SPECIAL GENERIC MAPS ON OPEN 4-MANIFOLDS

(OSAMU SAEKI)

Faculty of Mathematics, Kyushu University

ABSTRACT. We characterize those smooth 1-connected open 4-manifolds with certain finite type properties which admit proper special generic maps into 3-manifolds. As a corollary, we show that a smooth 4-manifold homeomorphic to $\mathbf{R}^4$ admits a proper special generic map into $\mathbf{R}^3$ if and only if it is diffeomorphic to $\mathbf{R}^4$. We also characterize those smooth 4-manifolds homeomorphic to $L \times \mathbf{R}$ for some closed orientable 3-manifold L which admit proper special generic maps into $\mathbf{R}^3$.

1. INTRODUCTION

A *special generic map* $f : M \to N$ between smooth manifolds is a smooth map with at most *definite fold singularities*, which have the normal form

$$(1.1) \qquad (x_1, x_2, \ldots, x_m) \mapsto (x_1, x_2, \ldots, x_{n-1}, x_n^2 + x_{n+1}^2 + \cdots + x_m^2),$$

where $m = \dim M \geq \dim N = n$. For some typical examples of special generic maps, refer to Fig. 1. Note also that the map $\mathbf{R}^m \to \mathbf{R}^n$ defined by (1.1) is itself a proper special generic map, where a continuous map is *proper* if the inverse image of a compact set is always compact. Submersions are also considered special generic maps.

It has been known as the Reeb Theorem [19] that if a smooth connected closed m-dimensional manifold admits a special generic map into $\mathbf{R}$, then it is homeomorphic to the m-sphere S^m. In [20, 21], the author has shown that a smooth connected closed m-dimensional manifold M admits a special generic map into $\mathbf{R}^n$ for every n with $1 \leq n \leq m$ if and only if M is diffeomorphic to the standard m-sphere S^m. In [23, 24] Sakuma and the author found some pairs of homeomorphic smooth closed 4-manifolds such that one of them admits a special generic map into $\mathbf{R}^3$, while the other does not. These show that special generic maps are sensitive to detecting distinct differentiable structures on a given topological manifold.

On the other hand, it has been known that a smooth m-dimensional manifold is homeomorphic to $\mathbf{R}^m$ if and only if it is diffeomorphic to the standard $\mathbf{R}^m$, provided $m \neq 4$ (see [15, 26]), while for $m = 4$, there exist uncountably many
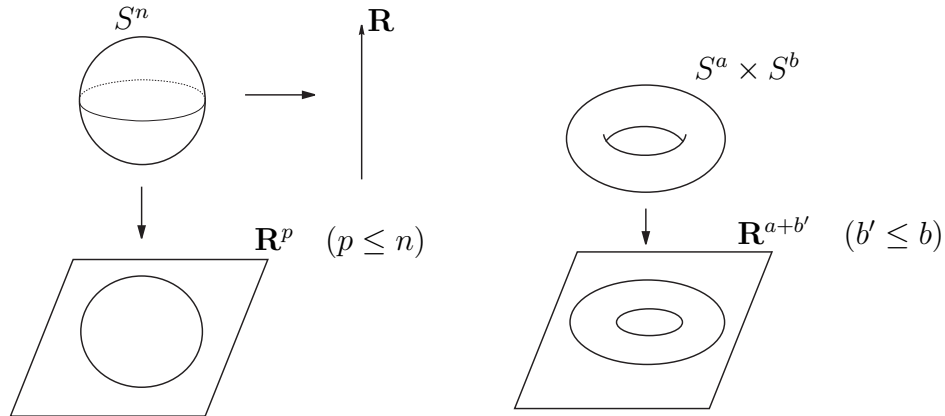
FIGURE 1. Examples of special generic maps

distinct differentiable structures on $\mathbf{R}^4$ (for example, see [4, 6, 8, 27]). In fact, it is known that most open 4-manifolds admit infinitely (and very often, uncountably) many distinct differentiable structures [1, 3, 5, 7].

