(k)$ and $\widetilde{\mathcal{M}}_{\mathrm{pr}}(k)^{\mathrm{ori}}$ are closed under suspension.) Then, by using Remark 9.20, we can show that

$$C^\kappa(\widetilde{\mathcal{M}}_{\mathrm{pr}}(-1)^{\mathrm{ori}}, \mathcal{R}^0_{-1}) = C^\kappa(\mathcal{M}_{\mathrm{pr}}(4,3)^{\mathrm{ori}}, \rho^0_{4,3})$$

for all $\kappa \leq 3$, and hence

$$H^\kappa(\widetilde{\mathcal{M}}_{\mathrm{pr}}(-1)^{\mathrm{ori}}, \mathcal{R}^0_{-1}) = H^\kappa(\mathcal{M}_{\mathrm{pr}}(4,3)^{\mathrm{ori}}, \rho^0_{4,3})$$

for $\kappa \leq 2$ (see also the paragraph just after Problem 9.17). Compare this with Problem 10.8.

10. Universal complex of singular fibers for stable maps of orientable 4-manifolds into 3-manifolds

Now, let us consider a more specific situation, i.e., the case of proper C^∞ stable maps of orientable 4-manifolds into 3-manifolds. Recall that a proper smooth map of a 4-manifold into a 3-manifold is C^∞ stable if and only if it is C^0 stable, as we have noted in Remark 4.2. In the following, we denote by $\mathcal{S}^0_{\mathrm{pr}}(n,p)^{\mathrm{ori}}$ the subset of $S^0_{\mathrm{pr}}(n,p)$ consisting of the proper C^0 stable maps of *orientable* manifolds of dimension n into manifolds of dimension p.

The universal complex of singular fibers $\mathcal{C}(\mathcal{S}^0_{\mathrm{pr}}(4,3)^{\mathrm{ori}}, \rho^0_{4,3})$ for proper C^0 stable maps of orientable 4-manifolds into 3-manifolds with respect to the C^0 (or C^∞) right-left equivalence $\rho^0_{4,3}$ can be described as follows.

For a positive integer ℓ, let us denote by I^0_ℓ the equivalence class of the singular fiber which is the disjoint union of the corresponding singular fiber I^0 as in Fig. 8 and some fibers of the trivial circle bundle such that its total number of connected components is equal to ℓ. We define I^1_ℓ $(\ell \geq 1)$, II^{00}_ℓ $(\ell \geq 2)$, etc. similarly. Furthermore, let $\mathbf{0}_\ell$ $(\ell \geq 0)$ denote the equivalence class of the regular fiber consisting of ℓ copies of a fiber of the trivial circle bundle.

Then, by the construction in §9, we obtain the complex of $\mathbf{Z}_2$-coefficients

$$\begin{aligned} 0 \longrightarrow C^0(\mathcal{S}^0_{\mathrm{pr}}(4,3)^{\mathrm{ori}}, \rho^0_{4,3}) &\xrightarrow{\delta_0} C^1(\mathcal{S}^0_{\mathrm{pr}}(4,3)^{\mathrm{ori}}, \rho^0_{4,3}) \\ &\xrightarrow{\delta_1} C^2(\mathcal{S}^0_{\mathrm{pr}}(4,3)^{\mathrm{ori}}, \rho^0_{4,3}) \xrightarrow{\delta_2} C^3(\mathcal{S}^0_{\mathrm{pr}}(4,3)^{\mathrm{ori}}, \rho^0_{4,3}) \longrightarrow 0, \end{aligned}$$

where $C^0(\mathcal{S}^0_{\mathrm{pr}}(4,3)^{\mathrm{ori}}, \rho^0_{4,3})$ is generated by $\mathbf{0}_\ell$, and $C^i(\mathcal{S}^0_{\mathrm{pr}}(4,3)^{\mathrm{ori}}, \rho^0_{4,3})$, $i = 1,2,3$, are generated by $\mathrm{I}^*_\ell, \mathrm{II}^*_\ell$ and III^*_ℓ, respectively, for various ℓ. Note that we have not specified any proper C^0 stable map $f : M \to N$ of an orientable 4-manifold into a 3-manifold. Hence, this complex can be regarded as a *universal complex of singular fibers for proper C^0 stable maps of orientable 4-manifolds into 3-manifolds*, in the sense that the corresponding complex for a specific C^0 stable map f is realized as a quotient complex of the universal complex (see Lemma 9.2).

This complex has the disadvantage that it has too many generators at each dimension and hence that it is a bit difficult to pursue a straightforward calculation of its cohomology groups. Thus, it seems reasonable to consider an equivalence relation weaker than the C^0 equivalence. For this, let us fix a positive integer m.

Definition 10.1. We say that two fibers of proper Thom maps between manifolds of dimensions $p+1$ and p, $p \geq 0$, are C^0 *equivalent modulo m circle components* if one of them is C^0 equivalent to the disjoint union of the other one and ℓm copies of a fiber of the trivial circle bundle for some nonnegative integer ℓ. We denote this equivalence relation by $\rho^0_{p+1,p}(m)$. Given a subset $\Gamma_{p+1,p}$ of $\mathcal{T}_{\mathrm{pr}}(p+1,p)$, we shall use the same notation $\rho^0_{p+1,p}(m)$ for the equivalence relation for $\Gamma^*_{p+1,p}$ induced by the above one, when there is no confusion (for the notation $\Gamma^*_{p+1,p}$, refer to Definition 9.21).

Lemma 10.2. *Let p be a nonnegative integer and m a positive integer. The C^0 equivalence modulo m circle components $\rho^0_{p+1,p}(m)$ is an admissible equivalence relation for the fibers of elements of $\mathcal{T}_{\mathrm{pr}}(p+1,p)$ and hence for $\Gamma^*_{p+1,p}$.*

Proof. By definition, we see easily that condition (1) of Definition 8.2 is satisfied. Suppose that two fibers are C^0 equivalent modulo m circle components. Then by definition, the corresponding nearby fibers are all C^0 equivalent modulo m circle components. Hence condition (2) of Definition 8.2 is also satisfied. This completes the proof. □

Remark 10.3. Furthermore, we can also show that the system of admissible equivalence relations $\mathcal{R}^0_{-1}(m) = \{\rho^0_{p+1,p}(m)\}_{p\geq 0}$ for the fibers of elements of $\widetilde{\mathcal{T}}_{\mathrm{pr}}(-1)$ is stable in the sense of Definition 9.5. Hence, for any subset $\widetilde{\Gamma}_{-1}$ of $\widetilde{\mathcal{T}}_{\mathrm{pr}}(-1)$ which is closed under suspension, the restriction of $\mathcal{R}^0_{-1}(m) = \{\rho^0_{p+1,p}(m)\}_{p\geq 0}$ to $\widetilde{\Gamma}^*_{-1}$ is also stable.

By Lemma 10.2, for a nonnegative integer p and a positive integer m, we can define the universal complex of singular fibers

$$\mathcal{C}(\mathcal{T}_{\mathrm{pr}}(p+1,p), \rho^0_{p+1,p}(m))$$

for proper Thom maps between manifolds of dimensions $p+1$ and p with respect to the C^0 equivalence modulo m circle components. More generally, for every subset $\Gamma_{p+1,p}$ of $\mathcal{T}_{\mathrm{pr}}(p+1,p)$, we can define the universal complex of singular fibers

$$\mathcal{C}(\Gamma_{p+1,p}, \rho^0_{p+1,p}(m))$$

for $\Gamma_{p+1,p}$ with respect to the C^0 equivalence modulo m circle components (see Definition 9.21). We call the universal complexes thus obtained the *universal complexes of singular fibers modulo m circle components.*

The argument in §5 can be elaborated to prove the following results. Details are left to the reader.

Proposition 10.4. *The cohomology groups of the universal complex of singular fibers modulo two circle components*

$$\mathcal{C}(\mathcal{S}^0_{\mathrm{pr}}(4,3)^{\mathrm{ori}}, \rho^0_{4,3}(2))$$

for proper C^0 stable maps of orientable 4-manifolds into 3-manifolds are given as follows:

$$\begin{aligned}
H^0(\mathcal{S}^0_{\mathrm{pr}}(4,3)^{\mathrm{ori}}, \rho^0_{4,3}(2)) &\cong \mathbf{Z}_2 \ (\textit{generated by } [\mathbf{0}_{\mathrm{o}} + \mathbf{0}_{\mathrm{e}}]),\\
H^1(\mathcal{S}^0_{\mathrm{pr}}(4,3)^{\mathrm{ori}}, \rho^0_{4,3}(2)) &\cong \mathbf{Z}_2 \ (\textit{generated by } [\mathrm{I}^0_{\mathrm{o}} + \mathrm{I}^1_{\mathrm{e}}] = [\mathrm{I}^0_{\mathrm{e}} + \mathrm{I}^1_{\mathrm{o}}]),\\
H^2(\mathcal{S}^0_{\mathrm{pr}}(4,3)^{\mathrm{ori}}, \rho^0_{4,3}(2)) &= 0,
\end{aligned}$$

where $\mathfrak{F}_{\mathrm{o}}$ (or $\mathfrak{F}_{\mathrm{e}}$) denotes the C^0 equivalence class modulo two circle components represented by $\mathfrak{F}_\ell$ with ℓ odd (resp. even), and $[*]$ denotes the cohomology class represented by the cocycle $*$.

Remark 10.5. We can apply Proposition 8.4 as follows. The $\mathbf{Z}_2$-homology class (of closed support) in the target 3-manifold represented by a cycle corresponding to a coboundary of the universal complex of singular fibers (modulo m circle components) always vanishes. For $m = 2$, the coboundary groups are generated by the cochains listed in Table 2.

κ	generator(s)
1	$(\mathrm{I}^0_{\mathrm{e}} + \mathrm{I}^1_{\mathrm{o}}) - (\mathrm{I}^0_{\mathrm{o}} + \mathrm{I}^1_{\mathrm{e}})$
2	$\mathrm{II}^{01}_{\mathrm{o}} + \mathrm{II}^{01}_{\mathrm{e}} + \mathrm{II}^{a}_{\mathrm{e}},\ \mathrm{II}^{01}_{\mathrm{o}} + \mathrm{II}^{01}_{\mathrm{e}} + \mathrm{II}^{a}_{\mathrm{o}}$

TABLE 2. Generators for the coboundary groups of $\mathcal{C}(\mathcal{S}^0_{\mathrm{pr}}(4,3)^{\mathrm{ori}}, \rho^0_{4,3}(2))$

For $\kappa = 3$, we can easily read off the generators from Table 1 given in §5. Note that these lead to the congruences modulo two obtained in Proposition 5.1.

Compare this with [50, 12.5.4, 12.6.5, 13.4.1] and [33].

Remark 10.6. By using the classification theorem of singular fibers for proper C^0 stable maps in $\mathcal{S}^0_{\mathrm{pr}}(3,2)^{\mathrm{ori}}$ and in $\mathcal{S}^0_{\mathrm{pr}}(2,1)^{\mathrm{ori}}$ (see Remark 4.14 and Theorem 3.1), we see that the $\mathbf{Z}_2$-linear maps

$$s_\kappa : C^\kappa(\mathcal{S}^0_{\mathrm{pr}}(4,3)^{\mathrm{ori}}, \rho^0_{4,3}(2)) \to C^\kappa(\mathcal{S}^0_{\mathrm{pr}}(3,2)^{\mathrm{ori}}, \rho^0_{3,2}(2))$$

for $\kappa \leq 2$ and

$$s_\kappa : C^\kappa(\mathcal{S}^0_{\mathrm{pr}}(4,3)^{\mathrm{ori}}, \rho^0_{4,3}(2)) \to C^\kappa(\mathcal{S}^0_{\mathrm{pr}}(2,1)^{\mathrm{ori}}, \rho^0_{2,1}(2))$$

for $\kappa \leq 1$ induced by the suspension are in fact isomorphisms. Hence, some of the above results are valid also for $\mathcal{C}(\mathcal{S}^0_{\mathrm{pr}}(3,2)^{\mathrm{ori}}, \rho^0_{3,2}(2))$ and $\mathcal{C}(\mathcal{S}^0_{\mathrm{pr}}(2,1)^{\mathrm{ori}}, \rho^0_{2,1}(2))$. For example, we have the following, where for an equivalence class of fibers, we use the same notation as the equivalence class of its suspension.

(1) For $f \in \mathcal{S}^0_{\mathrm{pr}}(3,2)^{\mathrm{ori}}$, we have

$$|\mathrm{II}^{01}_{\mathrm{o}}(f)| + |\mathrm{II}^{01}_{\mathrm{e}}(f)| + |\mathrm{II}^{a}_{\mathrm{e}}(f)| \equiv |\mathrm{II}^{01}_{\mathrm{o}}(f)| + |\mathrm{II}^{01}_{\mathrm{e}}(f)| + |\mathrm{II}^{a}_{\mathrm{o}}(f)| \equiv 0 \pmod 2.$$

(2) For $f \in \mathcal{S}^0_{\mathrm{pr}}(2,1)^{\mathrm{ori}}$, we have

$$|\mathrm{I}^0_{\mathrm{e}}(f)| + |\mathrm{I}^1_{\mathrm{o}}(f)| \equiv |\mathrm{I}^0_{\mathrm{o}}(f)| + |\mathrm{I}^1_{\mathrm{e}}(f)| \pmod 2.$$

We can also prove the following. For the notation, refer to Theorem 4.15 and Proposition 10.4.

Proposition 10.7. *The cohomology groups of the universal complex of singular fibers modulo two circle components*

$$\mathcal{C}(\mathcal{S}^0_{\mathrm{pr}}(3,2), \rho^0_{3,2}(2))$$

for proper C^0 stable maps of (not necessarily orientable) 3-manifolds into surfaces are given as follows:

$$\begin{aligned} H^0(\mathcal{S}^0_{\mathrm{pr}}(3,2), \rho^0_{3,2}(2)) &\cong \mathbf{Z}_2 \ (\textit{generated by } [\mathbf{0}_{\mathrm{o}} + \mathbf{0}_{\mathrm{e}}]), \\ H^1(\mathcal{S}^0_{\mathrm{pr}}(3,2), \rho^0_{3,2}(2)) &\cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \ (\textit{generated by } [\widetilde{\mathrm{I}}^0_{\mathrm{o}} + \widetilde{\mathrm{I}}^1_{\mathrm{e}}] = [\widetilde{\mathrm{I}}^0_{\mathrm{e}} + \widetilde{\mathrm{I}}^1_{\mathrm{o}}] \\ &\qquad \textit{and } [\widetilde{\mathrm{I}}^2_{\mathrm{o}} + \widetilde{\mathrm{I}}^2_{\mathrm{e}}]). \end{aligned}$$

The coboundary groups of the cochain complex $\mathcal{C}(\mathcal{S}^0_{\mathrm{pr}}(3,2), \rho^0_{3,2}(2))$ are generated by the cochains listed in Table 3 (see also Remark 5.5).

κ	generator(s)
1	$(\widetilde{\mathrm{I}}^0_{\mathrm{e}} + \widetilde{\mathrm{I}}^1_{\mathrm{o}}) - (\widetilde{\mathrm{I}}^0_{\mathrm{o}} + \widetilde{\mathrm{I}}^1_{\mathrm{e}})$
2	$\widetilde{\mathrm{II}}^{01}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{01}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{a}_{\mathrm{e}}$, $\widetilde{\mathrm{II}}^{01}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{01}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{a}_{\mathrm{o}}$, $\widetilde{\mathrm{II}}^{02}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{02}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{12}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{12}_{\mathrm{e}} + \widetilde{\mathrm{II}}^{6}_{\mathrm{o}} + \widetilde{\mathrm{II}}^{6}_{\mathrm{e}}$

TABLE 3. Generators for the coboundary groups of $\mathcal{C}(\mathcal{S}^0_{\mathrm{pr}}(3,2), \rho^0_{3,2}(2))$

By the same reason as in Remark 10.6, some of the above results hold also for $\mathcal{C}(\mathcal{S}^0_{\mathrm{pr}}(2,1), \rho^0_{2,1}(2))$ as well.

