研究集会「トポロジーと写像の特異点」

Singular Fibers of Differentiable Maps and 4-Dimensional Cobordism Group

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1 Cobordism of Manifolds

M^n, N^n: closed manifolds (possibly oriented)

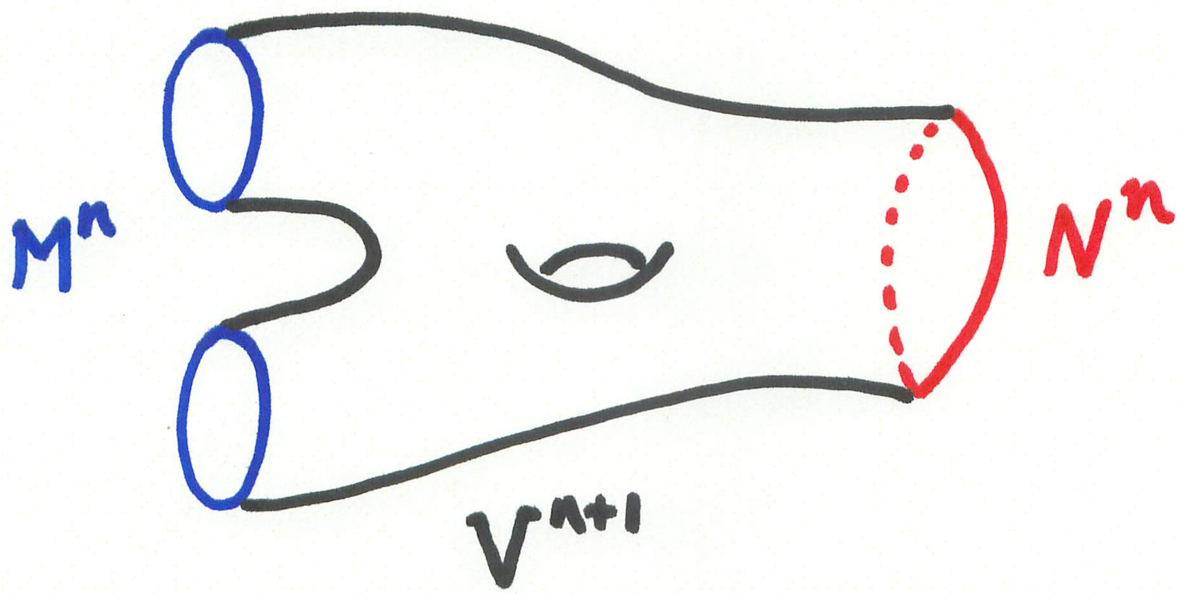
Def. 1 $M^n \sim_{\text{cob}} N^n$

cobordant (resp. oriented cobordant)

$\overset{\text{def.}}{\Longleftrightarrow}$

$\exists V^{n+1}$: compact manifold (resp. oriented)

s.t. $\partial V^{n+1} = M^n \cup N^n$ (resp. $M^n \cup (-N^n)$)



Cobordism group of manifolds

$\mathfrak{N}_n = \{[M] \mid \dim M = n\}$

$\Omega_n = \{[M]_{\text{ori}} \mid \dim M = n \text{ and } M \text{ is oriented}\}$

$\Uparrow$

additive groups

$[M] + [M'] = [M \cup M']$

Pontrjagin, Thom, Milnor, Wall, etc...

Detailed structures of $\mathfrak{N}_n$ and Ω_n are known.