In this paper, we characterize those smooth 1-connected open 4-manifolds of "finite type" which admit proper special generic maps into 3-manifolds, using the solution to the Poincaré Conjecture in dimension three (see [16, 17, 18] or [14], for example). Here, an open 4-manifold is of finite type if its homology is finitely generated and it has only finitely many ends, whose associated fundamental groups are stable and finitely presentable. As a corollary, we show that a smooth 4-manifold homeomorphic to $\mathbf{R}^4$ is diffeomorphic to the standard $\mathbf{R}^4$ if and only if it admits a proper special generic map into $\mathbf{R}^3$.

Furthermore, we show that if a smooth 4-manifold M is homeomorphic to $L \times \mathbf{R}$ for some connected closed orientable 3-manifold L and if M admits a proper special generic map into $\mathbf{R}^3$, then M is diffeomorphic to $L \times \mathbf{R}$ and the 3-manifold L admits a special generic map into $\mathbf{R}^2$.

All these results claim that among the (uncountably or infinitely) many distinct differentiable structures on a certain open topological 4-manifold, there is at most one smooth structure that allows the existence of a proper special generic map into a 3-manifold.

Throughout the paper, manifolds and maps between them are differentiable of class C^∞ unless otherwise indicated. The symbol "$\cong$" denotes a diffeomorphism between smooth manifolds or an appropriate isomorphism between algebraic objects.

The author would like to express his sincere gratitude to Kazuhiro Sakuma for stimulating discussions and invaluable comments.

2. PRELIMINARIES

Let us first recall the following notion of a Stein factorization, which will play an important role in this paper.

Definition 2.1. Let $f : M \to N$ be a smooth map between smooth manifolds. For two points $x, x' \in M$, we define $x \sim_f x'$ if $f(x) = f(x')(= y)$, and the points x and x' belong to the same connected component of $f^{-1}(y)$. We define
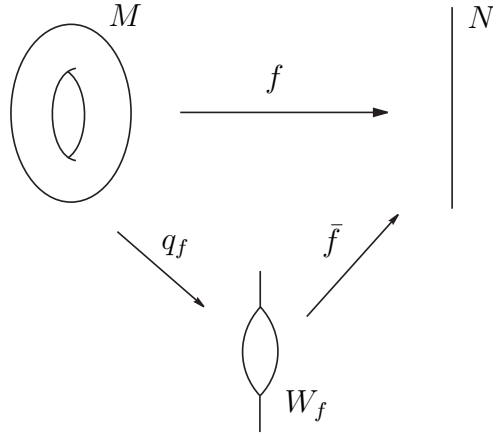
FIGURE 2. Stein factorization

$W_f = M/\sim_f$ to be the quotient space with respect to this equivalence relation, and denote by $q_f : M \to W_f$ the quotient map. Then we see easily that there exists a unique continuous map $\bar{f} : W_f \to N$ that makes the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\quad f \quad} & N \\
{\scriptstyle q_f} \searrow & & \nearrow {\scriptstyle \bar{f}} \\
& W_f &
\end{array}
$$

commutative. The above diagram is called the *Stein factorization* of f (see [13]). Refer to Fig. 2 for an example.

The Stein factorization is a very useful tool for studying topological properties of special generic maps. In fact, we can prove the following, which is folklore (for example, see [2, 20]).

Proposition 2.2. *Let $f : M \to N$ be a proper special generic map between smooth manifolds with $m = \dim M > \dim N = n$. Then we have the following.*

(1) *The set of singular points $S(f)$ of f is a regular submanifold of M of dimension $n - 1$, which is closed as a subset of M.*

(2) *The quotient space W_f has the structure of a smooth n-dimensional manifold possibly with boundary such that $\bar{f} : W_f \to N$ is an immersion.*

(3) *The quotient map $q_f : M \to W_f$ restricted to $S(f)$ is a diffeomorphism onto ∂W_f.*

(4) *If M is connected, then the quotient map q_f restricted to $M \setminus S(f)$ is a smooth fiber bundle over $\operatorname{Int} W_f$. Furthermore, if $S(f) \neq \emptyset$, then the fiber is the standard $(m - n)$-sphere S^{m-n}.*

See Fig. 3 for an illustrative explanation.