Problem 10.8. Is the natural map

$$H^\kappa(\widetilde{\mathcal{S}}^0_{\mathrm{pr}}(-1), \mathcal{R}^0_{-1}) \to H^\kappa(\mathcal{S}^0_{\mathrm{pr}}(4,3), \rho^0_{4,3})$$

an isomorphism for $\kappa \leq 2$? More generally, is the natural map

$$H^\kappa(\widetilde{\mathcal{S}}^0_{\mathrm{pr}}(-1), \mathcal{R}^0_{-1}) \to H^\kappa(\mathcal{S}^0_{\mathrm{pr}}(p+1,p), \rho^0_{p+1,p})$$

an isomorphism for $\kappa \leq p-1$? Compare this with Remark 9.20 and Example 9.23.

Now, let us introduce the following equivalence relation among the fibers weaker than the C^0 equivalence.

Definition 10.9. Let us consider the class of maps f in $\mathcal{T}_{\rm pr}(n,p)$ such that the restriction to its singular set $S(f)$, $f|_{S(f)}$, is finite-to-one. We say that two fibers of such maps are C^0 *multi-singularity equivalent* (or *multi-singularity equivalent*) if the associated multi-germs at their singular points are C^0 right-left equivalent to each other. It is easy to show that this defines an admissible equivalence relation for the fibers of the above class of maps. Here, we adopt the convention that if the fibers contain no singular points, then they are always multi-singularity equivalent. We denote the multi-singularity equivalence relation by $\rho^{\rm ms}_{n,p}$.

It is easy to see that if $n = p+1$, then

$$\rho^{\rm ms}_{p+1,p} \leq \rho^0_{p+1,p}(m) \leq \rho^0_{p+1,p}$$

for every positive integer m.

Remark 10.10. Note that the universal complex of singular fibers with respect to the multi-singularity equivalence corresponds to Vassiliev's universal complex of multi-singularities [50] (see also [19, 33]).

By using a characterization of C^0 stable maps of orientable 5-dimensional manifolds into 4-dimensional manifolds as in Proposition 4.1, we can easily obtain the following. The details are left to the reader.

Proposition 10.11. *The cohomology groups of the universal complex of singular fibers for proper C^0 stable maps of orientable 5-dimensional manifolds into 4-dimensional manifolds with respect to the multi-singularity equivalence*

$$\mathcal{C}(\mathcal{S}^0_{\rm pr}(5,4)^{\rm ori}, \rho^{\rm ms}_{5,4})$$

are given as follows:

$$\begin{aligned} H^0(\mathcal{S}^0_{\rm pr}(5,4)^{\rm ori}, \rho^{\rm ms}_{5,4}) &\cong \mathbf{Z}_2 \ (\textit{generated by } [\overline{\mathbf{0}}]), \\ H^1(\mathcal{S}^0_{\rm pr}(5,4)^{\rm ori}, \rho^{\rm ms}_{5,4}) &\cong \mathbf{Z}_2 \ (\textit{generated by } [\overline{\mathrm{I}}^0 + \overline{\mathrm{I}}^1]), \\ H^2(\mathcal{S}^0_{\rm pr}(5,4)^{\rm ori}, \rho^{\rm ms}_{5,4}) &= 0, \\ H^3(\mathcal{S}^0_{\rm pr}(5,4)^{\rm ori}, \rho^{\rm ms}_{5,4}) &= 0, \end{aligned}$$

where $\overline{\mathbf{0}}$ denotes the multi-singularity equivalence class of regular fibers, $\overline{\mathrm{I}}^0$ the multi-singularity equivalence class of the definite fold mono-germ, $\overline{\mathrm{I}}^1$ the multi-singularity equivalence class of the indefinite fold mono-germ, and $[*]$ denotes the cohomology class represented by the cocycle $*$.

The above proposition shows that if we consider Vassiliev's universal complex of multi-singularities, then a result like Theorem 6.1 cannot be obtained. In fact, although we have not included the computation of $H^3(\mathcal{S}^0_{\rm pr}(5,4)^{\rm ori}, \rho^0_{5,4}(2))$, we will see in Corollary 13.12 that it contains a nontrivial element which corresponds to the singular fiber of type III^8 as in Fig. 8 (see also Remark 11.12). We will also see that such a nontrivial element is closely related to the formula given in Theorem 6.1. This justifies our study of the universal complexes of singular fibers instead of multi-singularities.

11. Universal complex of co-orientable singular fibers

Let us now proceed to the construction of another universal complex corresponding to co-orientable strata.

11.1. **Universal complex with respect to the C^0 equivalence.** Let us begin by the following definition.

Definition 11.1. A C^0 equivalence class $\mathfrak{F}$ of fibers of proper Thom maps is *weakly co-orientable* if for any homeomorphisms $\widetilde{\varphi}$ and φ which make the diagram

$$\begin{array}{ccc} (f^{-1}(U_0), f^{-1}(y)) & \xrightarrow{\widetilde{\varphi}} & (f^{-1}(U_1), f^{-1}(y)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ (U_0, y) & \xrightarrow{\varphi} & (U_1, y) \end{array}$$

commutative, φ preserves the local orientation of the normal bundle of $\mathfrak{F}(f)$ at y, where f is a proper Thom map such that the fiber over y belongs to $\mathfrak{F}$, and U_i are open neighborhoods of y. We also call any fiber belonging to a weakly co-orientable C^0 equivalence class a *weakly co-orientable fiber*. In particular, if the codimension of $\mathfrak{F}$ coincides with the dimension of the target of f, then φ above should preserve the local orientation of the target at y.

Note that if $\mathfrak{F}$ is weakly co-orientable, then $\mathfrak{F}(f)$ has orientable normal bundle for every proper Thom map f. The author does not know whether the converse also holds or not.

Remark 11.2. Note that $\mathfrak{F}(f)$ is merely a C^0 submanifold of the target in general (see Lemma 8.1 and its proof) and we have to be careful when we talk about its normal bundle. However, as we have seen in the proof of Lemma 8.1, it is always locally flat and the orientability of its normal bundle is well-defined. For example, use the fact that $U_i \smallsetminus (U_i \cap \mathfrak{F}(f))$ is homotopy equivalent to $S^{\kappa-1} \times (U_i \cap \mathfrak{F}(f))$ for appropriate U_i, where κ is the codimension of $\mathfrak{F}$.

Note that a weakly co-orientable C^0 equivalence class $\mathfrak{F}$ of fibers has exactly two co-orientations corresponding to the two orientations of the normal bundle of $\mathfrak{F}(f)$ at a point y in the target, where f is a Thom map such that the fiber over y belongs to $\mathfrak{F}$ and that the target itself is a small neighborhood of y. When one of the co-orientations is fixed, we call it a *co-oriented* C^0 equivalence class of fibers.

Using the co-orientations, we can construct the universal complex of weakly co-orientable singular fibers with coefficients in $\mathbf{Z}$ as follows. Let us first fix a dimension pair (n, p) with $p - n = k$. For $\kappa \in \mathbf{Z}$, let $CO^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p), \rho^0_{n,p})$ be the free $\mathbf{Z}$-module consisting of all formal linear combinations with integer coefficients, which may possibly contain infinitely many terms, of the C^0 equivalence classes $\mathfrak{F}$ of weakly co-orientable and co-oriented fibers of proper Thom maps between manifolds of dimensions n and p with $\kappa(\mathfrak{F}) = \kappa$, where $\rho^0_{n,p}$ stands for the C^0 equivalence among the fibers of elements of $\mathcal{T}_{\mathrm{pr}}(n,p)$. In particular, $CO^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p), \rho^0_{n,p}) = 0$ for $\kappa > p$ and $\kappa < 0$. Here, we adopt the convention that -1 times a co-oriented C^0 equivalence class coincides with the C^0 equivalence class with reversed co-orientation. For two co-oriented C^0 equivalence classes of fibers $\mathfrak{F}$ and $\mathfrak{G}$ with $\kappa(\mathfrak{F}) = \kappa(\mathfrak{G}) - 1$, we define $[\mathfrak{F} : \mathfrak{G}] = n_{\mathfrak{F}}(\mathfrak{G}) \in \mathbf{Z}$, which is called the *incidence coefficient*, as in §8, where we take the co-orientations into account and the result is an integer. Then we define the homomorphism

$$\delta_\kappa : CO^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p), \rho^0_{n,p}) \to CO^{\kappa+1}(\mathcal{T}_{\mathrm{pr}}(n,p), \rho^0_{n,p})$$

by

$$\delta_\kappa(\mathfrak{F}) = \sum_{\kappa(\mathfrak{G}) = \kappa+1} [\mathfrak{F} : \mathfrak{G}]\mathfrak{G}, \tag{11.1}$$

for $\mathfrak{F}$ with $\kappa(\mathfrak{F}) = \kappa$. Note that the homomorphism δ_κ is well-defined (for details, see the remarks just after (9.1) in §9).

Then, we can prove $\delta_{\kappa+1}\circ\delta_\kappa = 0$ as in §9 or in [50, §8]. Therefore,

$$\mathcal{CO}(\mathcal{T}_{\mathrm{pr}}(n,p),\rho^0_{n,p}) = (CO^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p),\rho^0_{n,p}),\delta_\kappa)_\kappa$$

constitutes a complex and its cohomology groups

$$H^*(\mathcal{CO}(\mathcal{T}_{\mathrm{pr}}(n,p),\rho^0_{n,p}))$$

are well-defined. We call the complex the *universal complex of weakly co-orientable singular fibers for proper Thom maps between manifolds of dimensions n and p with respect to the C^0 equivalence.*

Remark 11.3. Let $\mathfrak{F}$ be a weakly co-orientable and co-oriented C^0 equivalence class of fibers and $\mathfrak{G}$ a C^0 equivalence class of fibers such that $\kappa(\mathfrak{F}) = \kappa(\mathfrak{G})-1$. Then, by using a "local co-orientation" for $\mathfrak{G}$, we can define the incidence coefficient $[\mathfrak{F}:\mathfrak{G}]$ as an integer. If this integer is non-zero, then we can show that $\mathfrak{G}$ is also weakly co-orientable. In other words, if $\mathfrak{G}$ is not weakly co-orientable, then the incidence coefficient $[\mathfrak{F}:\mathfrak{G}]$ vanishes.

By restricting the class of Thom maps that we consider, we can also obtain the universal complex of weakly co-orientable singular fibers for such a class of maps (for details, refer to §9.6). Such a complex is a quotient complex of the above constructed universal complex (see Lemma 9.19).

Example 11.4. For proper C^0 stable maps of orientable 4-manifolds into 3-manifolds, we see easily that $\mathbf{0}_\ell$, I^0_ℓ, I^1_ℓ, II^{01}_ℓ, II^a_ℓ, III^{0a}_ℓ, III^{1a}_ℓ, and III^b_ℓ are weakly co-orientable for every ℓ, and that the others are not weakly co-orientable. Using these weakly co-orientable fibers, we can construct the universal complex of weakly co-orientable singular fibers with coefficients in $\mathbf{Z}$ as follows:

$$\begin{aligned}0 \longrightarrow CO^0(\mathcal{S}^0_{\mathrm{pr}}(4,3)^{\mathrm{ori}},\rho^0_{4,3}) &\xrightarrow{\delta_0} CO^1(\mathcal{S}^0_{\mathrm{pr}}(4,3)^{\mathrm{ori}},\rho^0_{4,3})\\ &\xrightarrow{\delta_1} CO^2(\mathcal{S}^0_{\mathrm{pr}}(4,3)^{\mathrm{ori}},\rho^0_{4,3}) \xrightarrow{\delta_2} CO^3(\mathcal{S}^0_{\mathrm{pr}}(4,3)^{\mathrm{ori}},\rho^0_{4,3}) \longrightarrow 0.\end{aligned}$$

We call this the *universal complex of weakly co-orientable singular fibers for proper C^0 stable maps of orientable 4-manifolds into 3-manifolds.*

By a method similar to that in §9.3, for an integer k, we can also define the universal complex $\mathcal{CO}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k),\mathcal{R}^0_k)$ of weakly co-orientable singular fibers for proper Thom maps of codimension k with respect to the stable system of C^0 equivalence relations as the projective limit of the complexes

$$\mathcal{CO}(\mathcal{T}_{\mathrm{pr}}(p-k,p),\rho^0_{p-k,p}),$$

where $\mathcal{R}^0_k = \{\rho^0_{p-k,p}\}_p$. We write the associated cohomology group of dimension κ by $H^\kappa(\mathcal{CO}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k),\mathcal{R}^0_k))$. As in §9.4, we can also give another description of this universal complex. Furthermore, we can also define a similar universal complex of weakly co-orientable singular fibers for a given class of Thom maps, and show that it is a quotient complex of the above constructed universal complex for Thom maps.

11.2. **Universal complex with respect to an admissible equivalence.** Now, let us fix an admissible equivalence relation $\rho_{n,p}$ among the fibers of proper Thom maps between smooth manifolds of dimensions n and p. The following definition strongly depends on $\rho_{n,p}$.

Definition 11.5. An equivalence class $\widetilde{\mathfrak{F}}$ of fibers of proper Thom maps with respect to $\rho_{n,p}$ is *co-orientable* (or *strongly co-orientable*) if for any homeomorphism $\varphi:(U_0,y)\to(U_1,y)$ such that $\varphi(\widetilde{\mathfrak{G}}(f)\cap U_0) = \widetilde{\mathfrak{G}}(f)\cap U_1$ for every equivalence class $\widetilde{\mathfrak{G}}$, φ preserves the local orientation of the normal bundle of $\widetilde{\mathfrak{F}}(f)$ at y, where

f is a proper Thom map such that the fiber over y belongs to $\widetilde{\mathfrak{F}}$, and U_i are open neighborhoods of y. (Note that by Lemma 8.3, $\widetilde{\mathfrak{F}}(f)$ is a C^0 submanifold of the target.) In particular, if the codimension of $\widetilde{\mathfrak{F}}$ coincides with the dimension of the target, then φ should preserve the local orientation of the target at y. Note that if $\widetilde{\mathfrak{F}}$ is co-orientable, then $\widetilde{\mathfrak{F}}(f)$ has orientable normal bundle for every proper Thom map f, while the converse may not hold in general.

Remark 11.6. When the admissible equivalence relation $\rho_{n,p}$ is given by the C^0 equivalence, i.e., when $\rho_{n,p} = \rho^0_{n,p}$, we can show that a C^0 equivalence class of fibers is weakly co-orientable if it is strongly co-orientable. The author does not know whether the converse also holds or not.

Using Definition 11.5, we can naturally define the *universal complex of co-orientable singular fibers for proper Thom maps between manifolds of dimensions n and p with respect to the admissible equivalence relation* $\rho_{n,p}$, and we denote it by $\mathcal{CO}(\mathcal{T}_{\mathrm{pr}}(n,p), \rho_{n,p})$. Note that its cochain group at each dimension is a free $\mathbf{Z}$-module. We denote the corresponding cohomology group of dimension κ by $H^\kappa(\mathcal{CO}(\mathcal{T}_{\mathrm{pr}}(n,p), \rho_{n,p}))$.

If we are given a stable system of admissible equivalence relations

$$\mathcal{R}_k = \{\rho_{p-k,p}\}_p$$

for the fibers of proper Thom maps of codimension k, then we can also define the corresponding universal complex $\mathcal{CO}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}_k)$ as the projective limit of the complexes $\mathcal{CO}(\mathcal{T}_{\mathrm{pr}}(p-k,p), \rho_{p-k,p})$ as in §9.3. We denote the corresponding cohomology group of dimension κ by $H^\kappa(\mathcal{CO}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}_k))$. Note that if the equivalence class of the suspension of a fiber whose equivalence class has codimension κ is co-orientable of codimension κ, then the original equivalence class is necessarily co-orientable, and hence the cochain map

$$\mathcal{CO}(\mathcal{T}_{\mathrm{pr}}(n+\ell, p+\ell), \rho_{n+\ell,p+\ell}) \to \mathcal{CO}(\mathcal{T}_{\mathrm{pr}}(n,p), \rho_{n,p})$$

is well-defined for every (n,p) with $p-n=k$ and $\ell > 0$. We can also give another description of the universal complex $\mathcal{CO}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}_k)$ as in §9.4.

Let us consider two admissible equivalence relations $\rho_{n,p}$ and $\overline{\rho}_{n,p}$ for the fibers of elements of $\mathcal{T}_{\mathrm{pr}}(n,p)$. The following lemma is a direct consequence of Definition 11.5.

