Topics on singularities of differentiable maps

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Please choose some of the following problems (preferably ≥ 2), and submit their answers as a report. In addition to the answers, please do not forget to write your comments or impressions of my lectures. The deadline will be announced later.

Please refer to the references at the end, if necessary.

(1) In Example 1.2.2, compute $\varphi_- \circ \varphi_+^{-1}$ and check that it is of class C^∞. Furthermore, in Example 1.3.3, compute $\mathrm{id} \circ f \circ \varphi_+^{-1}$ and check that it is of class C^∞.

(2) Show that S^2 is a differentiable manifold of dimension 2.

(3) Show that any non-empty open subset of a differentiable manifold is again a differentiable manifold of the same dimension.

(4) Show that for differentiable manifolds, the relation "diffeomorphic" defines an equivalence relation.

(5) Check Definition 2.1.3 (product manifold).

(6) Show that every closed surface of odd Euler characteristic is non-orientable.

(7) Prove Theorem 2.4.1 (Classification theorem of closed surfaces), or prove that every connected closed surface is diffeomorphic to Σ_g or F_g for some g (refer to [2]).

(8) Show that a compact connected 1-dimensional differentiable manifold is necessarily diffeomorphic to S^1 (refer to [12]).

(9) Describe a rigorous definition of the tangent space of a differentiable manifold at a point, and show that it is a real vector space. Furthermore, based on the definition, describe a rigorous definition of the differential of a differentiable map between manifolds. Finally, show that it is a linear map (refer to [10]).

(10) For each integer n, explicitly construct a differentiable map $S^1 \to S^1$ with mapping degree n.

(11) Construct a differentiable map $f : \Sigma_3 \to \Sigma_2$ with mapping degree 2. (Enough to "draw" the behavior of the map as a figure.)

(12) Let $f : M \to N$ be a differentiable map between equi-dimensional manifolds. Show that if $p \in M$ is not a singular point of f, then there exist an open neighborhood U of p in M and an open neighborhood V of $f(p)$ in N such that $f(U) = V$ and $f|_U : U \to V$ is a diffeomorphism.

(13) Let $f : M \to N$ be a differentiable map between closed surfaces. Show that the singular set $S(f)$ of f is a closed subset of M.

(14) Let $f : M \to N$ be a differentiable map between closed surfaces and $q \in N$ a regular value. Show that then $f^{-1}(q)$ is a finite set (refer to [12]).

(15) Prove Proposition 4.4.2 (Homotopy invariance of the mapping degree) (refer to [12]).

(16) Prove Theorem 4.4.3 (Hopf's Theorem).

(17) Describe a rigorous definition of a vector field.

(18) Determine those connected closed surfaces that admit a vector field without singular points.

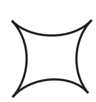
(19) Construct explicitly isolated singular points of a vector field of index ± 3.

(20) Construct explicitly vector fields with exactly one singular point on Σ_g and F_g for each g.

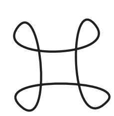
(21) Construct explicit vector fields on some closed surfaces and check that the Poincaré-Hopf theorem holds for them.

(22) Construct explicitly stable maps between closed oriented surfaces and check that Quine's theorem holds for them.

(23) Construct a stable map $f : T^2 \to S^2$ with $\deg(f) = 1$ whose apparent contour is either

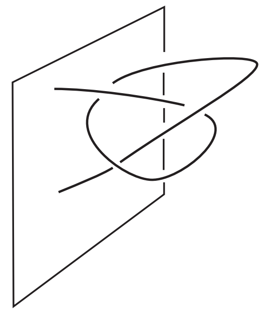


or



Furthermore, show that these are minimal contours.

(24) Show that the spun of the following knot has width smaller than or equal to 8.



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