Using the above proposition, the author proved the following [20].

Theorem 2.3 (Disk bundle theorem)**.** *Let $f : M \to N$ be a proper special generic map between smooth connected manifolds with $\dim M = m$ and $\dim N = n$. If $m - n = 1, 2, 3$ and $S(f) \neq \emptyset$, then M is diffeomorphic to the boundary of a D^{m-n+1}-bundle over W_f with $O(m - n + 1)$ as structure group.*
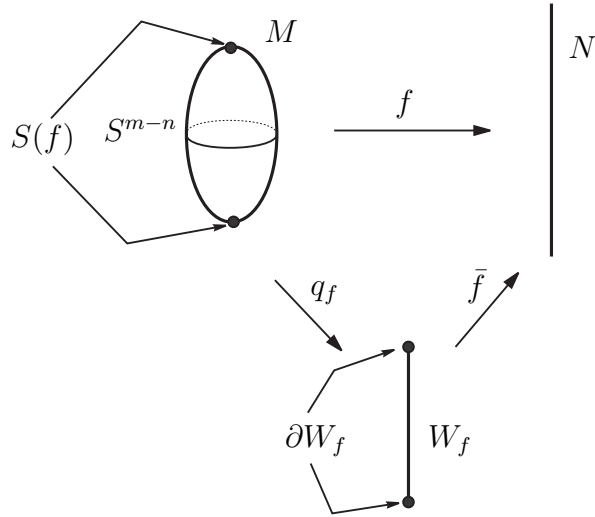
FIGURE 3. Proposition 2.2

In the following, we recall several notions concerning ends of manifolds. For details, the reader is referred to Siebenmann's thesis [25].

Definition 2.4. Let X be a Hausdorff space. Consider a collection ε of subsets of X with the following properties.

(i) Each $G \in \varepsilon$ is a connected open non-empty set with compact frontier $\overline{G} - G$,

(ii) If $G, G' \in \varepsilon$, then there exists $G'' \in \varepsilon$ with $G'' \subset G \cap G'$,

(iii) $\bigcap\limits_{G \in \varepsilon} \overline{G} = \emptyset$.

Adding to ε every connected open non-empty set $H \subset X$ with compact frontier such that $G \subset H$ for some $G \in \varepsilon$, we produce a collection satisfying (i), (ii) and (iii), which we call the *end* of X determined by ε.

An *end* of a Hausdorff space X is a collection ε of subsets of X which is maximal with respect to the properties (i), (ii) and (iii) above.

A *neighborhood* of an end ε is any set $N \subset X$ that contains some member of ε. (See Fig. 4.)

Definition 2.5. Let ε be an end of a topological manifold X. The fundamental group π_1 is *stable* at ε if there exists a sequence of path connected neighborhoods of ε, $X_1 \supset X_2 \supset \cdots$, with $\bigcap \overline{X}_i = \emptyset$ such that (with base points and base paths chosen) the sequence

$$\pi_1(X_1) \xleftarrow{\quad f_1 \quad} \pi_1(X_2) \xleftarrow{\quad f_2 \quad} \cdots$$

induced by the inclusions induces isomorphisms

$$\mathrm{Im}(f_1) \xleftarrow{\quad \cong \quad} \mathrm{Im}(f_2) \xleftarrow{\quad \cong \quad} \cdots .$$