Today's topic

Singular fibers of

generic differentiable maps

$\Downarrow$

$\mathfrak{N}_2 \cong \mathbb{Z}_2$, $\Omega_2 = \Omega_3 = 0$,

$\Omega_4 \cong \mathbb{Z}$

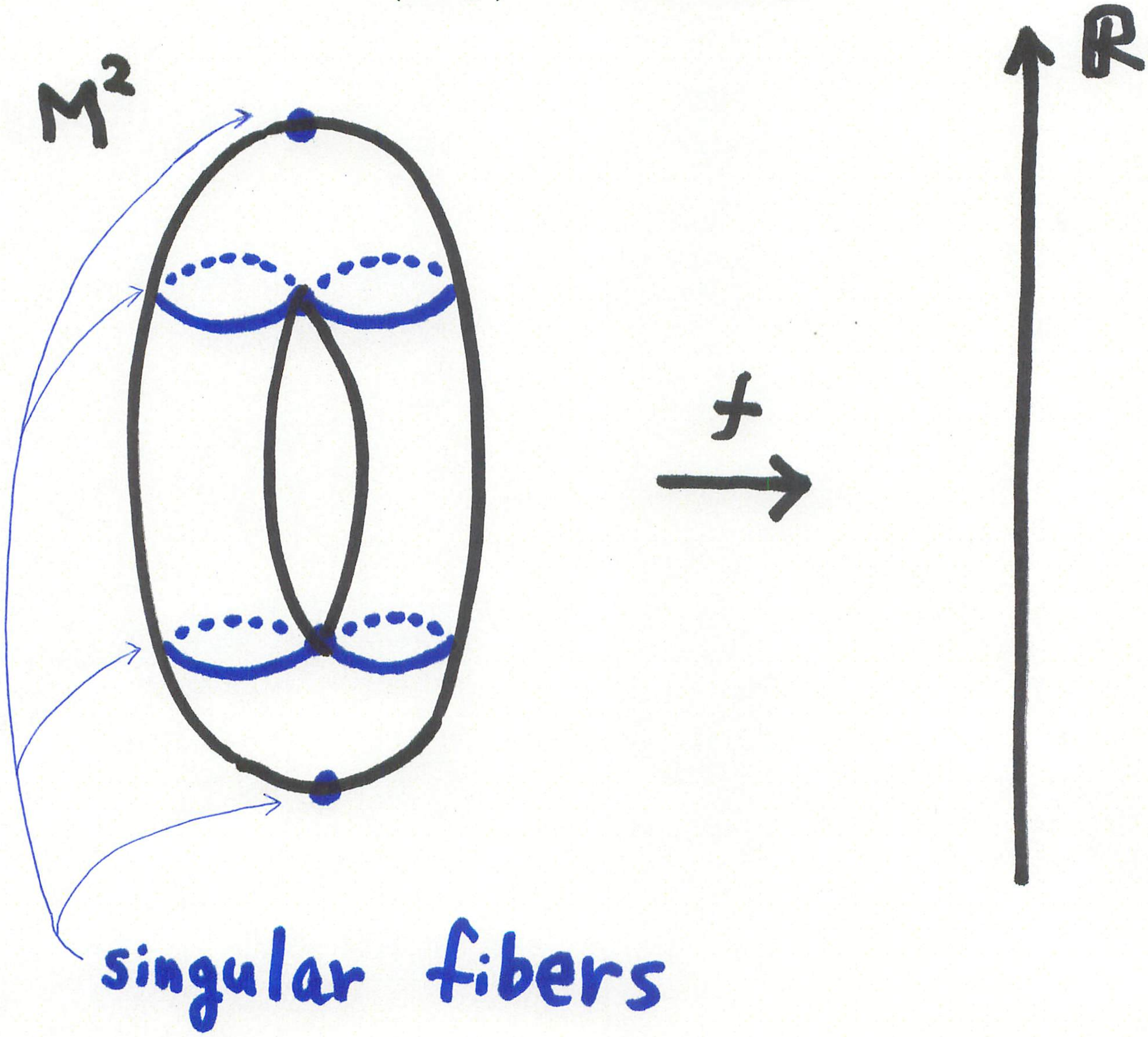
2 2-dimensional case

$\forall [M^2] \in \mathfrak{N}_2$

$\exists f : M^2 \to \mathbb{R}$ Morse function

Singularities of f : non-degenerate critical points

$$(x, y) \mapsto \pm x^2 \pm y^2$$



Def. 2 $f_i : M_i \to N_i$ smooth maps

$$y_i \in N_i \quad (i = 0, 1)$$

fibers over y_0 and y_1 are C^∞ equivalent

$\overset{\text{def.}}{\Longleftrightarrow}$ $\exists U_i \ni y_i$,

diffeomorphisms

$$\exists \widetilde{\varphi} : (f_0)^{-1}(U_0) \xrightarrow{\cong} (f_1)^{-1}(U_1)$$

$$\exists \varphi : U_0 \xrightarrow{\cong} U_1 \quad \text{with} \quad \varphi(y_0) = y_1$$

s.t.