Lemma 11.7. *Suppose $\rho_{n,p} \leq \overline{\rho}_{n,p}$. Furthermore, suppose that $\widetilde{\mathfrak{F}}$ and $\overline{\mathfrak{F}}$ are equivalence classes with respect to $\rho_{n,p}$ and $\overline{\rho}_{n,p}$ respectively such that $\widetilde{\mathfrak{F}} \supset \overline{\mathfrak{F}}$ and that they have the same codimension. If $\widetilde{\mathfrak{F}}$ is co-orientable, then so is $\overline{\mathfrak{F}}$.*

By virtue of the above lemma, the homomorphism

$$\varepsilon_{\rho_{n,p},\overline{\rho}_{n,p}} : \mathcal{CO}(\mathcal{T}_{\mathrm{pr}}(n,p), \rho_{n,p}) \to \mathcal{CO}(\mathcal{T}_{\mathrm{pr}}(n,p), \overline{\rho}_{n,p})$$

as in (9.7) is a well-defined cochain map. We can also define a similar cochain map for two stable systems of admissible equivalence relations.

Furthermore, as in §9.6, by restricting the class of Thom maps that we consider, we can also obtain the universal complex of co-orientable singular fibers for such a class of maps. Such a complex is a quotient complex of one of the above constructed universal complexes of co-orientable singular fibers for proper Thom maps.

11.3. **Universal complex of co-orientable singular fibers for stable maps of orientable 4-manifolds into 3-manifolds.** Now, let us consider proper C^0 stable maps of orientable 4-manifolds into 3-manifolds. By considering the C^0 equivalence modulo m circle components introduced in Definition 10.1, we get the corresponding universal complex of co-orientable singular fibers modulo m circle components. We denote the resulting complex by $\mathcal{CO}(\mathcal{S}^0_{\rm pr}(4,3)^{\rm ori}, \rho^0_{4,3}(m))$. For $m=2$, we see easily that $\mathbf{0}_\ell$, I^0_ℓ, I^1_ℓ, II^{01}_ℓ, II^a_ℓ, III^{0a}_ℓ, III^{1a}_ℓ, and III^b_ℓ are co-orientable for $\ell = \mathrm{o}, \mathrm{e}$, and that the others are not co-orientable, using the notation as in Proposition 10.4 (compare this with Example 11.4).

Then we easily get the following.

Proposition 11.8. *The cohomology groups of the universal complex*

$$\mathcal{CO}(\mathcal{S}^0_{\rm pr}(4,3)^{\rm ori}, \rho^0_{4,3}(m))$$

of co-orientable singular fibers modulo two circle components for proper C^0 stable maps of orientable 4-manifolds into 3-manifolds are given as follows:

$$\begin{aligned} H^0(\mathcal{CO}(\mathcal{S}^0_{\rm pr}(4,3)^{\rm ori}, \rho^0_{4,3}(m))) &\cong \mathbf{Z} \text{ (generated by } [\mathbf{0}_{\rm o} + \mathbf{0}_{\rm e}]), \\ H^1(\mathcal{CO}(\mathcal{S}^0_{\rm pr}(4,3)^{\rm ori}, \rho^0_{4,3}(m))) &\cong \mathbf{Z} \text{ (generated by } [\mathrm{I}^0_{\rm o} + \mathrm{I}^1_{\rm e}] = [\mathrm{I}^0_{\rm e} + \mathrm{I}^1_{\rm o}]), \\ H^2(\mathcal{CO}(\mathcal{S}^0_{\rm pr}(4,3)^{\rm ori}, \rho^0_{4,3}(m))) &= 0, \end{aligned}$$

where $[*]$ denotes the cohomology class represented by the cocycle $*$.

Remark 11.9. As in Remark 10.5, if the target 3-manifold is orientable, then we can show that the integral homology class (of closed support) in the target 3-manifold represented by a cycle corresponding to a coboundary of the universal complex of co-orientable singular fibers (modulo m circle components) always vanishes. For $m=2$, the coboundary groups are generated by the cochains listed in Table 4.

κ	generator(s)
1	$(\mathrm{I}^0_{\rm e} + \mathrm{I}^1_{\rm o}) - (\mathrm{I}^0_{\rm o} + \mathrm{I}^1_{\rm e})$
2	$\mathrm{II}^{01}_{\rm o} + \mathrm{I}^{01}_{\rm e} + \mathrm{II}^a_{\rm e}$, $\mathrm{II}^{01}_{\rm o} + \mathrm{II}^{01}_{\rm e} + \mathrm{II}^a_{\rm o}$
3	$\mathrm{III}^{0a}_{\rm o} + \mathrm{III}^{1a}_{\rm e} + \mathrm{III}^b_{\rm e}$, $\mathrm{III}^{0a}_{\rm e} + \mathrm{III}^{1a}_{\rm o} + \mathrm{III}^b_{\rm o}$

TABLE 4. Generators for the coboundary groups of $\mathcal{CO}(\mathcal{S}^0_{\rm pr}(4,3)^{\rm ori}, \rho^0_{4,3}(2))$

Thus, by the same reason as in Remark 10.6, we get the following proposition, where for an equivalence class of fibers, we use the same notation as the equivalence class of its suspension.

Proposition 11.10. (1) *Let $f : M \to N$ be a C^0 stable map of a closed orientable surface into a 1-dimensional manifold N. Then we have*

$$||\mathrm{I}^0_{\rm e}(f)|| + ||\mathrm{I}^1_{\rm o}(f)|| = ||\mathrm{I}^0_{\rm o}(f)|| + ||\mathrm{I}^1_{\rm e}(f)||,$$

where for a co-oriented equivalence class $\widetilde{\mathfrak{F}}$ of fibers, $||\widetilde{\mathfrak{F}}(f)||$ denotes the algebraic number of fibers of f of type $\widetilde{\mathfrak{F}}$.

(2) *Let $f : M \to N$ be a C^0 stable map of a closed orientable 3-manifold into an orientable surface N. Then we have*

$$||\mathrm{II}^{01}_{\rm o}(f)|| + ||\mathrm{II}^{01}_{\rm e}(f)|| + ||\mathrm{II}^a_{\rm e}(f)|| = ||\mathrm{II}^{01}_{\rm o}(f)|| + ||\mathrm{II}^{01}_{\rm e}(f)|| + ||\mathrm{II}^a_{\rm o}(f)|| = 0.$$

(3) *Let $f : M \to N$ be a C^0 stable map of a closed orientable 4-manifold into an orientable 3-manifold N. Then we have*

$$||\mathrm{III}^{0a}_{\rm o}(f)|| + ||\mathrm{III}^{1a}_{\rm e}(f)|| + ||\mathrm{III}^b_{\rm e}(f)|| = ||\mathrm{III}^{0a}_{\rm e}(f)|| + ||\mathrm{III}^{1a}_{\rm o}(f)|| + ||\mathrm{III}^b_{\rm o}(f)|| = 0.$$

Let us pose the following problem concerning §§9, 10 and 11.

Problem 11.11. Let us consider the homology class in the target represented by a cycle corresponding to a cocycle of the universal complex of (co-orientable) singular fibers representing a nontrivial cohomology class of the complex. Can it be written as a polynomial of some characteristic classes as in [50, 19, 33]? Can we find such a polynomial which is universal in a certain sense, like Thom polynomials for singularities? (For Thom polynomials, see [2, 16] for example.)

Note that the above problem is closely related to the homomorphism which will be defined in §12.

Remark 11.12. We have not included the calculation of the third cohomology groups of the universal complexes of (co-orientable) singular fibers modulo two circle components for proper C^0 stable maps of orientable 4-manifolds into 3-manifolds, since the corresponding complex terminates essentially at dimension three. In order to calculate the third cohomology groups which make sense, we have to calculate the third cohomology group of the complex $\mathcal{C}(\mathcal{S}^0_{\mathrm{pr}}(5,4)^{\mathrm{ori}}, \rho^0_{5,4}(2))$ (or $\mathcal{CO}(\mathcal{S}^0_{\mathrm{pr}}(5,4)^{\mathrm{ori}}, \rho^0_{5,4}(2))$). In other words, we have to classify the singular fibers of proper C^0 stable maps of orientable 5-manifolds into 4-manifolds.

Nevertheless, Theorem 6.1 indicates that the singular fiber III^8 might represent a generator of the third cohomology group and that the corresponding homology class for proper C^0 stable maps of orientable 4-manifolds into 3-manifolds can be written in terms of a polynomial of Stiefel-Whitney classes. In fact, this will be shown to be correct in §13 by using Theorem 6.1 (see Corollary 13.12).

In the case where the dimensions are smaller by two, we can check that the above expectations are affirmative as follows. As we have seen in Proposition 10.7, the singular fiber $\widetilde{\mathrm{I}}^2$ (or more precisely, the cocycle $\widetilde{\mathrm{I}}^2_{\mathrm{o}} + \widetilde{\mathrm{I}}^2_{\mathrm{e}}$) represents a generator of the first cohomology group of the universal complex $\mathcal{C}(\mathcal{S}^0_{\mathrm{pr}}(3,2), \rho^0_{3,2}(2))$ of singular fibers for proper C^0 stable maps of (not necessarily orientable) 3-manifolds into surfaces with respect to the C^0 equivalence modulo two circle components. Furthermore, by Corollary 3.4, the corresponding 0-dimensional homology class for Morse functions on surfaces is nothing but the Euler characteristic modulo two of the source surface, which coincides with its top Stiefel-Whitney class (for a more precise argument, see §15).

The following problem is closely related to Problem 5.6. See also Problem 12.14.

Problem 11.13. For each generator of the cohomology groups of the universal complex of (co-orientable) singular fibers for proper C^0 stable maps of orientable 4-manifolds into 3-manifolds, does there exist a C^0 stable map of a closed orientable 4-manifold into a 3-manifold whose corresponding cycle represents a nonzero element in the homology of the target?

Compare the above problem with [50, §17].

12. Homomorphism induced by a Thom map

In this section, we show that one can obtain a lot of information on the cohomology groups of the universal complexes of singular fibers by using concrete examples of Thom maps.

Let $\Gamma = \Gamma_{n,p}$ be a subset of $\mathcal{T}_{\mathrm{pr}}(n,p)$ and $\rho^\Gamma = \rho^\Gamma_{n,p}$ an admissible equivalence relation among the fibers of elements of $\Gamma_{n,p}$.

Definition 12.1. Let

$$c = \sum_{\kappa(\widetilde{\mathfrak{F}})=\kappa} n_{\widetilde{\mathfrak{F}}} \widetilde{\mathfrak{F}}$$

be a κ-dimensional cocycle of the complex $\mathcal{C}(\Gamma_{n,p}, \rho_{n,p}^{\Gamma})$, where $n_{\widetilde{\mathfrak{F}}} \in \mathbf{Z}_2$. For a Thom map $f : M \to N$ which is an element of $\Gamma^* = \Gamma_{n,p}^*$, we define $c(f)$ to be the closure of the set of points $y \in N$ such that the fiber over y belongs to some $\widetilde{\mathfrak{F}}$ with $n_{\widetilde{\mathfrak{F}}} \neq 0$. Since c is a cocycle, $c(f)$ is a $\mathbf{Z}_2$-cycle of closed support of codimension κ of the target manifold N. When M is closed and $\kappa > 0$, $c(f)$ is a $\mathbf{Z}_2$-cycle in the usual sense.

When c is a cocycle of the complex $\mathcal{CO}(\Gamma_{n,p}, \rho_{n,p}^{\Gamma})$, $c(f)$ is naturally a $\mathbf{Z}$-cycle, provided that the target manifold N is oriented.

Lemma 12.2. *If c and c' are κ-dimensional cocycles of the complex $\mathcal{C}(\Gamma_{n,p}, \rho_{n,p}^{\Gamma})$ which are cohomologous, then $c(f)$ and $c'(f)$ are homologous in N for every $f \in \Gamma_{n,p}^*$.*

Proof. There exists a $(\kappa-1)$-dimensional cochain d of the complex such that $c - c' = \delta_{\kappa-1} d$. Then we see easily that $c(f) - c'(f) = \partial d(f)$, where $d(f)$ is defined similarly. Hence the result follows. $\square$

Note that a similar result holds also for cocycles of the universal complex of co-orientable singular fibers.

Definition 12.3. Let α be a κ-dimensional cohomology class of the complex $\mathcal{C}(\Gamma_{n,p}, \rho_{n,p}^{\Gamma})$. For a proper Thom map $f : M \to N$ which is an element of $\Gamma_{n,p}^*$, we define $\alpha(f) \in H_{p-\kappa}^c(N; \mathbf{Z}_2)$ to be the homology class represented by the cycle $c(f)$ of closed support, where c is a cocycle representing α and $p = \dim N$. By Lemma 12.2, this is well-defined. When M is closed and $\kappa > 0$, we can also regard $\alpha(f)$ as an element of $H_{p-\kappa}(N; \mathbf{Z}_2)$.

Then we can define the map

$$\varphi_f : H^{\kappa}(\Gamma_{n,p}, \rho_{n,p}^{\Gamma}) \to H^{\kappa}(N; \mathbf{Z}_2)$$

by $\varphi_f(\alpha) = \alpha(f)^*$, where $\alpha(f)^* \in H^{\kappa}(N; \mathbf{Z}_2)$ is the Poincaré dual to $\alpha(f) \in H_{p-\kappa}^c(N; \mathbf{Z}_2)$. This is clearly a homomorphism and we call it the *homomorphism induced by the Thom map f*. When M is closed and $\kappa > 0$, we can also regard φ_f as a homomorphism into the cohomology group $H_c^{\kappa}(N; \mathbf{Z}_2)$ of compact support.

When the target manifold N is oriented, we can define

$$\varphi_f : H^{\kappa}(\mathcal{CO}(\Gamma_{n,p}, \rho_{n,p}^{\Gamma})) \to H^{\kappa}(N; \mathbf{Z})$$

similarly.

Suppose that $\rho^{\Gamma} = \rho_{n,p}^{\Gamma_{n,p}}$ and $\overline{\rho}^{\Gamma} = \overline{\rho}_{n,p}^{\Gamma_{n,p}}$ be two admissible equivalence relations for the fibers of elements of $\Gamma = \Gamma_{n,p} \subset \mathcal{T}_{\mathrm{pr}}(n,p)$ such that $\rho_{n,p}^{\Gamma_{n,p}} \leq \overline{\rho}_{n,p}^{\Gamma_{n,p}}$. Then the following diagram is clearly commutative for every element $f : M \to N$ of Γ^*:

$$\begin{tikzcd} H^{\kappa}(\Gamma, \rho^{\Gamma}) \arrow[rr, "\varepsilon_{\rho^{\Gamma}, \overline{\rho}^{\Gamma} *}"] \arrow[dr, "\varphi_f"'] & & H^{\kappa}(\Gamma, \overline{\rho}^{\Gamma}) \arrow[dl, "\varphi_f"] \\ & H^{\kappa}(N; \mathbf{Z}_2), & \end{tikzcd}$$

where $\varepsilon_{\rho^{\Gamma}, \overline{\rho}^{\Gamma} *} : H^{\kappa}(\Gamma, \rho^{\Gamma}) \to H^{\kappa}(\Gamma, \overline{\rho}^{\Gamma})$ is the homomorphism induced by the natural homomorphism $\varepsilon_{\rho^{\Gamma}, \overline{\rho}^{\Gamma}} : \mathcal{C}(\Gamma, \rho^{\Gamma}) \to \mathcal{C}(\Gamma, \overline{\rho}^{\Gamma})$ defined in §§9.5 and 9.6.

Furthermore, if $\Gamma \subset \Gamma' \subset \mathcal{T}_{\mathrm{pr}}(n,p)$, then for every element $f : M \to N$ of Γ^*, we have the commutative diagram

$$\begin{tikzcd} H^{\kappa}(\Gamma', \rho^{\Gamma'}) \arrow[rr, "\pi_{\Gamma', \Gamma *}"] \arrow[dr, "\varphi_f"'] & & H^{\kappa}(\Gamma, \rho^{\Gamma}) \arrow[dl, "\varphi_f"] \\ & H^{\kappa}(N; \mathbf{Z}_2), & \end{tikzcd}$$

where $\rho^{\Gamma'}$ is an admissible equivalence relation among the fibers of elements of Γ', ρ^{Γ} is its restriction to the fibers of elements of Γ, and $\pi_{\Gamma',\Gamma *}$ is the homomorphism induced by the natural homomorphism $\pi_{\Gamma',\Gamma} : \mathcal{C}(\Gamma', \rho^{\Gamma'}) \to \mathcal{C}(\Gamma, \rho^{\Gamma})$ defined in §9.6.

