Topology of Singular Fibers for Visualization

Osamu Saeki
(Institute of Mathematics for Industry, Kyushu Univ.)

Joint work with
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May 20, 2015, at TopoInVis2015

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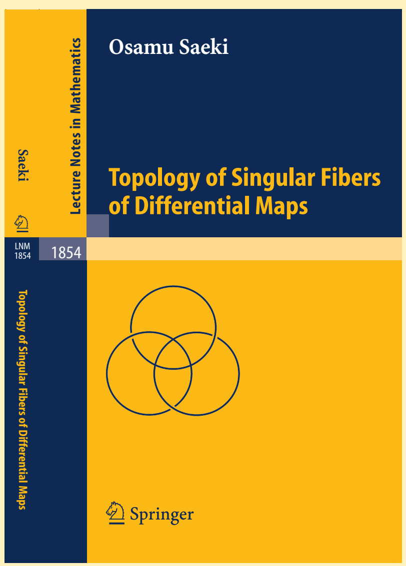
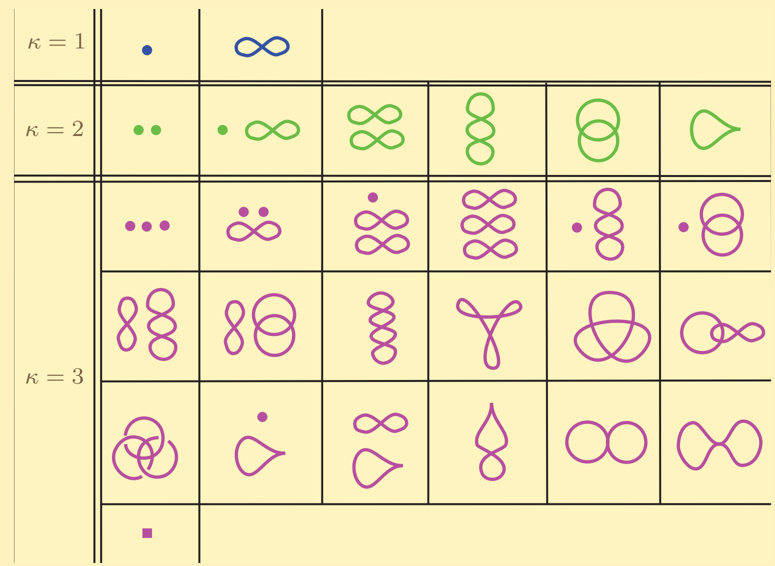
Main interest: Singularity Theory, 3- and 4-Dimensional Topology
Proposed the **Theory of Singular Fibers of Differentiable Maps**.





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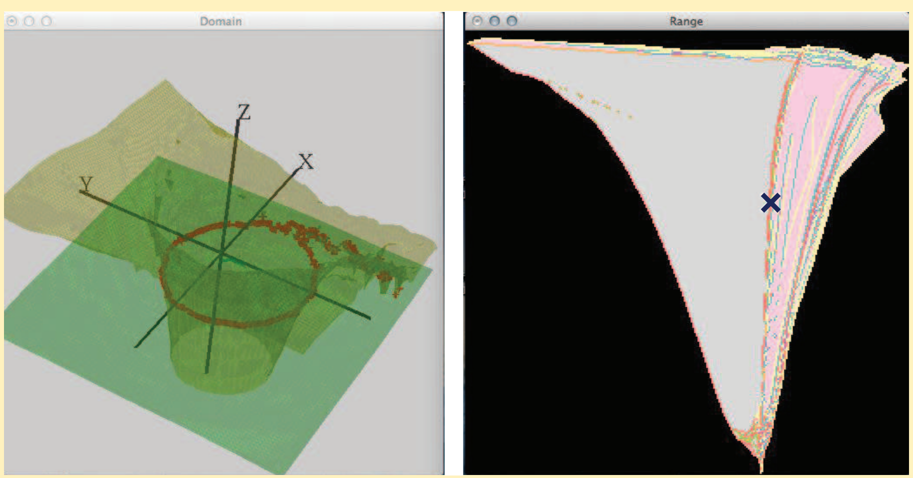
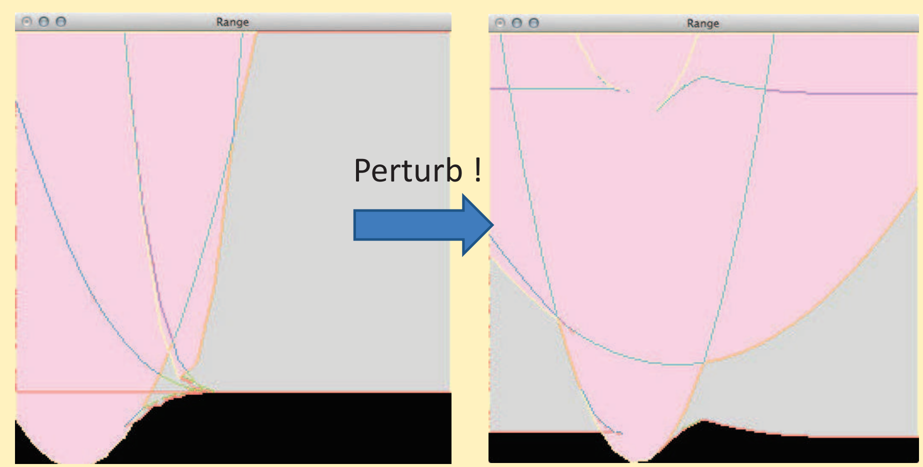
Kyushu University, Japan

Main idea of today's talk

"Topological Approach to Visualization of Scientific Data"

- Use techniques from **Differential Topology**, especially those of **Singularity Theory**: **Topology** is essential for extracting global features of given data.
- Visualize **Multi-fields**, instead of Scalar fields.
- Apply visualization techniques to **Mathematics** itself.