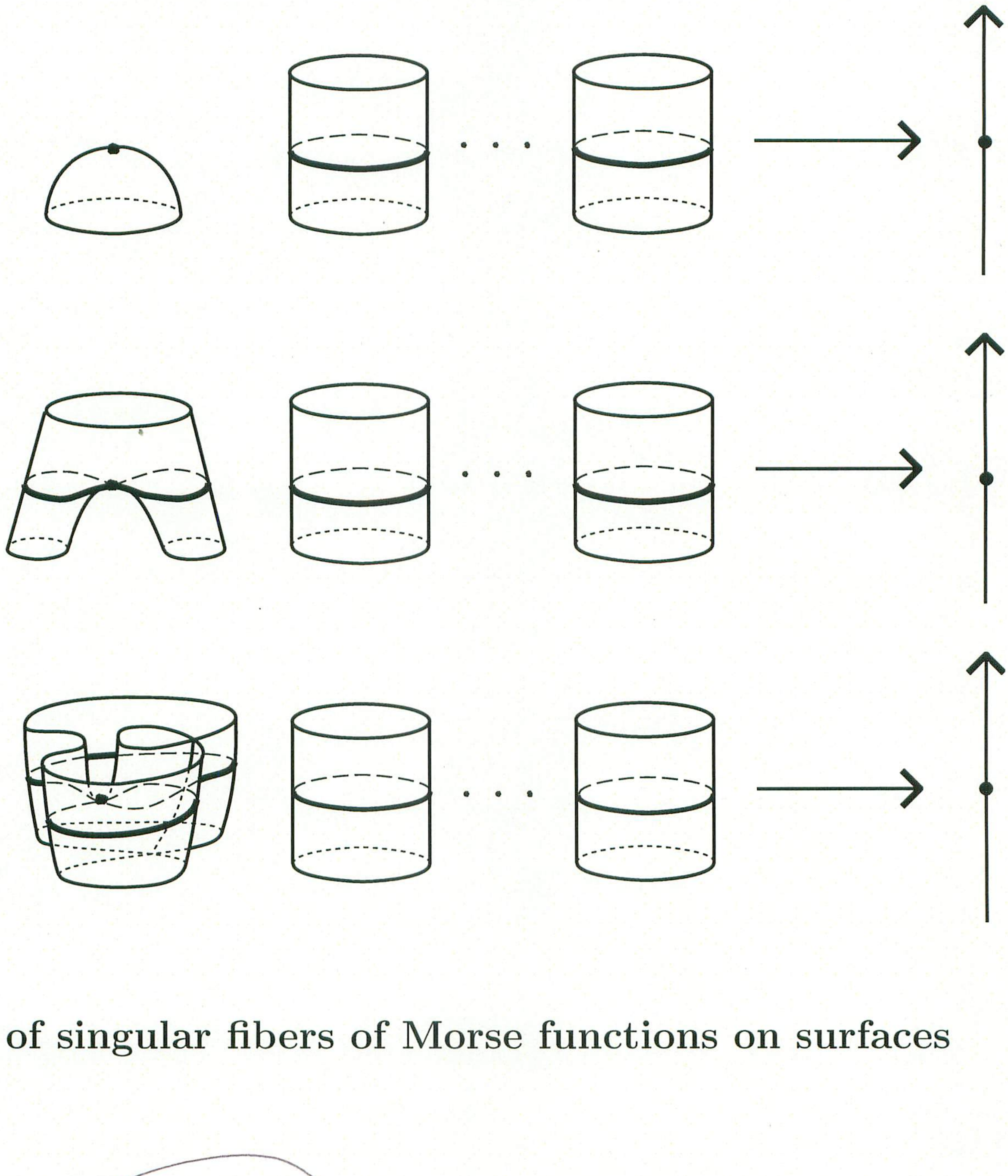
$$\begin{array}{ccc} (f_0)^{-1}(U_0) & \xrightarrow{\widetilde{\varphi}} & (f_1)^{-1}(U_1) \\ {\scriptstyle f_0}\downarrow & \circlearrowright & \downarrow{\scriptstyle f_1} \\ U_0 & \xrightarrow{\varphi} & U_1 \end{array}$$