In particular, we have the commutative diagram

$$\begin{array}{ccccc} H^{\kappa}(\Gamma_{n,p}, \rho^{\Gamma}_{n,p}) & & \rightarrow & & H^{\kappa}(\Gamma_{n,p}, \rho^{0}_{n,p}) \\ & {\scriptstyle \varphi_f}\searrow & & \swarrow{\scriptstyle \varphi_f} & \\ \downarrow & & H^{\kappa}(N; \mathbf{Z}_2) & & \downarrow \\ & {\scriptstyle \varphi_f}\nearrow & & \nwarrow{\scriptstyle \varphi_f} & \\ H^{\kappa}(f, \rho^{\Gamma}_{n,p}) & & \rightarrow & & H^{\kappa}(f, \rho^{0}_{n,p}) \end{array}$$

for every element $f : M \to N$ of $\Gamma^*_{n,p}$, where $\Gamma_{n,p} \subset \mathcal{T}_{\mathrm{pr}}(n,p)$, $\rho^{\Gamma}_{n,p}$ is an admissible equivalence relation for the fibers of elements of $\Gamma_{n,p}$, $\rho^0_{n,p}$ denotes the C^0 equivalence, and the vertical and the horizontal homomorphisms are the natural ones defined as above (see also (9.8)).

Now, as in §9.6, let

$$\widetilde{\Gamma} = \widetilde{\Gamma}_k = \bigcup_{p-n=k} \Gamma_{n,p}$$

be a set of C^0 equivalence classes of fibers of proper Thom maps of codimension k such that each $\Gamma_{n,p}$ is an ascending set of C^0 equivalence classes of fibers of elements of $\mathcal{T}_{\mathrm{pr}}(n,p)$, and that $\widetilde{\Gamma}$ is closed under suspension in the sense of Definition 9.4. Furthermore, let $\mathcal{R}^{\widetilde{\Gamma}}_k = \{\rho^{\Gamma_{p-k,p}}_{p-k,p}\}_p$ be a system of equivalence relations, where each $\rho^{\Gamma_{p-k,p}}_{p-k,p}$ is an admissible equivalence relation among the fibers of $\Gamma_{p-k,p}$-maps. We assume that the system $\mathcal{R}^{\widetilde{\Gamma}}_k$ of admissible equivalence relations is stable in the sense of Definition 9.5. Then, for every $f : M \to N$ in $\Gamma^*_{n,p}$ with $p - n = k$, we have the natural and well-defined homomorphism

$$\widetilde{\varphi}_f : H^{\kappa}(\widetilde{\Gamma}_k, \mathcal{R}^{\widetilde{\Gamma}}_k) \to H^{\kappa}(N; \mathbf{Z}_2)$$

which is defined as the composition of the homomorphism

$$\Phi^{\kappa}_{n,p*} : H^{\kappa}(\widetilde{\Gamma}_k, \mathcal{R}^{\widetilde{\Gamma}}_k) \to H^{\kappa}(\Gamma_{n,p}, \rho^{\Gamma_{n,p}}_{n,p})$$

induced by the cochain map $\Phi^{\kappa}_{n,p} : C^{\kappa}(\widetilde{\Gamma}_k, \mathcal{R}^{\widetilde{\Gamma}}_k) \to C^{\kappa}(\Gamma_{n,p}, \rho^{\Gamma_{n,p}}_{n,p})$ as in Remark 9.20 and the homomorphism

$$\varphi_f : H^{\kappa}(\Gamma_{n,p}, \rho^{\Gamma_{n,p}}_{n,p}) \to H^{\kappa}(N; \mathbf{Z}_2)$$

defined above. We can also use the other description of the universal complex as in §9.4 and the definition as in Definition 12.3 in order to define $\widetilde{\varphi}_f$.

Remark 12.4. In the above situation, it is easy to verify that for every $f : M \to N$ in $\Gamma^*_{n,p}$ and a positive integer ℓ, the following diagram is commutative:

$$\begin{array}{ccc} H^{\kappa}(\Gamma_{n+\ell,p+\ell}, \rho^{\Gamma_{n+\ell,p+\ell}}_{n+\ell,p+\ell}) & \xrightarrow{s_{\kappa *}} & H^{\kappa}(\Gamma_{n,p}, \rho^{\Gamma_{n,p}}_{n,p}) \\ {\scriptstyle \varphi_{\tilde{f}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \varphi_f} \\ H^{\kappa}(N \times \mathbf{R}^{\ell}; \mathbf{Z}_2) & \cong & H^{\kappa}(N; \mathbf{Z}_2), \end{array}$$

where $s_{\kappa *}$ is the homomorphism induced by the suspension, $\tilde{f} : M \times \mathbf{R}^{\ell} \to N \times \mathbf{R}^{\ell}$ is the ℓ-th suspension of f, and the last horizontal isomorphism is the natural one.

As a direct consequence of the above definitions, we have the following.

Proposition 12.5. *In the above situations, we have*

$$\mathrm{rank}_{\mathbf{Z}_2} \varphi_f \leq \dim_{\mathbf{Z}_2} H^{\kappa}(\Gamma_{n,p}, \rho^{\Gamma_{n,p}}_{n,p})$$

and

$$\mathrm{rank}_{\mathbf{Z}_2} \widetilde{\varphi}_f \leq \dim_{\mathbf{Z}_2} H^{\kappa}(\widetilde{\Gamma}_k, \mathcal{R}^{\widetilde{\Gamma}}_k).$$

Remark 12.6. It is sometimes difficult to directly calculate the cohomology group $H^*(\Gamma_{n,p}, \rho^0_{n,p})$ with respect to the C^0 equivalence. However, the above argument shows that if we have an element α of the cohomology group $H^*(\Gamma_{n,p}, \rho_{n,p}^{\Gamma_{n,p}})$ with respect to an admissible equivalence relation $\rho_{n,p}^{\Gamma_{n,p}}$ such that $\varphi_f(\alpha) \neq 0$ for some $f \in \Gamma^*_{n,p}$, then the image of α in $H^*(\Gamma_{n,p}, \rho^0_{n,p})$ does not vanish. In other words, by calculating the cohomology group with respect to an admissible equivalence relation, which is often much easier than that with respect to the C^0 equivalence, and by constructing explicit examples, we can find nontrivial elements of the cohomology group with respect to the C^0 equivalence. This justifies our study developed in §§10 and 7.

Let us prepare some lemmas, which will be used later. For this, let us introduce the following definitions.

Definition 12.7. Let $f : M \to N$ be a proper Thom map and $g : V \to N$ a smooth map which is transverse to f and to all the strata of N. Put

$$\widetilde{V} = \{(x, y) \in M \times V \, : \, f(x) = g(y)\} \subset M \times V$$

and consider the following commutative diagram:

$$\begin{array}{ccc} \widetilde{V} & \xrightarrow{\widetilde{g}} & M \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{g} & N, \end{array}$$

where $\widetilde{g}$ and $\widetilde{f}$ are the restrictions of the projections to the first and the second factors respectively. Note that $\widetilde{V}$ is a smooth manifold of dimension $\dim V + \dim M - \dim N$ and that $\widetilde{f}$ is a proper Thom map. We call $\widetilde{f}$ the *pull-back of f by g* and say that *$\widetilde{f}$ is obtained by pulling back f by g*.

Definition 12.8. Suppose that $\Gamma_{n,p} \subset \mathcal{T}_{\mathrm{pr}}(n, p)$ and $\Gamma_{n+\ell,p+\ell} \subset \mathcal{T}_{\mathrm{pr}}(n + \ell, p + \ell)$ are given with $\ell > 0$ such that the ℓ-th suspension of an element of $\Gamma_{n,p}$ always belong to $\Gamma_{n+\ell,p+\ell}$. Let $f : M \to N$ be an arbitrary element of $\Gamma_{n+\ell,p+\ell}$ and $g : \operatorname{Int} D^p \to N$ an arbitrary smooth map which is transverse to f and to all the strata of N. Note that the pull-back $\widetilde{f}$ of f by g is then an element of $\mathcal{T}_{\mathrm{pr}}(n, p)$. If the fibers of g always belong to $\Gamma^*_{n,p}$, then we say that $\Gamma_{n,p}$ is *transversely complete with respect to $\Gamma_{n+\ell,p+\ell}$*.

Furthermore, we say that

$$\widetilde{\Gamma}_k = \bigcup_{p-n=k} \Gamma_{n,p} \subset \widetilde{\mathcal{T}}_{\mathrm{pr}}(k)$$

is *transversely complete* if it is closed under suspension and if $\Gamma_{n,p}$ is transversely complete with respect to $\Gamma_{n+\ell,p+\ell}$ for all n, p and ℓ.

Note that the set $\widetilde{\mathcal{T}}_{\mathrm{pr}}(k)$ is clearly transversely complete.

The following lemma can be proved by the same argument as in the proof of Lemma 9.6. Details are left to the reader.

Lemma 12.9. *If $\Gamma_{n,p}$ is transversely complete with respect to $\Gamma_{n+\ell,p+\ell}$, then the natural $\mathbf{Z}_2$-linear map*

$$s_\kappa : C^\kappa(\Gamma_{n+\ell,p+\ell}, \rho_{n+\ell,p+\ell}^{\Gamma_{n+\ell,p+\ell}}) \to C^\kappa(\Gamma_{n,p}, \rho_{n,p}^{\Gamma_{n,p}})$$

induced by the suspension is a monomorphism for every $\kappa \leq p$, where $\rho_{n+\ell,p+\ell}^{\Gamma_{n+\ell,p+\ell}}$ and $\rho_{n,p}^{\Gamma_{n,p}}$ are admissible equivalence relations for the fibers of elements of $\Gamma_{n+\ell,p+\ell}$ and $\Gamma_{n,p}$, respectively, which are stable in a sense similar to Definition 9.5.

In the following lemma, we assume that each $\Gamma_{n,p}$, $p-n=k$, is a subset of $\mathcal{T}_{\rm pr}(n,p)$ and that $\widetilde{\Gamma}_k = \cup_{p-n=k}\Gamma_{n,p}$ is closed under suspension. Furthermore, $\{\rho_{n,p}^{\Gamma_{n,p}}\}_{p-n=k}$ is a stable system of admissible equivalence relations for the fibers of elements of $\widetilde{\Gamma}_k$, where each $\rho_{n,p}^{\Gamma_{n,p}}$ is an admissible equivalence relation for $\Gamma_{n,p}^*$. Recall that $\Gamma_{n,p}^*$ denotes the set of C^0 equivalence classes of fibers of elements of $\Gamma_{n,p}$ and, when there is no confusion, it also denotes the set of all $\Gamma_{n,p}^*$-maps (see §9.6).

Lemma 12.10. *Let $\alpha \in H^p(\Gamma_{n+\ell,p+\ell}, \rho_{n+\ell,p+\ell}^{\Gamma_{n+\ell,p+\ell}})$ be a cohomology class such that $\varphi_f(s_{p*}\alpha) = 0$ in $H^p(N;\mathbf{Z}_2)$ for every Thom map $f : M \to N$ in $\Gamma_{n,p}^*$ with both M and N being closed, where*

$$s_{p*} : H^p(\Gamma_{n+\ell,p+\ell}, \rho_{n+\ell,p+\ell}^{\Gamma_{n+\ell,p+\ell}}) \to H^p(\Gamma_{n,p}, \rho_{n,p}^{\Gamma_{n,p}})$$

is the homomorphism induced by the suspension, and

$$\varphi_f : H^p(\Gamma_{n,p}, \rho_{n,p}^{\Gamma_{n,p}}) \to H^p(N;\mathbf{Z}_2)$$

is the homomorphism induced by f. Furthermore, suppose that $\Gamma_{n,p}$ is transversely complete with respect to $\Gamma_{n+\ell,p+\ell}$. Then, for every $g : M' \to N'$ in $\Gamma_{n+\ell,p+\ell}^*$, we have $\varphi_g(\alpha) = 0$ in $H^p(N';\mathbf{Z}_2)$.

Proof. Let c be a cocycle of $C^p(\Gamma_{n+\ell,p+\ell}, \rho_{n+\ell,p+\ell}^{\Gamma_{n+\ell,p+\ell}})$ which represents α. We have only to show that the homology class $[c(g)] \in H_\ell^c(N';\mathbf{Z}_2)$ represented by $c(g)$ vanishes. For this, it suffices to prove that the intersection number $[c(g)]\cdot\xi$ vanishes for all $\xi \in H_p(N';\mathbf{Z}_2)$ by Poincaré duality.

By [48], there exists a closed p-dimensional manifold V and a smooth map $h : V \to N'$ such that $h_*[V]_2 = \xi$, where $[V]_2 \in H_p(V;\mathbf{Z}_2)$ is the fundamental class of V. We may assume that h is transverse to g and to all the strata of N'. Let us consider the pull-back $\widetilde{g} : \widetilde{V} \to V$ of g by h (see Definition 12.7). Since $\Gamma_{n,p}$ is transversely complete with respect to $\Gamma_{n+\ell,p+\ell}$ by our assumption, we see that $\widetilde{g}$ is an element of $\Gamma_{n,p}^*$. Furthermore, both the source and the target manifolds of $\widetilde{g}$ are closed. Therefore, by our assumption, $\varphi_{\widetilde{g}}(s_{p*}\alpha) = 0$; in other words, $(s_p c)(\widetilde{g})$ consists of an even number of points in V. Hence, the intersection number of $[c(g)]$ and ξ vanishes. This completes the proof. $\square$

In fact, we have the following.

Lemma 12.11. *Suppose that $\Gamma_{n,p}$ is transversely complete with respect to $\Gamma_{n+\ell,p+\ell}$. Then, for a cohomology class $\alpha \in H^p(\Gamma_{n+\ell,p+\ell}, \rho_{n+\ell,p+\ell}^{\Gamma_{n+\ell,p+\ell}})$, the following two are equivalent to each other.*

1. *For every proper Thom map $f : M \to N$ in $\Gamma_{n,p}^*$, we have $\varphi_f(s_{p*}\alpha) = 0$ in $H^p(N;\mathbf{Z}_2)$.*
2. *For every proper Thom map $g : M' \to N'$ in $\Gamma_{n+\ell,p+\ell}^*$, we have $\varphi_g(\alpha) = 0$ in $H^p(N';\mathbf{Z}_2)$.*

Proof. We have already proved that (1) implies (2). Suppose that (2) holds. For a given proper Thom map $f : M \to N$ in $\Gamma_{n,p}^*$, consider the commutative diagram given in Remark 12.4 with $\kappa = p$. Since the ℓ-th suspension $\widetilde{f} : M \times \mathbf{R}^\ell \to N \times \mathbf{R}^\ell$ is a $\Gamma_{n+\ell,p+\ell}^*$-map, we have $\varphi_{\widetilde{f}}(\alpha) = 0$ by our assumption. Hence (1) follows. $\square$

Remark 12.12. In fact, we can prove the following, without assuming that $\Gamma_{n,p}$ is transversely complete with respect to $\Gamma_{n+\ell,p+\ell}$. For a cohomology class $\beta \in \operatorname{Im} s_{p*}$, the following two are equivalent to each other, where

$$s_{p*} : H^p(\Gamma_{n+\ell,p+\ell}, \rho_{n+\ell,p+\ell}^{\Gamma_{n+\ell,p+\ell}}) \to H^p(\Gamma_{n,p}, \rho_{n,p}^{\Gamma_{n,p}})$$

is the homomorphism induced by the suspension.

(1) For every proper Thom map $f : M \to N$ in $\Gamma^*_{n,p}$, we have $\varphi_f(\beta) = 0$ in $H^p(N; \mathbf{Z}_2)$.
(2) For every Thom map $f : M \to N$ in $\Gamma^*_{n,p}$ with both M and N being closed, we have $\varphi_f(\beta) = 0$ in $H^p(N; \mathbf{Z}_2)$.