The following lemma is proved in [25].
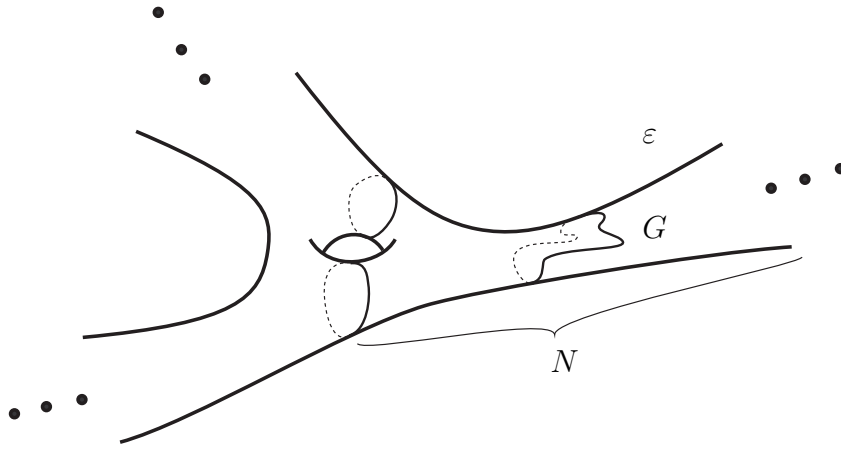
FIGURE 4. Ends of a manifold

Lemma 2.6. *If π_1 is stable at ε and $Y_1 \supset Y_2 \supset \cdots$ is any path connected sequence of neighborhoods of ε such that $\bigcap \overline{Y}_i = \emptyset$, then for any choice of base points and base paths, the inverse sequence*

$$\mathcal{G}: \qquad \pi_1(Y_1) \xleftarrow{\quad g_1 \quad} \pi_1(Y_2) \xleftarrow{\quad g_2 \quad} \cdots$$

induced by the inclusions is stable, i.e. there exists a subsequence

$$\pi_1(Y_{i_1}) \xleftarrow{\quad h_1 \quad} \pi_1(Y_{i_2}) \xleftarrow{\quad h_2 \quad} \cdots$$

inducing isomorphisms

$$\mathrm{Im}(h_1) \xleftarrow{\quad \cong \quad} \mathrm{Im}(h_2) \xleftarrow{\quad \cong \quad} \cdots ,$$

where each h_j is a suitable composition of g_i's.

Definition 2.7. When π_1 is stable at an end ε, we define $\pi_1(\varepsilon)$ to be the projective limit $\varprojlim \mathcal{G}$ for some fixed system $\mathcal{G}$ as above. According to [25], $\pi_1(\varepsilon)$ is well defined up to isomorphism.

Let us introduce the following definition.

Definition 2.8. An open manifold M is of *finite type* if

   (i) M has finitely many ends,
   (ii) for each end ε, π_1 is stable at ε with $\pi_1(\varepsilon)$ being finitely presentable, and
  (iii) $H_*(M; \mathbf{Z}_2)$ is finitely generated.

We will need the following result due to Husch–Price [11, 12].

Lemma 2.9 (Husch–Price, 1970). *Let W be an open orientable 3-manifold of finite type. Then there exists a compact orientable 3-manifold $\widetilde{W}$ and an embedding $h: W \to \widetilde{W}$ such that $h(\mathrm{Int}\, W) = \mathrm{Int}\, \widetilde{W}$.*

5

3. Open 4-manifolds that admit special generic maps

In the following, a manifold is *open* if it has no boundary and each of its component is non-compact, while a manifold is *closed* if it has no boundary and is compact.

Theorem 3.1. *Let M be a smooth 1-connected open 4-manifold of finite type. Then there exists a proper special generic map $f : M \to N$ into a smooth 3-manifold N with $S(f) \neq \emptyset$ if and only if M is diffeomorphic to the connected sum of a finite number of copies of the following 4-manifolds:*

(1) $\mathbf{R}^4$,
(2) *the interior of the boundary connected sum of a finite number of copies of $S^2 \times D^2$,*
(3) *the total space of a 2-plane bundle over S^2,*
(4) *the total space of an S^2-bundle over S^2,*

where at least one manifold of the form (1), (2) or (3) should appear in the connected sum.

Sketch of proof. Let $f : M \to N$ be a proper special generic map into a 3-manifold N. Then we can prove that the quotient space W_f in the Stein factorization of f is an open 3-manifold of finite type. Since M is 1-connected, so is W_f. By the solution to the Poincaré Conjecture together with the Husch–Price Lemma (Lemma 2.9), we see that $W_f \cong D^3 \setminus F$ or $\natural^k(S^2 \times [0,1]) \setminus F$, where F is a compact surface (possibly with boundary) contained in the boundary. On the other hand, M is diffeomorphic to the boundary of a D^2-bundle over W_f by the Disk bundle theorem, Theorem 2.3. Then we easily get the desired conclusion.