Classification of singular fibers

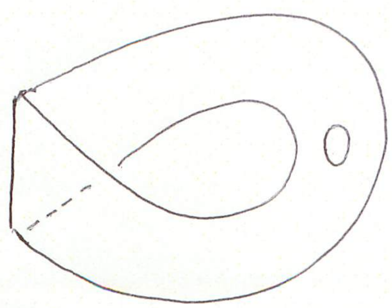
(1)

(2)

(3)



List of singular fibers of Morse functions on surfaces



$f : M^2 \to \mathbb{R}$ Morse function

$\Downarrow$

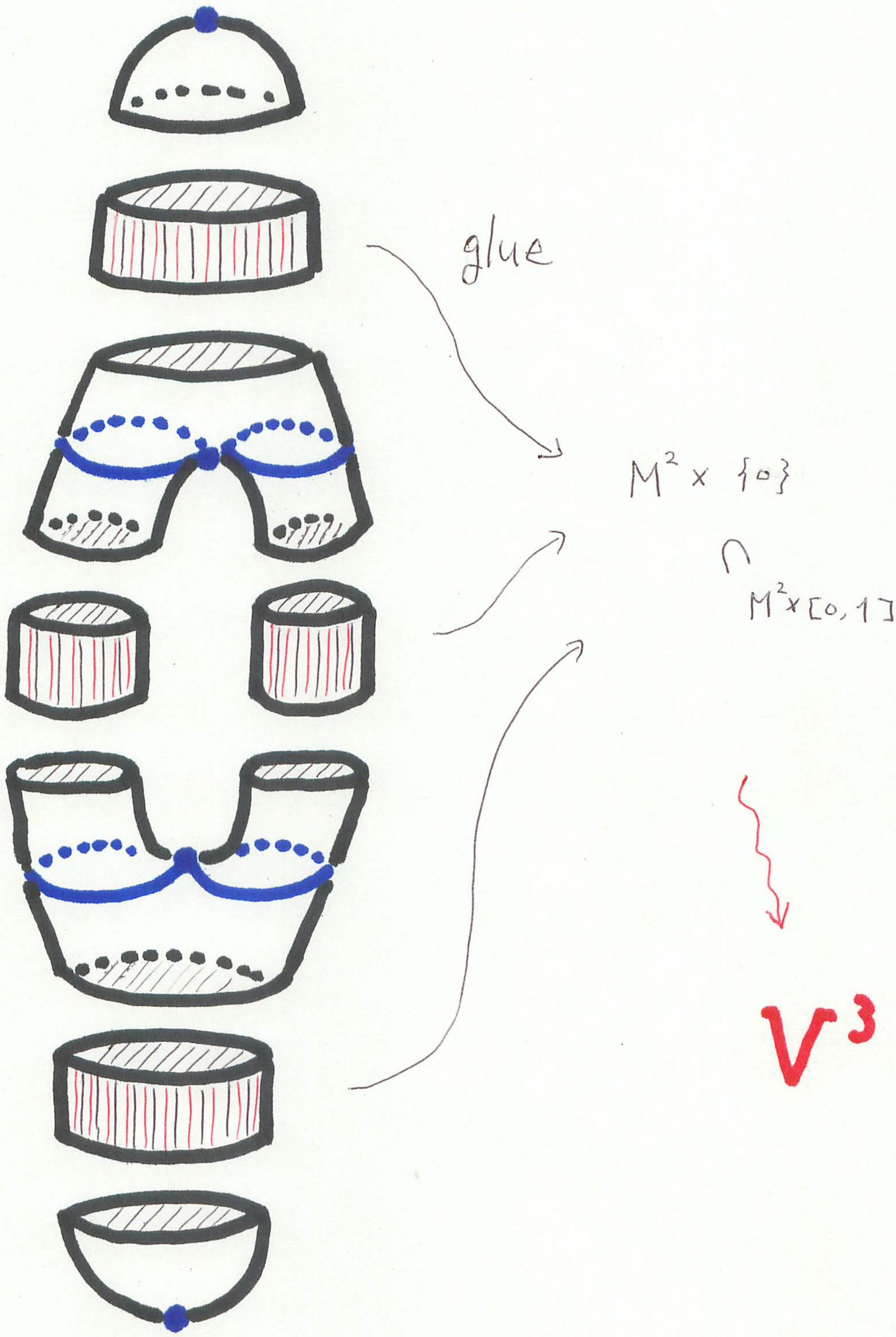
construct $\exists V^3$ from $M^2 \times [0, 1]$

by

gluing 2-disks along regular S^1-fibers of

$f : M^2 \times \{0\} \to \mathbb{R}$.