The proof goes as follows. Suppose (2) holds and take f as in (1). Consider the commutative diagram given in Remark 12.4 with $\kappa = p$. Then apply an argument similar to that in the proof of Lemma 12.10, using a smooth map h of a closed p-dimensional manifold into $N \times \mathbf{R}^\ell$. Since $\widetilde{f}$ is the ℓ-th suspension of f, the pull-back of $\widetilde{f}$ by h is a $\Gamma^*_{n,p}$-map. Hence, from (2), (1) follows.

Note that all the results in this section hold also for the universal complexes of co-orientable singular fibers and the cohomology groups with $\mathbf{Z}$-coefficients, provided that the target manifolds are oriented.

Problem 12.13. Are the cohomology groups

$$H^\kappa(\mathcal{T}_{\mathrm{pr}}(n,p), \rho^0_{n,p}), \tag{12.1}$$
$$H^\kappa(\mathcal{S}^0_{\mathrm{pr}}(n,p), \rho^0_{n,p}), \tag{12.2}$$
$$H^\kappa(\mathcal{CO}(\mathcal{T}_{\mathrm{pr}}(n,p), \rho^0_{n,p})), \tag{12.3}$$
$$H^\kappa(\mathcal{CO}(\mathcal{S}^0_{\mathrm{pr}}(n,p), \rho^0_{n,p})), \tag{12.4}$$
$$H^\kappa(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}^0_k), \tag{12.5}$$
$$H^\kappa(\widetilde{\mathcal{S}}^0_{\mathrm{pr}}(k), \mathcal{R}^0_k), \tag{12.6}$$
$$H^\kappa(\mathcal{CO}(\widetilde{\mathcal{T}}_{\mathrm{pr}}(k), \mathcal{R}^0_k)), \tag{12.7}$$
$$H^\kappa(\mathcal{CO}(\widetilde{\mathcal{S}}^0_{\mathrm{pr}}(k), \mathcal{R}^0_k)) \tag{12.8}$$

finitely generated for all κ?

The following is a generalization of Problem 11.13.

Problem 12.14. Let α be an element of one of the cohomology groups (12.1)–(12.8). If $\alpha \neq 0$, then does there exist a smooth map f (in the relevant class) such that $\varphi_f(\alpha)$ does not vanish? In other words, if $\varphi_f(\alpha) = 0$ for all f, then does α vanish?

As to an interpretation of the above problem, see Remark 13.14.

13. Cobordism invariance

In this section, we define cobordisms of singular maps with a given set of singular fibers and show that the homomorphism φ_f induced by a Thom map f defined in §12 is a cobordism invariant of f when restricted to a certain subgroup. We also apply this notion of cobordisms to give a necessary and sufficient condition for a certain cochain of the universal complex to be a cocycle.

13.1. **Invariance under cobordism of the homomorphism induced by a specific singular map.**
As in §9.6, let

$$\widetilde{\Gamma} = \widetilde{\Gamma}_k = \bigcup_{p-n=k} \Gamma_{n,p}$$

be a set of C^0 equivalence classes of fibers of proper Thom maps of codimension k such that each $\Gamma_{n,p}$ is an ascending set of C^0 equivalence classes of fibers of elements of $\mathcal{T}_{\mathrm{pr}}(n,p)$, and that $\widetilde{\Gamma}$ is closed under suspension in the sense of Definition 9.4. Recall that a proper Thom map $f : M \to N$ of codimension k is a $\widetilde{\Gamma}_k$-map if its fibers all lie in $\widetilde{\Gamma}_k$. If M is a manifold with boundary, then we also suppose that $f(\partial M) \subset \partial N$ and for collar neighborhoods $C = \partial M \times [0,1)$ and $C' = \partial N \times [0,1)$ of ∂M and ∂N respectively, we have $f|_C = (f|_{\partial M}) \times \mathrm{id}_{[0,1)}$.

Definition 13.1. For a smooth manifold N, two $\widetilde{\Gamma}_k$-maps $f_0 : M_0 \to N$ and $f_1 : M_1 \to N$ of closed manifolds M_0 and M_1 are said to be *$\widetilde{\Gamma}_k$-cobordant* if there exist a compact manifold W with boundary the disjoint union of M_0 and M_1, and a $\widetilde{\Gamma}_k$-map $F : W \to N \times [0,1]$ such that $f_i = F|_{M_i} : M_i \to N \times \{i\}$, $i = 0, 1$. We call F a *$\widetilde{\Gamma}_k$-cobordism* between f_0 and f_1.

When M_i are oriented and W can be taken to be oriented so that $\partial W = (-M_0) \amalg M_1$, then we say that f_0 and f_1 are *oriented $\widetilde{\Gamma}_k$-cobordant*.

Remark 13.2. The notion of $\widetilde{\Gamma}_k$-maps and that of $\widetilde{\Gamma}_k$-cobordisms were essentially introduced by Rimányi and Szűcs [35], although they considered only the nonnegative codimension case and they called them τ-maps and τ-cobordisms respectively. Note that if the codimension is nonnegative, then a fiber of a proper generic map is always a finite set of points and that map-germs along the fibers are nothing but multi-germs. In the nonnegative codimension case, Rimányi and Szűcs constructed a universal $\widetilde{\Gamma}_k$-map and this gives rise to a lot of $\widetilde{\Gamma}_k$-cobordism invariants. Our aim in this section is to construct invariants of $\widetilde{\Gamma}_k$-cobordisms even in the negative codimension case.

Remark 13.3. In Definition 13.1, when the dimensions of the source manifolds M_0 and M_1 are equal to n, we have only to give $\Gamma_{n,p}$ and $\Gamma_{n+1,p+1}$ instead of the whole $\widetilde{\Gamma}_k$ in order to define the notion of $\widetilde{\Gamma}_k$-cobordisms. For this reason, we will sometimes talk about $\widetilde{\Gamma}_k$-cobordisms even when only $\Gamma_{n,p}$ and $\Gamma_{n+1,p+1}$ are given.

Let

$$s_{\kappa *} : H^\kappa(\Gamma_{n+1,p+1}, \rho_{n+1,p+1}^{\Gamma_{n+1,p+1}}) \to H^\kappa(\Gamma_{n,p}, \rho_{n,p}^{\Gamma_{n,p}})$$

be the homomorphism induced by the suspension, where $\mathcal{R}_k^{\widetilde{\Gamma}} = \{\rho_{p-k,p}^{\Gamma_{p-k,p}}\}_p$ is a stable system of admissible equivalence relations for $\widetilde{\Gamma}$.

Lemma 13.4. *Let $f_i : M_i \to N$, $i = 0, 1$, be Thom maps which are elements of $\Gamma_{n,p}$ and are $\widetilde{\Gamma}_k$-maps, where we assume that M_i are closed. If they are $\widetilde{\Gamma}_k$-cobordant, then for every κ we have*

$$\varphi_{f_0}|_{\operatorname{Im} s_{\kappa *}} = \varphi_{f_1}|_{\operatorname{Im} s_{\kappa *}} : \operatorname{Im} s_{\kappa *} \to H^\kappa(N; \mathbf{Z}_2).$$

Proof. Let $F : W \to N \times [0,1]$ be a $\widetilde{\Gamma}_k$-cobordism between f_0 and f_1. Let c be an arbitrary κ-dimensional cocycle of the complex $\mathcal{C}(\Gamma_{n+1,p+1}, \rho_{n+1,p+1}^{\Gamma_{n+1,p+1}})$ and set $\overline{c} = s_\kappa(c) \in C^\kappa(\Gamma_{n,p}, \rho_{n,p}^{\Gamma_{n,p}})$. Then we see easily that $\partial c(F) = \overline{c}(f_1) \times \{1\} - \overline{c}(f_0) \times \{0\}$, since c is a cocycle (for the notation, refer to Definition 12.1). Then the result follows immediately. □

Remark 13.5. In Lemma 13.4, if $\kappa \geq 1$, then φ_{f_i} can be regarded as homomorphisms into $H_c^\kappa(N; \mathbf{Z}_2)$, since M_i are closed. In this case, we can prove that

$$\varphi_{f_0}|_{\operatorname{Im} s_{\kappa *}} = \varphi_{f_1}|_{\operatorname{Im} s_{\kappa *}} : \operatorname{Im} s_{\kappa *} \to H_c^\kappa(N; \mathbf{Z}_2).$$

Definition 13.6. The pairs $\{\Gamma_{n+1,p+1}, \rho_{n+1,p+1}^{\Gamma_{n+1,p+1}}\}$ and $\{\Gamma_{n,p}, \rho_{n,p}^{\Gamma_{n,p}}\}$ are said to be *compatible at dimension κ* if the homomorphism

$$s_{\kappa *} : H^\kappa(\Gamma_{n+1,p+1}, \rho_{n+1,p+1}^{\Gamma_{n+1,p+1}}) \to H^\kappa(\Gamma_{n,p}, \rho_{n,p}^{\Gamma_{n,p}})$$

is surjective.

Lemma 13.7. *The pairs $\{\Gamma_{n+1,p+1}, \rho_{n+1,p+1}^{\Gamma_{n+1,p+1}}\}$ and $\{\Gamma_{n,p}, \rho_{n,p}^{\Gamma_{n,p}}\}$ are compatible at dimension κ if the following conditions hold.*

(1) *Every fiber in $\Gamma_{n+1,p+1}$ of codimension $\kappa + 1$ with respect to $\rho_{n+1,p+1}^{\Gamma_{n+1,p+1}}$ is a suspension of a fiber in $\Gamma_{n,p}$ of the same codimension with respect to $\rho_{n,p}^{\Gamma_{n,p}}$.*

(2) *If an equivalence class of fibers in* $\Gamma_{n,p}$ *with respect to* $\rho_{n,p}^{\Gamma_{n,p}}$ *has codimension* κ*, then the equivalence class of their suspensions with respect to* $\rho_{n+1,p+1}^{\Gamma_{n+1,p+1}}$ *has also codimension* κ.
(3) *Two fibers in* $\Gamma_{n,p}$ *whose equivalence classes with respect to* $\rho_{n,p}^{\Gamma_{n,p}}$ *have codimension* κ *are equivalent with respect to* $\rho_{n,p}^{\Gamma_{n,p}}$ *if and only if their suspensions are equivalent with respect to* $\rho_{n+1,p+1}^{\Gamma_{n+1,p+1}}$.

Proof. If $\kappa < 0$ or $\kappa > p$, then the result is trivial. When $0 \le \kappa \le p$, by an argument similar to that in Remark 9.20, we see that if conditions (2) and (3) are satisfied, then

$$s_\kappa : C^\kappa(\Gamma_{n+1,p+1}, \rho_{n+1,p+1}^{\Gamma_{n+1,p+1}}) \to C^\kappa(\Gamma_{n,p}, \rho_{n,p}^{\Gamma_{n,p}})$$

is an epimorphism. Furthermore, condition (1) implies that $s_{\kappa+1}$ is a monomorphism (see also Remark 9.20). Thus the homomorphism $s_{\kappa *}$ induced on the κ-dimensional cohomology is an epimorphism, and hence the compatibility follows. □

Corollary 13.8. *Consider the case where* $\Gamma_{n,p} = \mathcal{T}_{\mathrm{pr}}(n,p)$ *for all* (n,p) *with* $p-n = k$*, and put* $\rho_{n,p} = \rho_{n,p}^{\Gamma_{n,p}}$*. We suppose that* $\kappa + 1 \le p$*. Then the pairs* $\{\mathcal{T}_{\mathrm{pr}}(n+1,p+1), \rho_{n+1,p+1}\}$ *and* $\{\mathcal{T}_{\mathrm{pr}}(n,p), \rho_{n,p}\}$ *are compatible at dimension* κ *if the following two conditions hold.*

(1) *If an equivalence class of fibers of elements of* $\mathcal{T}_{\mathrm{pr}}(n,p)$ *with respect to* $\rho_{n,p}$ *has codimension* κ*, then the equivalence class of their suspensions with respect to* $\rho_{n+1,p+1}$ *has also codimension* κ.
(2) *Two fibers of elements of* $\mathcal{T}_{\mathrm{pr}}(n,p)$ *whose equivalence classes with respect to* $\rho_{n,p}$ *have codimension* κ *are equivalent with respect to* $\rho_{n,p}$ *if and only if their suspensions are equivalent with respect to* $\rho_{n+1,p+1}$.

Proof. Recall that if $\kappa + 1 \le p$, then by Lemma 9.6,

$$s_{\kappa+1} : C^{\kappa+1}(\mathcal{T}_{\mathrm{pr}}(n+1,p+1), \rho_{n+1,p+1}) \to C^{\kappa+1}(\mathcal{T}_{\mathrm{pr}}(n,p), \rho_{n,p})$$

is always a monomorphism. Then the result follows from Lemma 13.7. □

Corollary 13.9. *Suppose that the pairs* $\{\Gamma_{n+1,p+1}, \rho_{n+1,p+1}^{\Gamma_{n+1,p+1}}\}$ *and* $\{\Gamma_{n,p}, \rho_{n,p}^{\Gamma_{n,p}}\}$ *are compatible at dimension* κ*. Let* $f_i : M_i \to N$*,* $i = 0, 1$*, be Thom maps which are elements of* $\Gamma_{n,p}$ *and are* $\widetilde{\Gamma}_k$*-maps, where we assume that* M_i *are closed. If they are* $\widetilde{\Gamma}_k$*-cobordant, then we have*

$$\varphi_{f_0} = \varphi_{f_1} : H^\kappa(\Gamma_{n,p}, \rho_{n,p}^{\Gamma_{n,p}}) \to H^\kappa(N; \mathbf{Z}_2).$$

If $\kappa \ge 1$*, then we also have*

$$\varphi_{f_0} = \varphi_{f_1} : H^\kappa(\Gamma_{n,p}, \rho_{n,p}^{\Gamma_{n,p}}) \to H_c^\kappa(N; \mathbf{Z}_2).$$

By using a natural generalization of Proposition 9.15 to certain subsets of $\widetilde{\mathcal{T}}_{\mathrm{pr}}(k)$ together with an argument similar to that in the proof of Lemma 13.4, we get the following as well.

Corollary 13.10. *Let* $\mathcal{R}_k^{\widetilde{\Gamma}} = \{\rho_{p-k,p}^{\Gamma_{p-k,p}}\}_p$ *be a stable system of admissible equivalence relations for* $\widetilde{\Gamma}$*. Let* $f_i : M_i \to N$*,* $i = 0, 1$*, be* $\widetilde{\Gamma}$*-maps with* $\dim M_i = n$ *and* $\dim N = p$*, where we assume that* M_i *are closed. If they are* $\widetilde{\Gamma}$*-cobordant, then for every* κ *we have*

$$\widetilde{\varphi}_{f_0} = \widetilde{\varphi}_{f_1} : H^\kappa(\widetilde{\Gamma}, \mathcal{R}_k^{\widetilde{\Gamma}}) \to H^\kappa(N; \mathbf{Z}_2).$$

If $\kappa \ge 1$*, then we also have*

$$\widetilde{\varphi}_{f_0} = \widetilde{\varphi}_{f_1} : H^\kappa(\widetilde{\Gamma}, \mathcal{R}_k^{\widetilde{\Gamma}}) \to H_c^\kappa(N; \mathbf{Z}_2).$$

When the manifold N is oriented, we get similar results in coefficients in $\mathbf{Z}$ by using the universal complex of co-orientable singular fibers. Details are left to the reader.

13.2. **A characterization of cocycles.** In this subsection, we shall give a necessary and sufficient condition for a certain cochain of the universal complex to be a cocycle in terms of the homomorphism induced by Thom maps.

Let $\widetilde{\Gamma} = \widetilde{\Gamma}_k$ be as in the previous subsection, and let $\mathcal{R}_k^{\widetilde{\Gamma}} = \{\rho_{p-k,p}^{\Gamma_{p-k,p}}\}_p$ be a stable system of admissible equivalence relations for $\widetilde{\Gamma}$.