Conversely, it is easy to construct explicitly a proper special generic map into a 3-manifold for each 4-manifold in the list. $\square$

Remark 3.2. Every 4-manifold as in Theorem 3.1 admits infinitely many (or uncountably many) distinct smooth structures. Theorem 3.1 implies that among them there is exactly one structure that allows the existence of a proper special generic map into a 3-manifold.

In particular, we have the following.

Corollary 3.3. *Let M be a smooth 4-manifold homeomorphic to $\mathbf{R}^4$. Then there exists a proper special generic map $f : M \to \mathbf{R}^3$ if and only if M is diffeomorphic to the standard $\mathbf{R}^4$.*

We also have the following[1].

Theorem 3.4. *Let L be a smooth connected closed orientable 3-manifold. A smooth 4-manifold M homeomorphic to $L \times \mathbf{R}$ admits a proper special generic map into $\mathbf{R}^3$ if and only if M is diffeomorphic to $L \times \mathbf{R}$ and L is a smooth closed 3-manifold that admits a special generic map into $\mathbf{R}^2$.*

---

[1]Theorem 3.4 was first conjectured by Kazuhiro Sakuma to whom the author would like to express his sincere gratitude.

Sketch of proof. Suppose M is homeomorphic to $L \times \mathbf{R}$ and let $f : M \to N$ be a proper special generic map into a 3-manifold N. Then one can show that W_f is of finite type and has exactly two ends $F_i \times [0, \infty)$, $i = 1, 2$, for some surfaces F_i. Furthermore, the inclusions $F_i \times \{0\} \hookrightarrow W_f$ induce isomorphisms of fundamental groups. By the standard theory of 3-manifolds together with the solution to the Poincaré Conjecture and the Husch–Price Lemma, we see that $W_f \cong (F_1 \times \mathbf{R}) \natural (\natural^k D^3)$ (for example, see [10]). Since M is homeomorphic to $L \times \mathbf{R}$, we see that $W_f \cong F_1 \times \mathbf{R}$. Therefore, M is diffeomorphic to $L' \times \mathbf{R}$ for some 3-manifold L'. Note that $\pi_1(L') \cong \pi_1(L)$ is free. Therefore, $L' \cong L \cong \natural^\ell(S^1 \times S^2)$, and hence there exists a special generic map $g : L \to \mathbf{R}^2$ by a result of Burlet–de Rham [2].

Conversely, if L admits a special generic map $g : L \to \mathbf{R}^2$, then

$$g \times \mathrm{id}_{\mathbf{R}} : L \times \mathbf{R} \to \mathbf{R}^2 \times \mathbf{R}$$

is a proper special generic map, where $\mathrm{id}_{\mathbf{R}}$ denotes the identity map of $\mathbf{R}$. $\square$

Conjecture 3.5. Let M be a topological 4-manifold. Then there exists at most one smooth structure on M that allows the existence of a proper special generic map into $\mathbf{R}^3$.

Remark 3.6. In the above conjecture, the *properness* of the special generic map is essential. Let $f : M \to N$ be a special generic map of an open 4-manifold and assume that M' is homeomorphic to M. Then there exists a "formal solution" over M' on the jet level for the open differential relation corresponding to special generic maps. Therefore, M' admits a special generic map by the Gromov h-principle for open manifolds [9]. Note that even if f is proper, the resulting special generic map on M' may not be proper.

Compare this with the following: if a smooth 4-manifold M is homeomorphic to $\mathbf{R}^4$, then there exists a proper special generic map $g : M \to \mathbf{R}^4$. In the equidimensional case, the C^0 dense h-principle holds and the properness can be preserved (see [9]).

References

[1] Ž. Bižaca and J. Etnyre, *Smooth structures on collarable ends of 4-manifolds*, Topology **37** (1998), 461–467.