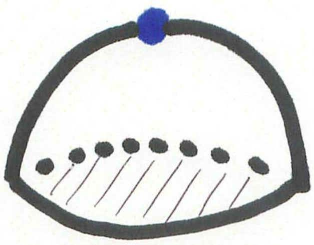
More precisely, glue 2-disk bundles over arcs.


glue
$M^2 \times \{0\}$
$\cap$
$M^2 \times [0,1]$
V^3

$$\partial V^3 = (M^2 \times \{1\}) \cup \left(\bigcup_i F_i^2 \right)$$

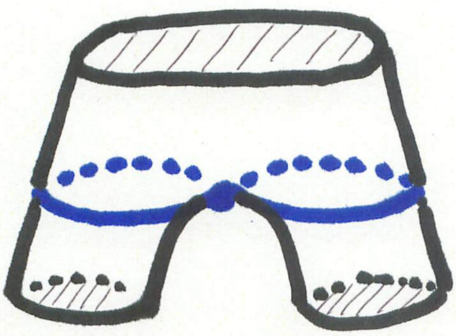
each $F_i^2 \longleftrightarrow$ singular fiber

(1)



$F_i^2 \cong S^2 \quad (= \partial D^3)$

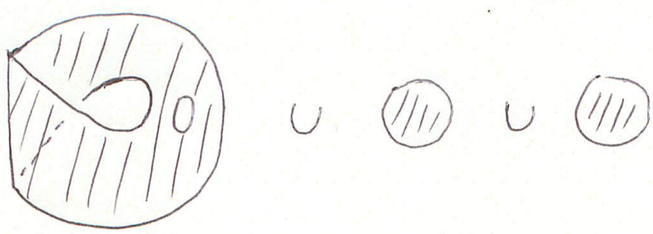
(2)



$F_i^2 \cong S^2 \quad (= \partial D^3)$

(3)

$F_i^2 \cong \mathbb{R}P^2$



Lemma 3

$$\forall M^2 \sim_{\mathrm{cob}} \bigcup_j \mathbb{R}P^2$$

Define the homomorphism

$$\varphi : \mathbb{Z}_2 \longrightarrow \mathfrak{N}_2$$

by $\varphi(1) = [\mathbb{R}P^2]$.

$\mathbb{R}P^2 \cup \mathbb{R}P^2 = \partial(\mathbb{R}P^2 \times [0,1])$

$\Rightarrow \varphi$ is well-defined

φ is surjective by Lemma 3.

Consider the composition:

$$\mathbb{Z}_2 \xrightarrow{\varphi} \mathfrak{N}_2 \xrightarrow{\chi} \mathbb{Z}_2$$

χ: Euler characteristic mod 2

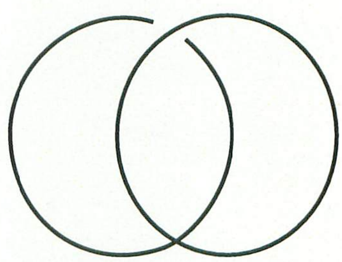
This is the identity map. $\Rightarrow \varphi$ is injective.

Thm. 4 $\mathfrak{N}_2 \cong \mathbb{Z}_2$

The projective plane $\mathbb{R}P^2$ is
a natural generator of
$\mathfrak{N}_2 \cong \mathbb{Z}_2$.

Cor. 5 M^2: closed surface

$f : M^2 \to \mathbb{R}$ Morse function

$\Longrightarrow \quad \chi(M^2) \equiv \sharp\Bigg($  $\Bigg)$ (mod 2)

Similarly, we have $\Omega_2 = 0$.