Let c be an arbitrary cochain in $C^\kappa(\Gamma_{n,p}, \rho_{n,p}^{\Gamma_{n,p}})$ with $0 < \kappa < p$. Set $\lambda = \kappa - k$. Since we always have $C^{\kappa+1}(\Gamma_{\lambda,\kappa}, \rho_{\lambda,\kappa}^{\Gamma_{\lambda,\kappa}}) = 0$,

$$\delta_\kappa : C^\kappa(\Gamma_{\lambda,\kappa}, \rho_{\lambda,\kappa}^{\Gamma_{\lambda,\kappa}}) \to C^{\kappa+1}(\Gamma_{\lambda,\kappa}, \rho_{\lambda,\kappa}^{\Gamma_{\lambda,\kappa}})$$

is the zero homomorphism, and hence $s_\kappa c \in C^\kappa(\Gamma_{\lambda,\kappa}, \rho_{\lambda,\kappa}^{\Gamma_{\lambda,\kappa}})$ is a cocycle of the complex $\mathcal{C}(\Gamma_{\lambda,\kappa}, \rho_{\lambda,\kappa}^{\Gamma_{\lambda,\kappa}})$, where

$$s_\kappa : C^\kappa(\Gamma_{n,p}, \rho_{n,p}^{\Gamma_{n,p}}) \to C^\kappa(\Gamma_{\lambda,\kappa}, \rho_{\lambda,\kappa}^{\Gamma_{\lambda,\kappa}})$$

is the homomorphism induced by the $(p-\kappa)$-th suspension. Therefore, for a $\Gamma_{\lambda,\kappa}^*$-map $f : M \to N$, the homology class $[s_\kappa c(f)] \in H_0^c(N; \mathbf{Z}_2)$ represented by $s_\kappa c(f)$ is well-defined. Note that its Poincaré dual in $H^\kappa(N; \mathbf{Z}_2)$ coincides with $\varphi_f([s_\kappa c])$, where $[s_\kappa c] \in H^\kappa(\Gamma_{\lambda,\kappa}, \rho_{\lambda,\kappa}^{\Gamma_{\lambda,\kappa}})$ is the cohomology class represented by the cocycle $s_\kappa c$, and

$$\varphi_f : H^\kappa(\Gamma_{\lambda,\kappa}, \rho_{\lambda,\kappa}^{\Gamma_{\lambda,\kappa}}) \to H^\kappa(N; \mathbf{Z}_2)$$

is the homomorphism induced by f. Furthermore, when the source manifold M is closed, $[s_\kappa c(f)]$ is well-defined as an element of $H_0(N; \mathbf{Z}_2)$.

Proposition 13.11. *Suppose that $\Gamma_{\lambda,\kappa}$ is transversely complete with respect to $\Gamma_{n,p}$, where $0 < \kappa < p$ and $p - n = \kappa - \lambda = k$. Then a cochain c in $C^\kappa(\Gamma_{n,p}, \rho_{n,p}^{\Gamma_{n,p}})$ is a cocycle of the complex $\mathcal{C}(\Gamma_{n,p}, \rho_{n,p}^{\Gamma_{n,p}})$ if and only if $[s_\kappa c(f)] = 0 \in H_0(N; \mathbf{Z}_2)$ (or equivalently, $\varphi_f([s_\kappa c]) = 0$ in $H^\kappa(N; \mathbf{Z}_2)$) for every $\Gamma_{\lambda,\kappa}^*$-map $f : M \to N$ such that both M and N are closed and that f is $\widetilde{\Gamma}_k$-cobordant to a nonsingular map.*

Proof. If c is a cocycle, then the cohomology class represented by $s_\kappa c$ lies in the image of

$$s_{\kappa *} : H^\kappa(\Gamma_{\lambda+1,\kappa+1}, \rho_{\lambda+1,\kappa+1}^{\Gamma_{\lambda+1,\kappa+1}}) \to H^\kappa(\Gamma_{\lambda,\kappa}, \rho_{\lambda,\kappa}^{\Gamma_{\lambda,\kappa}}).$$

Therefore, we have

$$[s_\kappa c(f)] = [s_\kappa c(f')] \in H_0(N; \mathbf{Z}_2) \tag{13.1}$$

for every f that is $\widetilde{\Gamma}_k$-cobordant to a nonsingular map f' by Lemma 13.4 (see also Remark 13.5). We see easily that (13.1) always vanishes, since $\kappa > 0$ and for a nonsingular map f', we have $s_\kappa c(f') = \emptyset$.

Conversely, suppose that $[s_\kappa c(f)] = 0 \in H_0(N; \mathbf{Z}_2)$ for every f as in the proposition. Let $\widetilde{\mathfrak{F}}$ be an arbitrary equivalence class of fibers in $\Gamma_{n,p}$ of codimension $\kappa + 1$ with respect to $\rho_{n,p}^{\Gamma_{n,p}}$, and $g : M' \to N'$ be an element of $\mathcal{T}_{\mathrm{pr}}(n,p)$ such that the fiber of g over a point $y \in N'$ belongs to $\widetilde{\mathfrak{F}}$. By the proof of Lemma 8.3, we may assume that the stratum Σ containing y is of codimension $\kappa + 1$. Let N be the boundary of a sufficiently small $(\kappa+1)$-dimensional disk B in N' centered at y and transverse to Σ such that N is transverse to g and to all the strata of N'. Note that B corresponds to B_Σ in the argument just after Lemma 8.3. Then $f = g|_M : M \to N$ with $M = g^{-1}(N)$ is an element of $\mathcal{T}_{\mathrm{pr}}(\lambda, \kappa)$. Furthermore, since $\Gamma_{\lambda,\kappa}$ is transversely complete with respect to $\Gamma_{n,p}$ by our assumption, we see that f is a $\Gamma^*_{\lambda,\kappa}$-map.

It is easy to see that B contains a regular value y_0 of g with $y_0 \in B \setminus N$. Set $C = B - \mathrm{Int}\, B_0$, where B_0 is a closed disk neighborhood of y_0 in $B \setminus N$ consisting only of regular values of g and $\mathrm{Int}\, B_0$ denotes its interior as a subspace of B. Note that C is diffeomorphic to $S^\kappa \times [0, 1]$. Then, we see that $g|_{\widetilde{C}} : \widetilde{C} \to C$ with $\widetilde{C} = g^{-1}(C)$ gives a $\widetilde{\Gamma}_k$-cobordism between f and a nonsingular map. Hence, by our assumption, $s_\kappa c(f)$ consists of an even number of points. This means that the coefficient of $\widetilde{\mathfrak{F}}$ in $\delta_\kappa(s_\kappa c)$ is zero (see (9.2)). Since this holds for an arbitrary $\widetilde{\mathfrak{F}}$ of codimension $\kappa + 1$, we have $\delta_\kappa(s_\kappa c) = 0$. This completes the proof. □

Now, let us apply the above proposition to a specific but important situation as follows.

Corollary 13.12. *Let us consider the complex*

$$\mathcal{C}(\mathcal{S}^0_{\mathrm{pr}}(5, 4)^{\mathrm{ori}}, \rho^0_{5,4}(2)) \tag{13.2}$$

of singular fibers for proper C^0 stable maps of orientable 5-dimensional manifolds into 4-dimensional manifolds with respect to the C^0 equivalence modulo two circle components. Let $\widehat{\mathrm{III}}^8_{\mathrm{o}}$ (or $\widehat{\mathrm{III}}^8_{\mathrm{e}}$) be the C^0 equivalence class modulo two circle components of the suspension of $\mathrm{III}^8_{\mathrm{o}}$ (resp. $\mathrm{III}^8_{\mathrm{e}}$). Then $\widehat{\mathrm{III}}^8_{\mathrm{o}} + \widehat{\mathrm{III}}^8_{\mathrm{e}}$ is a 3-cocycle of the complex (13.2) and represents a nontrivial cohomology class in $H^3(\mathcal{S}^0_{\mathrm{pr}}(5, 4)^{\mathrm{ori}}, \rho^0_{5,4}(2))$.

For notations, refer to Fig. 8 and Proposition 10.4.

Proof. As in Proposition 4.1, we can obtain a similar characterization of proper C^∞ stable maps of 5-dimensional manifolds into 4-dimensional manifolds. Using this and Proposition 4.1 itself, we can show that $\mathcal{S}^0_{\mathrm{pr}}(4, 3)^{\mathrm{ori}}$ is transversely complete with respect to $\mathcal{S}^0_{\mathrm{pr}}(5, 4)^{\mathrm{ori}}$. Furthermore, an argument similar to that of the proof of Corollary 4.9 shows that two elements of $\mathcal{S}^0_{\mathrm{pr}}(4, 3)^{\mathrm{ori}*}$ are C^0 equivalent modulo two circle components if and only if so are their suspensions in $\mathcal{S}^0_{\mathrm{pr}}(5, 4)^{\mathrm{ori}*}$. Hence, we have $s_3(\widehat{\mathrm{III}}^8_{\mathrm{o}} + \widehat{\mathrm{III}}^8_{\mathrm{e}}) = \mathrm{III}^8_{\mathrm{o}} + \mathrm{III}^8_{\mathrm{e}}$.

Now, suppose that a C^0 stable map $f : M \to N$ of a closed orientable 4-manifold into a closed 3-manifold is $\widetilde{\mathcal{S}}^0_{\mathrm{pr}}(-1)$-cobordant to a nonsingular map. Since the source manifold of a nonsingular map always has zero Euler characteristic, we see that the Euler characteristic of M should be even. Hence, by Theorem 6.1, the number of elements in the set $(\mathrm{III}^8_{\mathrm{o}} + \mathrm{III}^8_{\mathrm{e}})(f)$ should be even, and hence it represents the trivial homology class in $H_0(N; \mathbf{Z}_2)$. Then, by Proposition 13.11, we see that $\mathrm{III}^8_{\mathrm{o}} + \mathrm{III}^8_{\mathrm{e}}$ is a cocycle of the complex (13.2).

Note that there does exist a closed orientable 4-manifold whose Euler characteristic is odd. Let $g : M' \to N'$ be a C^0 stable map of such a 4-manifold M' into a 3-manifold N'. Then, again by Theorem 6.1, we see that, for the homomorphism

$$\varphi_g : H^3(\mathcal{S}^0_{\mathrm{pr}}(5, 4)^{\mathrm{ori}}, \rho^0_{5,4}(2)) \to H^3_c(N'; \mathbf{Z}_2)$$

induced by g, we have $\varphi_g([\mathrm{III}^8_{\mathrm{o}} + \mathrm{III}^8_{\mathrm{e}}]) \neq 0$. This shows that the cohomology class $[\mathrm{III}^8_{\mathrm{o}} + \mathrm{III}^8_{\mathrm{e}}]$ is nontrivial. This completes the proof. □

The above corollary justifies the prediction given in Remark 11.12.

Let us end this section by the following proposition concerning Problem 12.14.

Proposition 13.13. *Suppose that $\Gamma_{\lambda,\kappa}$ is transversely complete with respect to $\Gamma_{n,p}$, where $0 < \kappa < p$ and $p - n = \kappa - \lambda = k$. Then the following two are equivalent.*

(1) *A cochain* $c \in C^\kappa(\Gamma_{n,p}, \rho_{n,p}^{\Gamma_{n,p}})$ *is a coboundary if and only if* $[s_\kappa c(f)] = 0 \in H_0(N; \mathbf{Z}_2)$ *(or equivalently,* $\varphi_f([s_\kappa c]) = 0$ *in* $H^\kappa(N; \mathbf{Z}_2)$*) for every* $\Gamma^*_{\lambda,\kappa}$*-map* $f : M \to N$ *such that both* M *and* N *are closed.*

(2) *If* $\alpha \in H^\kappa(\Gamma_{n,p}, \rho_{n,p}^{\Gamma_{n,p}})$ *is nonzero, then there exists a* $\Gamma^*_{n,p}$*-map* $g : M' \to N'$ *such that* $\varphi_g(\alpha) \neq 0$ *in* $H^\kappa(N'; \mathbf{Z}_2)$*.*

Proof. (1) $\Longrightarrow$ (2). Suppose that $\varphi_g(\alpha) = 0$ in $H^\kappa(N'; \mathbf{Z}_2)$ for all $\Gamma^*_{n,p}$-map $g : M' \to N'$. Then, by Lemma 12.11, for every $\Gamma^*_{\lambda,\kappa}$-map $f : M \to N$, we have $\varphi_f(s_{\kappa *}\alpha) = 0$ in $H^\kappa(N; \mathbf{Z}_2)$. Now item (1) implies that $\alpha = 0$. This is a contradiction.

(2) $\Longrightarrow$ (1). Suppose that c is a coboundary. Then $\varphi_f([s_\kappa c]) = 0$ in $H^\kappa(N; \mathbf{Z}_2)$, since $[s_\kappa c] = s_{\kappa *}[c] = 0$. Conversely, suppose that $[s_\kappa c(f)] = 0 \in H_0(N; \mathbf{Z}_2)$ for every $\Gamma^*_{\lambda,\kappa}$-map $f : M \to N$ such that both M and N are closed. By Proposition 13.11, c is a cocycle. Then, by Lemma 12.10, $\varphi_g([c]) = 0$ for every $\Gamma^*_{n,p}$-map g. Then item (2) implies that $[c] = 0$; i.e. c is a coboundary. This completes the proof. $\square$

Note that all the results in this subsection hold also for the universal complexes of co-orientable singular fibers and the cohomology groups with $\mathbf{Z}$-coefficients, provided that the target manifolds are oriented, except for Corollary 13.12.

Remark 13.14. Recall that $\mathcal{T}_{\mathrm{pr}}(\lambda, \kappa)$ is always transversely complete with respect to $\mathcal{T}_{\mathrm{pr}}(n, p)$. Thus, in view of Proposition 13.13, a special case of Problem 12.14 can be interpreted as follows at least for Thom maps. A cochain $c \in C^\kappa(\mathcal{T}_{\mathrm{pr}}(n, p), \rho^0_{n,p})$ with $0 < \kappa < p$ of the universal complex is a cocycle if and only if $s_\kappa c(f)$ is null-homologous for all f cobordant to a nonsingular map. Is it true that a cocycle c is a coboundary if and only if $s_\kappa c(f)$ is null-homologous for all f?

14. Cobordism of maps with restricted local singularities

In this section, we consider another cobordism relation which is slightly different from the one given in the previous section.

Let us consider a (mono-)germ $\eta : (\mathbf{R}^n, 0) \to (\mathbf{R}^{n+k}, 0)$ of a C^∞ stable codimension k map. We define its *suspension* $\Sigma\eta : (\mathbf{R}^{n+1}, 0) \to (\mathbf{R}^{n+1+k}, 0)$ by $\Sigma\eta(u, t) = (\eta(u), t)$ for $u \in \mathbf{R}^n$ and $t \in \mathbf{R}$. For a fixed $k \in \mathbf{Z}$, let us consider the set of germs of C^∞ stable codimension k maps, and the equivalence relation generated by the C^∞ right-left equivalence and the suspension. We call such an equivalence class a *singularity type* (see [35]).

There is a hierarchy of singularity types. A singularity type A is said to be *under* another singularity type B if for a representative $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^{n+k}, 0)$ of A, there is a germ of B arbitrary close to f, in the sense that there are points x arbitrary close to the origin of $\mathbf{R}^n$ such that the germ of f at x belongs to B. In this case, we also say that B is *over* A. (Compare this with Definition 9.18.)

Let τ be an ascending set of singularity types.

Definition 14.1. We say that a smooth map $f : M \to N$ between smooth manifolds is a *τ-map* if its singularities (as mono-germs) in the source manifold M all lie in τ. If M is a manifold with boundary, then we also suppose that $f(\partial M) \subset \partial N$ and for collar neighborhoods C and C' of ∂M and ∂N respectively, we have $f|_C = \Sigma(f|_{\partial M})$.

Definition 14.2. For a smooth manifold N, two τ-maps $f_0 : M_0 \to N$ and $f_1 : M_1 \to N$ of closed manifolds M_0 and M_1 are said to be *τ-cobordant* if there exist a compact manifold W with boundary the disjoint union $M_0 \amalg M_1$, and a τ-map $F : W \to N \times [0, 1]$ such that $f_i = F|_{M_i} : M_i \to N \times \{i\}$, $i = 0, 1$. We call F a *τ-cobordism* between f_0 and f_1.