[2] O. Burlet and G. de Rham, *Sur certaines applications génériques d'une variété close à 3 dimensions dans le plan*, Enseignement Math. (2) **20** (1974), 275–292.

[3] F. Ding, *Smooth structures on some open 4-manifolds*, Topology **36** (1997), 203–207.

[4] S.K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geom. **18** (1983), 279–315.

[5] F. Fang, *Embedding 3-manifolds and smooth structures of 4-manifolds*, Topology Appl. **76** (1997), 249–259.

[6] M.H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982), 357–453.

[7] R.E. Gompf, *An exotic menagerie*, J. Differential Geom. **37** (1993), 199–223.

[8] R.E. Gompf and A.I. Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, Vol. 20, American Mathematical Society, Providence, RI, 1999.

[9] M. Gromov, *Partial differential relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Vol. 9, Springer-Verlag, Berlin, 1986.

[10] J. Hempel, *3-Manifolds*, Ann. of Math. Studies, No. 86, Princeton Univ. Press, Princeton, N.J.; Univ. of Tokyo Press, Tokyo, 1976.

[11] L.S. Husch and T.M. Price, *Finding a boundary for a 3-manifold*, Ann. of Math. (2) **91** (1970), 223–235.

[12] L.S. Husch and T.M. Price, *Addendum to: "Finding a boundary for a 3-manifold"*, Ann. of Math. (2) **93** (1971), 486–488.

[13] H. Levine, *Classifying immersions into $\mathbf{R}^4$ over stable maps of 3-manifolds into $\mathbf{R}^2$*, Lecture Notes in Math., Vol. 1157, Springer-Verlag, Berlin, 1985.

[14] J. Morgan and G. Tian, *Ricci flow and the Poincaré conjecture*, Clay Math. Monographs, Vol. 3, Amer. Math. Soc., Providence, RI; Clay Math. Institute, Cambridge, MA, 2007.

[15] J. Munkres, *Obstructions to the smoothing of piecewise-differentiable homeomorphisms*, Ann. of Math. (2) **72** (1960), 521–554.

[16] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, preprint, arXiv:math/0211159v1 [math.DG].

[17] G. Perelman, *Ricci flow with surgery on three-manifolds*, preprint, arXiv:math/0303109v1 [math.DG].

[18] G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, preprint, arXiv:math/0307245v1 [math.DG].

[19] G. Reeb, *Sur certaines propriétés topologiques des variétés feuilletées*, Actualités Scientifiques et Industrielles 1183, Hermann, Paris, 1952, pp. 91–154.

[20] O. Saeki, *Topology of special generic maps of manifolds into Euclidean spaces*, Topology Appl. **49** (1993), 265–293.

[21] O. Saeki, *Topology of special generic maps into $\mathbf{R}^3$*, Workshop on Real and Complex Singularities (São Carlos, 1992), Mat. Contemp. **5** (1993), 161–186.

[22] O. Saeki, *Special generic maps on open 4-manifolds*, J. of Singularities **1** (2010), 1–12.

[23] O. Saeki and K. Sakuma, *On special generic maps into $\mathbf{R}^3$*, Pacific J. Math. **184** (1998), 175–193.

[24] O. Saeki and K. Sakuma, *Special generic maps of 4-manifolds and compact complex analytic surfaces*, Math. Ann. **313** (1999), 617–633.

[25] L. Siebenmann, *The obstruction to finding a boundary for an open manifold of dimension greater than five*, Doctoral dissertation, Princeton University, 1965.

[26] J. Stallings, *The piecewise-linear structure of Euclidean space*, Proc. Cambridge Philos. Soc. **58** (1962), 481–488.

[27] C.H. Taubes, *Gauge theory on asymptotically periodic 4-manifolds*, J. Differential Geom. **25** (1987), 363–430.

FACULTY OF MATHEMATICS, KYUSHU UNIVERSITY, MOTOOKA 744, NISHI-KU, FUKUOKA 819-0395, JAPAN

E-mail address: saeki@math.kyushu-u.ac.jp