3 3-dimensional case

$\forall [M^3] \in \Omega_3$ (M^3: oriented)

$\exists f : M^3 \to \mathbb{R}^2$ C^∞ stable map

Singularities of f:

$$(x, y, z) \mapsto (x, y^2 \pm z^2) \qquad \text{fold point}$$

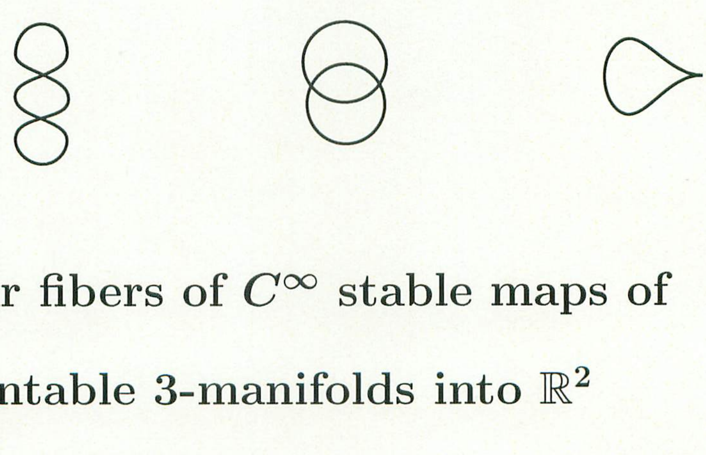
$$(x, y, z) \mapsto (x, y^3 + xy - z^2) \qquad \text{cusp}$$

Classification of singular fibers

(Kushner-Levine-Porto 1984)

$\kappa = 1$

$\kappa = 2$



Singular fibers of C^∞ stable maps of orientable 3-manifolds into $\mathbb{R}^2$

$$f : M^3 \times \{0\} \to \mathbb{R}^2 \qquad C^\infty \text{ stable map}$$

$$\Downarrow$$

Construct $\exists V^4$ from $M^3 \times [0, 1]$

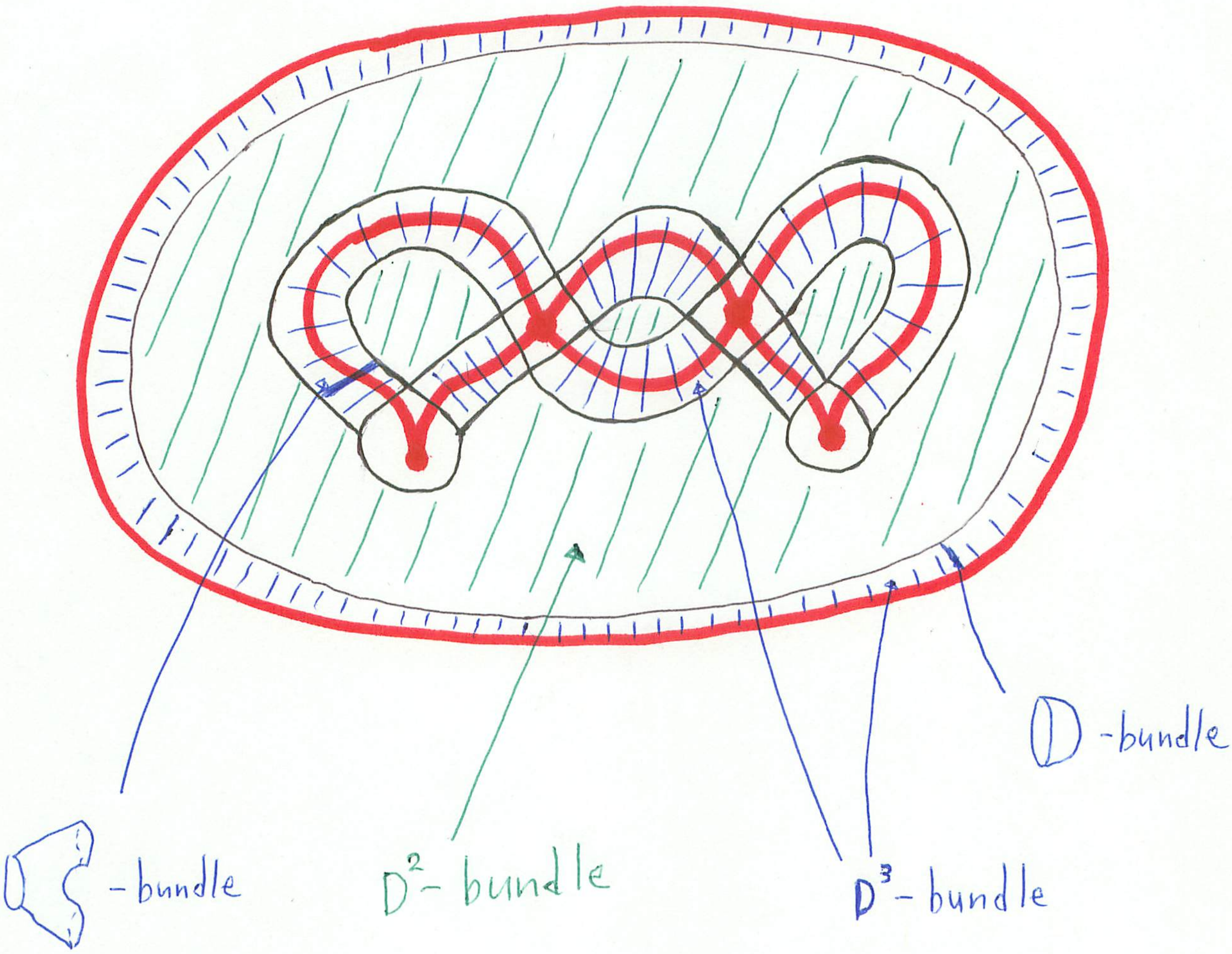
(1) by attaching 2-disks along regular S^1-fibers