When M_i are oriented and W can be taken to be oriented so that $\partial W = (-M_0) \amalg M_1$, then we say that f_0 and f_1 are *oriented τ-cobordant.*

Lemma 14.3. *Every τ-map of a closed manifold is τ-cobordant to a τ-map which is a Thom map.*

Proof. Suppose that a τ-map $f : M \to N$ is given, where M is a closed manifold. Then there exists a τ-map $\widetilde{f} : M \to N$ which is a Thom map and which is sufficiently close to f in the mapping space $C^\infty(M, N)$, since the set of Thom maps is dense in $C^\infty(M, N)$ and the local singularities of f are all C^∞ stable (and hence the set of all τ-maps is open in the mapping space). In particular, we may assume that f and $\widetilde{f}$ are homotopic through τ-maps and hence are τ-cobordant. This completes the proof. $\square$

Remark 14.4. Suppose that two τ-maps $f_i : M_i \to N$, $i = 0, 1$, of closed manifolds are Thom maps. If they are τ-cobordant, then a τ-cobordism between them can be chosen as a Thom map. This is proved by first taking any τ-cobordism and then by approximating it by a Thom map.

In what follows, we fix the codimension $k \in \mathbf{Z}$. For an ascending set τ of singularity types of codimension k and for a dimension pair (n, p) with $p - n = k$, let us denote by $\tau(n, p)$ the set of all proper Thom maps which are τ-maps. Furthermore, we set

$$\widetilde{\tau}(k) = \bigcup_p \tau(p - k, p),$$

and let us consider a stable system of admissible equivalence relations $\mathcal{R}_k^\tau = \{\rho_{p-k,p}^\tau\}_p$ for the fibers of elements of $\widetilde{\tau}(k)$. Note that the set $\widetilde{\tau}(k)$ is closed under suspension.

Definition 14.5. Let $f : M \to N$ be an arbitrary τ-map, which may not necessarily be a Thom map, where we assume that M is closed. Then by Lemma 14.3, f is τ-cobordant to a τ-map $\widetilde{f} : M \to N$ which is a Thom map. Then we define

$$\varphi_f : \operatorname{Im} s_{\kappa *} \to H^\kappa(N; \mathbf{Z}_2)$$

by $\varphi_f = \varphi_{\widetilde{f}}|_{\operatorname{Im} s_{\kappa *}}$, where

$$s_{\kappa *} : H^\kappa(\tau(n+1, p+1), \rho_{n+1,p+1}^\tau) \to H^\kappa(\tau(n, p), \rho_{n,p}^\tau)$$

is the homomorphism induced by the suspension, and

$$\varphi_{\widetilde{f}} : H^\kappa(\tau(n, p), \rho_{n,p}^\tau) \to H^\kappa(N; \mathbf{Z}_2)$$

is the homomorphism induced by the Thom map $\widetilde{f}$. The homomorphism φ_f is well-defined by virtue of Lemma 13.4 together with Remark 14.4.

By Lemma 13.4, we see that if f_0 and f_1 are τ-maps of closed manifolds into a p-dimensional manifold N which are τ-cobordant, then $\varphi_{f_0} = \varphi_{f_1}$. In other words, the correspondence $f \mapsto \varphi_f$ defines a τ-cobordism invariant of τ-maps into N.

Remark 14.6. If τ is big enough, or more precisely, if the space of τ-maps is always dense in the corresponding mapping space, then for every smooth map $f : M \to N$ of a closed manifold, we can define φ_f to be $\varphi_{\widetilde{f}}$, where $\widetilde{f}$ is an approximation of f which is a τ-map. Then, we can show that this is well-defined, and that it defines a *bordism invariant* of smooth maps into N, where two smooth maps $f_0 : M_0 \to N$ and $f_1 : M_1 \to N$ of closed manifolds M_0 and M_1 are said to be *bordant* if there exist a compact manifold W with boundary the disjoint union $M_0 \amalg M_1$, and a smooth map $F : W \to N \times [0, 1]$ such that $f_i = F|_{M_i} : M_i \to N \times \{i\}$, $i = 0, 1$ (for details, see [5]). In particular, if N is contractible, it defines a cobordism invariant of the source manifold.

Remark 14.7. So far, we have considered Thom maps which are τ-maps. It is easy to see that we could as well consider C^0 stable maps which are Thom maps (or C^∞ stable maps for nice dimension pairs (n,p) in the sense of Mather [27]) instead of Thom maps, since the corresponding sets are dense in the mapping spaces. Let us denote by $\tau^0(n,p)$ the set of C^0 stable maps in $\mathcal{T}_{\rm pr}(n,p)$ which are τ-maps. Then, for a τ-map $f : M \to N$ with M being closed, we can define the homomorphism

$$\varphi_f : \operatorname{Im} s^0_{\kappa *} \to H^\kappa(N; \mathbf{Z}_2),$$

which is a τ-cobordism invariant, where

$$s^0_{\kappa *} : H^\kappa(\tau^0(n+1,p+1), \rho^\tau_{n+1,p+1}) \to H^\kappa(\tau^0(n,p), \rho^\tau_{n,p})$$

is the homomorphism induced by the suspension.

In fact, we can show that the diagram

$$\begin{array}{ccc} H^\kappa(\tau(n+1,p+1), \rho^\tau_{n+1,p+1}) & \xrightarrow{s_{\kappa *}} & H^\kappa(\tau(n,p), \rho^\tau_{n,p}) \\ {\scriptstyle \pi_{\tau(n+1,p+1),\tau^0(n+1,p+1)*}}\downarrow & & \downarrow{\scriptstyle \pi_{\tau(n,p),\tau^0(n,p)*}} \\ H^\kappa(\tau^0(n+1,p+1), \rho^\tau_{n+1,p+1}) & \xrightarrow{s^0_{\kappa *}} & H^\kappa(\tau^0(n,p), \rho^\tau_{n,p}) \end{array}$$

is commutative, and that

$$\varphi_f : \operatorname{Im} s_{\kappa *} \to H^\kappa(N; \mathbf{Z}_2)$$

coincides with the composition of the natural homomorphism induced by the projection

$$\pi_{\tau(n,p),\tau^0(n,p)*}|_{\operatorname{Im} s_{\kappa *}} : \operatorname{Im} s_{\kappa *} \to \operatorname{Im} s^0_{\kappa *}$$

and

$$\varphi_f : \operatorname{Im} s^0_{\kappa *} \to H^\kappa(N; \mathbf{Z}_2).$$

Let us consider τ-maps into $N = N' \times \mathbf{R}$, where N' is a $(p-1)$-dimensional manifold. Then, the set of all τ-cobordism classes of τ-maps of closed manifolds into N, denoted by $\operatorname{Cob}_\tau(N)$, forms an abelian group with respect to the "far away disjoint union". (When we take the orientations into account, we denote the corresponding abelian group by $\operatorname{Cob}^{\rm ori}_\tau(N)$.) More precisely, for two τ-maps $f_i : M_i \to N$ of closed manifolds M_i, $i = 0, 1$, there exists a real number r such that $f_0(M_0) \cap (T_r \circ f_1(M_1)) = \emptyset$, where the diffeomorphism $T_r : N' \times \mathbf{R} \to N' \times \mathbf{R}$ is defined by $T_r(x,t) = (x, t+r)$. Then, it is not difficult to show that the τ-cobordism class of the disjoint union of the two maps f_0 and $T_r \circ f_1$ depends only on the τ-cobordism classes of f_0 and f_1. Furthermore, the resulting τ-cobordism class does not change even if we exchange f_0 and f_1. Thus, we define $[f_0]+[f_1] = [f_0 \amalg (T_r \circ f_1)]$, where $[*]$ denotes the τ-cobordism class of $*$. The neutral element is the map of the empty set, and the inverse element of a τ-map $f : M \to N' \times \mathbf{R}$ is given by $-f : M \to N' \times \mathbf{R}$ defined by $-f = R \circ f$, where $R : N' \times \mathbf{R} \to N' \times \mathbf{R}$ is the diffeomorphism defined by $R(x,t) = (x,-t)$. (When we take the orientations into account, the source manifold of $-f$ is understood to be $-M$.)

Then, the following is a direct consequence of the above definitions.

Proposition 14.8. *In the above situation, the map*

$$\Phi_\kappa : \operatorname{Cob}_\tau(N) \to \operatorname{Hom}(\operatorname{Im} s_{\kappa *}, H^\kappa(N; \mathbf{Z}_2))$$

defined by $\Phi_\kappa([f]) = \varphi_f$ for a τ-maps f of a closed manifold into N is a homomorphism of abelian groups for every κ, where

$$s_{\kappa *} : H^\kappa(\tau(n+1,p+1), \rho^\tau_{n+1,p+1}) \to H^\kappa(\tau(n,p), \rho^\tau_{n,p})$$

is the homomorphism induced by the suspension.

Note that a similar map

$$\Phi_\kappa : \mathrm{Cob}_\tau(N) \to \mathrm{Hom}\,(\mathrm{Im}\, s^0_{\kappa *}, H^\kappa(N;\mathbf{Z}_2))$$

can also be defined and is a homomorphism of abelian groups for every κ (see Remark 14.7).

We do not know if the homomorphism $\oplus_\kappa \Phi_\kappa$ is injective or not for some $\rho^\tau_{n,p}$ and $\rho^\tau_{n+1,p+1}$.

Remark 14.9. Note that the above proposition holds also for $\widetilde{\Gamma}_k$-maps in the sense of §9.6 or §13. However, we do not know if the group operation defined on the set of cobordism classes is commutative or not. If $N = N'' \times \mathbf{R}^2$ for some $(p-2)$-dimensional manifold N'', then we can show that the resulting group is abelian.

Note that all the results in this section hold also for the universal complexes of co-orientable singular fibers and the cohomology groups with $\mathbf{Z}$-coefficients, provided that the target manifolds are oriented.

15. Examples of cobordism invariants

In this section, we shall construct explicit cobordism invariants in specific situations following the procedure introduced in the previous sections. Throughout the section, the codimension will always be equal to -1. Furthermore, we shall work only with nice dimension pairs, and we shall consider C^0 stable maps instead of Thom maps following Remark 14.7.

15.1. **Cobordism of stable maps.** Let τ be the set of singularity types corresponding to a regular point and a Morin singularity [29], i.e., a fold point, a cusp point, a swallowtail, etc. Note that if the dimension of the source manifold is less than or equal to 4, this set is big enough in the sense of Remark 14.6.

Let us consider C^0 stable maps of surfaces and 3-manifolds. By Proposition 10.7, the first cohomology group of the universal complex of singular fibers

$$\mathcal{C}(\mathcal{S}^0_{\mathrm{pr}}(3,2), \rho^0_{3,2}(2)) = \mathcal{C}(\tau^0(3,2), \rho^0_{3,2}(2))$$

with respect to the C^0 equivalence modulo two circle components for $\tau^0(3,2)$-maps is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ and is generated by $\alpha_1 = [\widetilde{\mathrm{I}}^0_{\mathrm{o}} + \widetilde{\mathrm{I}}^1_{\mathrm{e}}] = [\widetilde{\mathrm{I}}^0_{\mathrm{e}} + \widetilde{\mathrm{I}}^1_{\mathrm{o}}]$ and $\alpha_2 = [\widetilde{\mathrm{I}}^2_{\mathrm{o}} + \widetilde{\mathrm{I}}^2_{\mathrm{e}}]$. In the following, let

$$s^0_{1*} : H^1(\tau^0(3,2), \rho^0_{3,2}(2)) \to H^1(\tau^0(2,1), \rho^0_{2,1}(2))$$

be the homomorphism induced by the suspension.

Let us first consider $\alpha_2 = [\widetilde{\mathrm{I}}^2_{\mathrm{o}} + \widetilde{\mathrm{I}}^2_{\mathrm{e}}]$. For a C^∞ stable map $f : M \to N$ of a closed surface into a connected 1-dimensional manifold N, $(s^0_{1*}\alpha_2)(f) \in H_0(N;\mathbf{Z}_2) \cong \mathbf{Z}_2$ is nothing but the number modulo two of the singular fibers as depicted in Fig. 3 (3). By Lemma 13.4 and Remark 14.6, this is a bordism invariant. On the other hand, by Corollary 3.4 and Remark 3.7, the number modulo two coincides with the parity of the Euler characteristic of the source surface M. Thus, $(s^0_{1*}\alpha_2)(f)$ coincides with the parity of the Euler characteristic of its source surface. When $N = \mathbf{R}$, this is a complete bordism (or τ-cobordism) invariant for (τ-)maps of closed surfaces into N.

For $\alpha_1 = [\widetilde{\mathrm{I}}^0_{\mathrm{o}} + \widetilde{\mathrm{I}}^1_{\mathrm{e}}] = [\widetilde{\mathrm{I}}^0_{\mathrm{e}} + \widetilde{\mathrm{I}}^1_{\mathrm{o}}]$, we have the following.

Lemma 15.1. *For every $\tau^0(2,1)$-map $f : M \to N$ of a closed surface M into a 1-dimensional manifold N, $(s^0_{1*}\alpha_1)(f) \in H_0(N;\mathbf{Z}_2)$ vanishes.*

Proof. Let $\mathbf{0}_{\mathrm{o}}(f)$ be the set

$$\{y \in N \,:\, y \text{ is a regular value of } f \text{ and } b_0(f^{-1}(y)) \text{ is odd}\}.$$

Since every 1-dimensional manifold is orientable, we give an orientation to N. Then each connected component of $\overline{\mathbf{0}_{\mathrm{o}}(f)}$, which is either an arc or a circle, has an induced orientation. Note that the end points of the arc components of $\overline{\mathbf{0}_{\mathrm{o}}(f)}$ correspond to

$$(\widetilde{\mathrm{I}}^0_{\mathrm{o}} + \widetilde{\mathrm{I}}^1_{\mathrm{e}} + \widetilde{\mathrm{I}}^0_{\mathrm{e}} + \widetilde{\mathrm{I}}^1_{\mathrm{o}})(f).$$

For an arc component, we say that it is of type $++$ (or $--$) if the number of connected components of a regular fiber of f increases (resp. decreases) by one when the target point passes through its starting point and also when it passes through its terminal point. We say that it is of type $+-$ (or $-+$) if the number increases (resp. decreases) by one when the target point passes through its starting point and it decreases (resp. increases) by one when it passes through its terminal point. In this way, the arc components of $\overline{\mathbf{0}_{\mathrm{o}}(f)}$ can be classified into these four types. We denote by $n(++)$, $n(--)$, $n(+-)$ and $n(-+)$ the numbers of arc components of types $++$, $--$, $+-$ and $-+$ respectively.

Then, it is easy to show that

$$\begin{aligned} |\widetilde{\mathrm{I}}^0_{\mathrm{o}}(f)| + |\widetilde{\mathrm{I}}^1_{\mathrm{e}}(f)| &= n(++) + n(--) + 2n(+-), \\ |\widetilde{\mathrm{I}}^0_{\mathrm{e}}(f)| + |\widetilde{\mathrm{I}}^1_{\mathrm{o}}(f)| &= n(++) + n(--) + 2n(-+). \end{aligned}$$

Since we should have $n(++) = n(--)$, we obtain

$$|\widetilde{\mathrm{I}}^0_{\mathrm{o}}(f)| + |\widetilde{\mathrm{I}}^1_{\mathrm{e}}(f)| \equiv |\widetilde{\mathrm{I}}^0_{\mathrm{e}}(f)| + |\widetilde{\mathrm{I}}^1_{\mathrm{o}}(f)| \equiv 0 \pmod 2.$$

This implies that $(s^0_{1*}\alpha_1)(f) = 0$ in $H_0(N; \mathbf{Z}_2)$. This completes the proof. □

Remark 15.2. Note that $s^0_{1*}\alpha_1$ does not vanish as an element of $H^1(\tau^0(2,1), \rho^0_{2,1}(2))$. Hence, the above lemma shows that even if we take a nontrivial cohomology class of the universal complex with respect to an admissible equivalence relation, the corresponding homology class in the target manifold can be trivial. Hence, the answer to the problem mentioned in Problem 12.14 is negative in general, if we replace the C^0 equivalence relation $\rho^0_{n,p}$ with an arbitrary admissible equivalence relation, at least for the cohomology group (12.2). See also Remark 13.14.

Let us consider the homomorphism

$$\varepsilon_{\rho^0_{2,1}(2), \rho^0_{2,1}*} : H^1(\mathcal{S}^0_{\mathrm{pr}}(2,1), \rho^0_{2,1}(2)) \to H^1(\mathcal{S}^0_{\mathrm{pr}}(2,1), \rho^0_{2,1})$$

induced by the cochain map

$$\varepsilon_{\rho^0_{2,1}(2), \rho^0_{2,1}} : \mathcal{C}(\mathcal{S}^0_{\mathrm{pr}}(2,1), \rho^0_{2,1}(2)) \to \mathcal{C}(\mathcal{S}^0_{\mathrm{pr}}(2,1), \rho^0_{2,1})$$

defined in §9.6. If the image of $s^0_{1*}\alpha_1 \in H^1(\mathcal{S}^0_{\mathrm{pr}}(2,1), \rho^0_{2,1}(2))$ by $\varepsilon_{\rho^0_{2,1}(2), \rho^0_{2,1}*}$ is nontrivial, then the problem mentioned in Problem 12.14 is negatively solved. The author conjectures that $\varepsilon_{\rho^0_{2,1}(2), \rho^0_{2,1}*}(s^0_{1*}\alpha_1) \neq 0$.

In [53], Yamamoto considers an equivalence relation among the fibers of a given map which takes into account their positions from a global viewpoint. In other words, even if two fibers are C^0 equivalent, if their positions are different from each other in a certain global sense, then one considers them to be nonequivalent. Probably, we can construct universal complexes of singular fibers with respect to such "global" equivalence relations. Then, the author conjectures that for such a universal complex with respect to a certain global equivalence relation, the answer to the problem mentioned in Problem 12.14 should be positive.

If we consider a C^0 stable map $f : M \to N$ of a closed 3-manifold into a surface, then $\alpha_1(f)$ and $\alpha_2(f)$ are defined as elements of $H_1(N; \mathbf{Z}_2)$. We see that $\alpha_2(f)$ can be nontrivial by the example constructed as follows.

Let $g : \mathbf{R}P^2 \to \mathbf{R}$ be an arbitrary Morse function. Note that $(s^0_{1*}\alpha_2)(g)$ is nontrivial by Corollary 3.4. We define $f = g \times \mathrm{id}_{S^1} : \mathbf{R}P^2 \times S^1 \to \mathbf{R} \times S^1$. Then we see that $\alpha_2(f)$ does not vanish in $H_1(\mathbf{R}\times S^1;\mathbf{Z}_2)$. (This implies, for example, that $f:\mathbf{R}P^2\times S^1\to\mathbf{R}\times S^1$ is not bordant to a constant map.)

On the other hand, $\alpha_1(f)$ always vanishes. This follows from Lemma 12.10, since $(s^0_{1*}\alpha_1)(h)$ always vanishes for a $\tau^0(2,1)$-map h of a closed surface into a 1-dimensional manifold as mentioned above. Note that $\tau^0(2,1)$ is transversely complete with respect to $\tau^0(3,2)$.

15.2. **Cobordism of fold maps.** Let us now consider an example of τ which is not big in the sense of Remark 14.6. Let τ be the set of singularity types corresponding to a regular point and a fold point. In this case, a τ-map is called a *fold map*. (Recall that this notion was already introduced in §7). In the following, we denote by $\tau^0(n,p)^{\mathrm{ori}}$ the set of all C^0 equivalence classes of fibers for proper C^0 stable τ-maps in $\mathcal{T}_{\mathrm{pr}}(n,p)$ of orientable n-dimensional manifolds.

Then the following proposition can be proved. Details are left to the reader.

Proposition 15.3. *The cohomology groups of the universal complex*

$$\mathcal{CO}(\tau^0(3,2)^{\mathrm{ori}},\rho^0_{3,2}(2))$$

of co-orientable singular fibers for proper C^0 stable fold maps of orientable 3-manifolds into surfaces with respect to the C^0 equivalence modulo two circle components are given as follows:

$$\begin{aligned}
H^0(\mathcal{CO}(\tau^0(3,2)^{\mathrm{ori}},\rho^0_{3,2}(2))) &\cong \mathbf{Z}\ (\textit{generated by } [\mathbf{0}_{\mathrm{o}}+\mathbf{0}_{\mathrm{e}}]),\\
H^1(\mathcal{CO}(\tau^0(3,2)^{\mathrm{ori}},\rho^0_{3,2}(2))) &\cong \mathbf{Z}\oplus\mathbf{Z}\ (\textit{generated by } \alpha_1=[\widetilde{\mathrm{I}}^0_{\mathrm{o}}+\widetilde{\mathrm{I}}^1_{\mathrm{e}}]=[\widetilde{\mathrm{I}}^0_{\mathrm{e}}+\widetilde{\mathrm{I}}^1_{\mathrm{o}}],\\
&\qquad \alpha_2=[\widetilde{\mathrm{I}}^0_{\mathrm{o}}+\widetilde{\mathrm{I}}^0_{\mathrm{e}}],\ \textit{and } \alpha_3=[\widetilde{\mathrm{I}}^1_{\mathrm{o}}+\widetilde{\mathrm{I}}^1_{\mathrm{e}}]\\
&\qquad \textit{with } 2\alpha_1=\alpha_2+\alpha_3),
\end{aligned}$$

where for a C^0 equivalence class $\mathfrak{F}$ of fibers, $\mathfrak{F}_{\mathrm{o}}$ (or $\mathfrak{F}_{\mathrm{e}}$) denotes the C^0 equivalence class modulo two circle components represented by $\mathfrak{F}_\ell$ with ℓ odd (resp. even), and $[*]$ denotes the cohomology class represented by the cocycle $*$.

Let $f:M\to\mathbf{R}$ be a Morse function, which is a fold map, of a closed oriented surface M. Then $(s^0_{1*}\alpha_2)(f)\in H_0(\mathbf{R};\mathbf{Z})\cong\mathbf{Z}$ coincides with $\max(f)-\min(f)$, where $\max(f)$ (or $\min(f)$) is the number of local maxima (resp. minima) of the Morse function f. Furthermore, $(s^0_{1*}\alpha_3)(f)$ coincides with the τ-cobordism invariant introduced in [18]. Since we can show that $(s^0_{1*}\alpha_1)(f)$ always vanishes as in Lemma 15.1, we have $(s^0_{1*}\alpha_2)(f)=-(s^0_{1*}\alpha_3)(f)$.

Note that by [18], two Morse functions f_0 and f_1 on closed oriented surfaces are oriented τ-cobordant if and only if $(s^0_{1*}\alpha_2)(f_0)=(s^0_{1*}\alpha_2)(f_1)$. In other words, the cohomology class $s^0_{1*}\alpha_2$ of the universal complex of co-orientable singular fibers with respect to the C^0 equivalence modulo two circle components gives a complete invariant for τ-cobordisms of τ-maps of oriented surfaces into $\mathbf{R}$.

16. Applications

In this section, we give some applications of the ideas developed in §8 to the topology of generic maps.

First, we prepare some lemmas.

Lemma 16.1. *Let W be a compact m-dimensional manifold such that its boundary is a disjoint union of open and closed subsets V_0 and V_1. If there exists a Morse function $g:W\to\mathbf{R}$ such that $g(W)=[a,b]$ for some $a<b$, $V_0=g^{-1}(a)$, $V_1=g^{-1}(b)$, and that g has a unique critical point in the interior of W, then the difference between the Euler characteristics of V_0 and V_1 is equal to ± 2, provided that m is odd.*

Proof. Let λ be the index of the critical point. Then by Morse theory, we see that V_1 is diffeomorphic to

$$(V_0 \setminus \mathrm{Int}(S^{\lambda-1} \times D^{m-\lambda})) \cup (D^\lambda \times S^{m-\lambda-1}).$$

Then the result follows immediately. □

Definition 16.2. Let V_0 and V_1 be closed oriented $(4k+1)$-dimensional manifolds with $k \geq 0$. Suppose that there exists an oriented cobordism W between V_0 and V_1. Then, we define $d(V_0, V_1)$ to be the Euler characteristic modulo two of W. Since every closed orientable $(4k+2)$-dimensional manifold has even Euler characteristic, $d(V_0, V_1) \in \mathbf{Z}_2$ does not depend of the choice of W. In fact, $d(V_0, V_1)$ coincides with the difference between the semi-characteristics $\chi^*(V_0)$ and $\chi^*(V_1)$ with respect to any coefficient field (see [26]).

Then the following lemma follows from the very definition.

Lemma 16.3. *Let W be a compact $(4k+2)$-dimensional oriented manifold such that its boundary is a disjoint union of open and closed subsets V_0 and V_1. If there exists a Morse function $g : W \to \mathbf{R}$ such that $g(W) = [a, b]$ for some $a < b$, $V_0 = g^{-1}(a)$, $V_1 = g^{-1}(b)$, and that g has a unique critical point in the interior of W, then $d(V_0, V_1)$ defined above is equal to $1 \in \mathbf{Z}_2$.*

With the help of the above lemmas, we prove the following. Recall that a smooth map between smooth manifolds is a *Boardman map* if its jet extensions are transverse to all the Thom-Boardman subbundles (see [3] and [13, Chapter VI, §5]). Furthermore, such a map satisfies the *normal crossing condition* if its restrictions to the Thom-Boardman strata intersect in general position (for more details, see [13, Chapter VI, §5]).

Proposition 16.4. *Let $f : M \to N$ be a Boardman map of a closed n-dimensional manifold M into a p-dimensional manifold with $n \geq p$. Suppose either that $n - p$ is even, or that $n - p \equiv 1 \pmod 4$ and M is orientable. Then $f_*[S(f)]_2 = 0 \in H_{p-1}(N; \mathbf{Z}_2)$, where $[S(f)]_2 \in H_{p-1}(M; \mathbf{Z}_2)$ is the $\mathbf{Z}_2$-homology class represented by the singular set $S(f)$ of f.*

Proof. We may assume that N is connected. We may also assume that f satisfies the normal crossing condition by perturbing f slightly. When $n - p \equiv 1 \pmod 4$, we fix an orientation of M. Take a regular value $y_0 \in N$ of f and fix it, where we take $y_0 \in N \setminus f(M)$ if N is open. Let R be the closure of the set of points $y \in N \setminus f(S(f))$ such that

$$\frac{\chi(f^{-1}(y)) - \chi(f^{-1}(y_0))}{2}$$

is odd for $n - p \equiv 0 \pmod 2$ and that

$$d(f^{-1}(y_0), f^{-1}(y_1)) \equiv 1 \pmod 2$$

for $n - p \equiv 1 \pmod 4$. Note that if A is an embedded arc connecting y and y_0 transverse to f, then $f^{-1}(A)$ gives a (oriented) cobordism between $f^{-1}(y)$ and $f^{-1}(y_0)$, and hence $\chi(f^{-1}(y)) - \chi(f^{-1}(y_0))$ is always an even integer for $n - p \equiv 0 \pmod 2$ and $d(f^{-1}(y_0), f^{-1}(y_1)) \in \mathbf{Z}_2$ is well-defined for $n - p \equiv 1 \pmod 4$. Then it is easy to see that R is compact.

Since f is a Boardman map, $S(f)$ is naturally stratified into the Thom-Boardman strata, and the top dimensional strata of $S(f)$ consist of fold points. Let J be an arc embedded in N such that J intersects $f(S(f))$ transversely at a unique interior point z such that $f^{-1}(z) \cap S(f)$ consists of a fold point. Then by applying Lemmas 16.1 and 16.3 to the (oriented) cobordism $f^{-1}(J)$ and the Morse function $f|_{f^{-1}(J)} : f^{-1}(J) \to J$, we see that exactly one end point of J belongs to R.

Therefore, $f_*[S(f)]_2$ coincides with the $\mathbf{Z}_2$-homology class represented by ∂R, since f satisfies the normal crossing condition. Thus the result follows. $\square$

Remark 16.5. Proposition 16.4 does not hold for general Thom maps. For example, let $f: S^1 \to S^1$ be a C^∞ homeomorphism such that f is equivalent to the function $x \mapsto x^3$ at a point. Then, f is a Thom map, but $S(f)$ consists exactly of one point.

By Thom [49], the Poincaré dual of $[S(f)]_2 \in H_{p-1}(M; \mathbf{Z}_2)$ coincides with the $(n-p+1)$-st Stiefel-Whitney class $w_{n-p+1}(TM - f^*TN)$ of the difference bundle $TM - f^*TN$. Since every continuous map between smooth manifolds is homotopic to a Boardman map, we obtain the following.

Corollary 16.6. *Let $f: M \to N$ be a continuous map of a smooth closed n-dimensional manifold M into a smooth p-dimensional manifold with $n \geq p$. Suppose either that $n-p$ is even, or that $n-p \equiv 1 \pmod 4$ and M is orientable. Then we have $f_! w_{n-p+1}(TM - f^*TN) = 0 \in H^1(N; \mathbf{Z}_2)$, where*

$$f_!: H^{n-p+1}(M; \mathbf{Z}_2) \to H^1(N; \mathbf{Z}_2)$$

denotes the Gysin homomorphism induced by f.

As another corollary to Proposition 16.4, we have the following.

Corollary 16.7. *Let $f: M \to N$ be a C^∞ stable map of a closed n-dimensional manifold M into a p-dimensional manifold N with $n \geq p$ such that f has only fold points as its singularities. Suppose either that n and p are odd, or that $n-p \equiv 1 \pmod 4$, $p \equiv 1 \pmod 2$ and M is orientable. Then the Euler characteristic of $f(S(f))$ is even.*

The above corollary follows from the fact that in the above situation, $S(f)$ is a $(p-1)$-dimensional closed submanifold of M and that $f|_{S(f)}$ is an immersion with normal crossings (for example, see [13, Chapter III, §4]), together with [32, Corollary 7.3].

Now, let $f: M \to N$ be a C^∞ stable map of a closed n-dimensional manifold M into a p-dimensional manifold N such that f has only fold points as its singularities. For $m \geq 0$, we put

$$\Sigma_m(f) = \{y \in N : f^{-1}(y) \cap S(f) \text{ consists exactly of } m \text{ points}\},$$

and for $m \geq 1$, we put

$$\widetilde{\Sigma}_m(f) = f^{-1}(\Sigma_m(f)) \cap S(f).$$

Note that $\Sigma_m(f)$ is a regular submanifold of N of dimension $p-m$, and that $\widetilde{\Sigma}_m(f)$ is a regular submanifold of M of dimension $p-m$.

Then we have the following.

Proposition 16.8. *Let $f: M \to N$ be a C^∞ stable map of a closed n-dimensional manifold M into a p-dimensional manifold N with $n \geq p$ such that f has only fold points as its singularities. Suppose that $n-p$ is even. Then, the $\mathbf{Z}_2$-homology class*

$$[\overline{\Sigma_m(f)}]_2 \in H_{p-m}(\overline{\Sigma_{m-1}(f)}; \mathbf{Z}_2)$$

represented by $\overline{\Sigma_m(f)}$ vanishes for m odd. Furthermore, the $\mathbf{Z}_2$-homology class

$$[\overline{\widetilde{\Sigma}_m(f)}]_2 \in H_{p-m}(\overline{\widetilde{\Sigma}_{m-1}(f)}; \mathbf{Z}_2)$$

represented by $\overline{\widetilde{\Sigma}_m(f)}$ vanishes for m even.

Proof. Take a point $y_0 \in \Sigma_{m-1}(f)$. Let $R \subset \overline{\Sigma_{m-1}(f)}$ be the the closure of the set of points $y \in \Sigma_{m-1}(f)$ such that

$$\frac{\chi(f^{-1}(y)) - \chi(f^{-1}(y_0))}{2}$$

is odd. Then by an argument similar to that in the proof of Proposition 16.4, we see that $[\overline{\Sigma_m(f)}]_2$ coincides with the $\mathbf{Z}_2$-homology class represented by ∂R, since m is odd. Hence the first half of the proposition follows. The second half follows from a similar argument. □

The above proposition shows, for example, that the singular value set $f(S(f))$ of the C^∞ stable map $f : \mathbf{C}P^2 \sharp 2\overline{\mathbf{C}P^2} \to \mathbf{R}^3$ constructed in §7 cannot be realized as the singular value set of a C^∞ stable map of a closed n-dimensional manifold into $\mathbf{R}^3$ for $n \geq 3$ odd.

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