(more precisely, attach 2-disk bundles over surfaces), and

(2) by attaching 3-disks along singular fibers of $\kappa = 1$

(cf. 2-dimensional case, $\Omega_2 = 0$)

(more precisely, attach 3-disk bundles over arcs).


D-bundle
-bundle
D^2- bundle
D^3- bundle

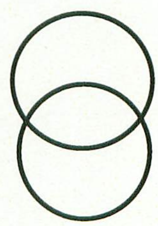
$$\partial V^4 = (-M^3) \cup \left(\bigcup_j F_j^3 \right)$$

each $F_j^3 \longleftrightarrow$ singular fiber of $\kappa = 2$

Prop. 6 (Costantino–D. Thurston 2006)

$F_j^3 \cong S^3 \quad (= \partial D^4)$

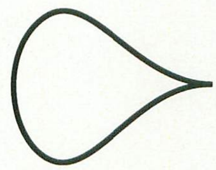
for



Prop. 7

$F_j^3 \cong S^3 \quad (= \partial D^4)$

for



$\Downarrow$

$\Omega_3 = 0$

Remark 1 (Kalmár 2007)

$\forall M^3$ admits a fold map $f : M^3 \to \mathbb{R}^2$.

(1) f is "oriented cobordant" to a simple fold map.

(2) Any simple fold map is "oriented null-cobordant".

$\Longrightarrow \quad \Omega_3 = 0$

4 4-dimensional case

$\forall [M^4] \in \Omega_4$ (M^4 : oriented)

$\exists f : M^4 \to \mathbb{R}^3$ C^∞ stable map

Singularities of f:

$(x, y, z, w) \mapsto (x, y, z^2 \pm w^2)$ fold point

$(x, y, z, w) \mapsto (x, y, z^3 + xz - w^2))$ cusp

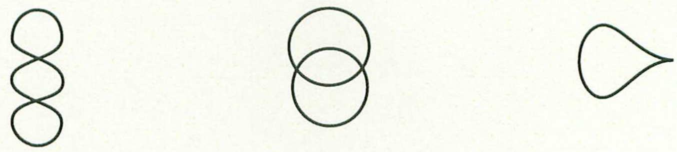
$(x, y, z, w) \mapsto (x, y, z^4 + xz^2 + yz + w^2)$

swallow-tail

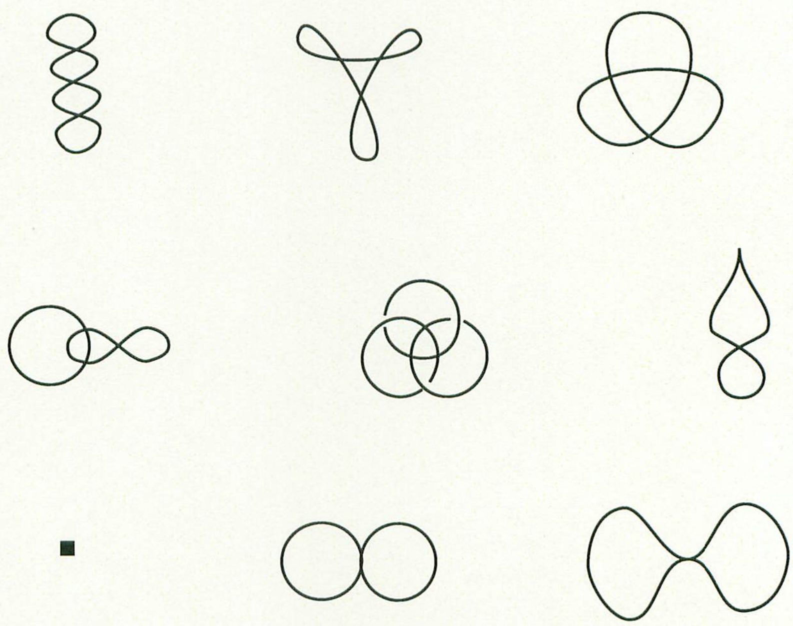
Classification of singular fibers (S. 1999)

$\kappa = 1$

$\kappa = 2$



$\kappa = 3$



Singular fibers of C^∞ stable maps of

orientable 4-manifolds into $\mathbb{R}^3$

Construct $\exists V^5$ from $M^4 \times [0,1]$

(1) by attaching 2-disks along regular S^1-fibers

(more precisely, attach 2-disk bundles over 3-manifolds),

(2) by attaching 3-disks along

singular fibers of $\kappa = 1$

(cf. 2-dimensional case, $\Omega_2 = 0$)

(more precisely, attach 3-disk bundles over surfaces), and

(3) by attaching 4-disks along

singular fibers of $\kappa = 2$

(cf. 3-dimensional case, $\Omega_3 = 0$)

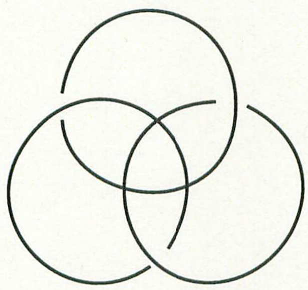
(more precisely, attach 4-disk bundles over arcs).

$$\partial V^5 = (-M^4) \cup \left(\bigcup_j F_j^4 \right)$$

each $F_j^4 \longleftrightarrow$ singular fiber of $\kappa = 3$

Prop. 8

We have $F_j^4 \cong S^4$ except for



For this singular fiber, we have $F_j^4 \cong \pm \mathbb{C}P^2$.

Cor. 9 $\forall M^4 \sim_{\text{cob}} \cup(\pm \mathbb{C}P^2)$

Define the homomorphism

$$\varphi : \mathbb{Z} \to \Omega_4$$

by $\varphi(1) = [\mathbb{C}P^2]$.

φ is surjective by the above Corollary.

Consider the composition:

$$\mathbb{Z} \xrightarrow{\varphi} \Omega_4 \xrightarrow{\sigma} \mathbb{Z}$$

σ : signature

This is the identity map. $\Rightarrow$ φ is injective.

Thm. 10 $\Omega_4 \cong \mathbb{Z}$

The complex projective plane $\mathbb{C}P^2$

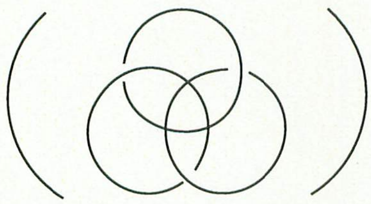
is a natural generator of

$\Omega_4 \cong \mathbb{Z}$.

Cor. 11 (T. Yamamoto-S. 2006)

M^4 : closed oriented 4-manifold

$f : M^4 \to \mathbb{R}^3$ $\quad C^\infty$ stable map

$\Longrightarrow \quad \sigma(M^4) = \sharp$ 

$\uparrow$

counted with signs ± 1

Cor. 12 $f : M \to N$ smooth map

M: closed oriented manifold

$\dim M = \dim N + 1 \geq 4$

s.t. singular fibers of f are of codimension ≤ 3,

not of the above type

$\Longrightarrow M$ is oriented null cobordant.