§1. Visualizing Scalar Field Data

Level set

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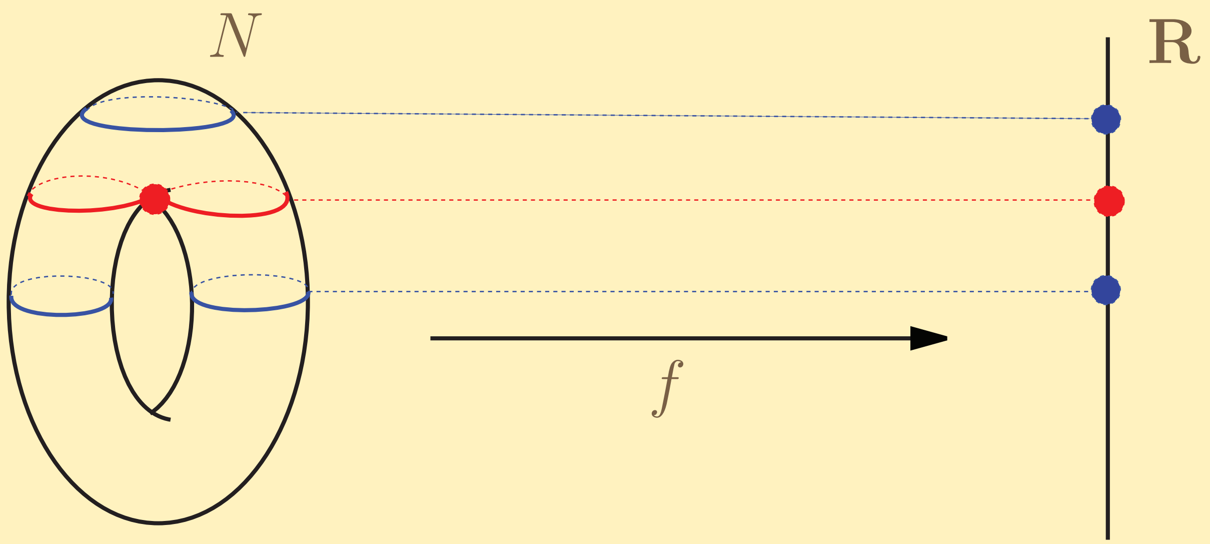
In general, a level set is of dimension $n-1$ (but may not be a manifold).
For $n = 2$, it is a curve; for $n = 3$, it is a surface, etc.

Example 1.2 Altitude from the sea level (height function):
level set $=$ contour line

Example of level sets



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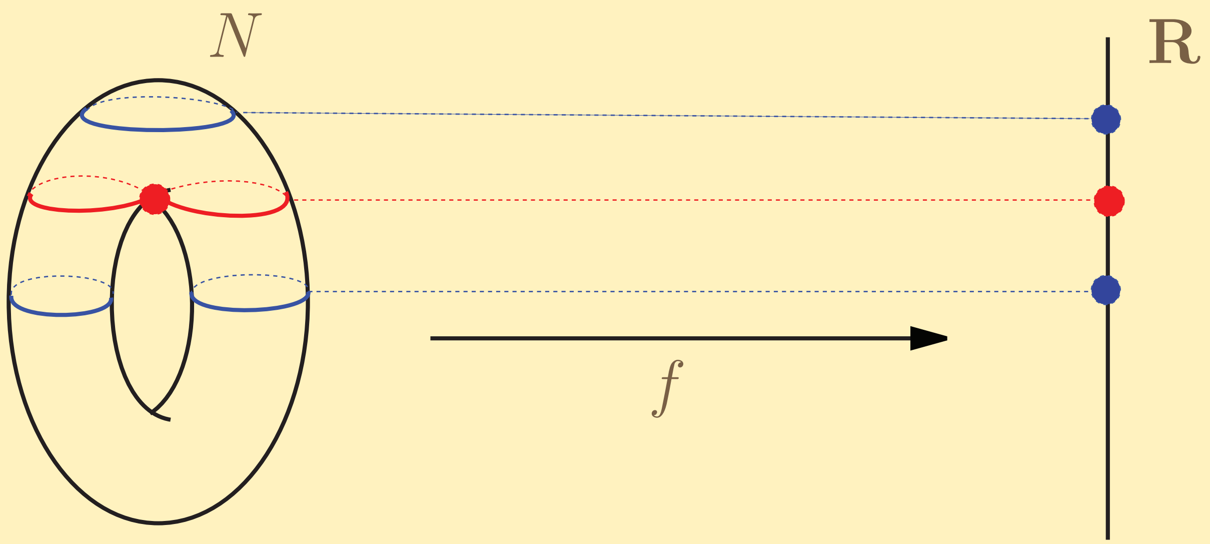

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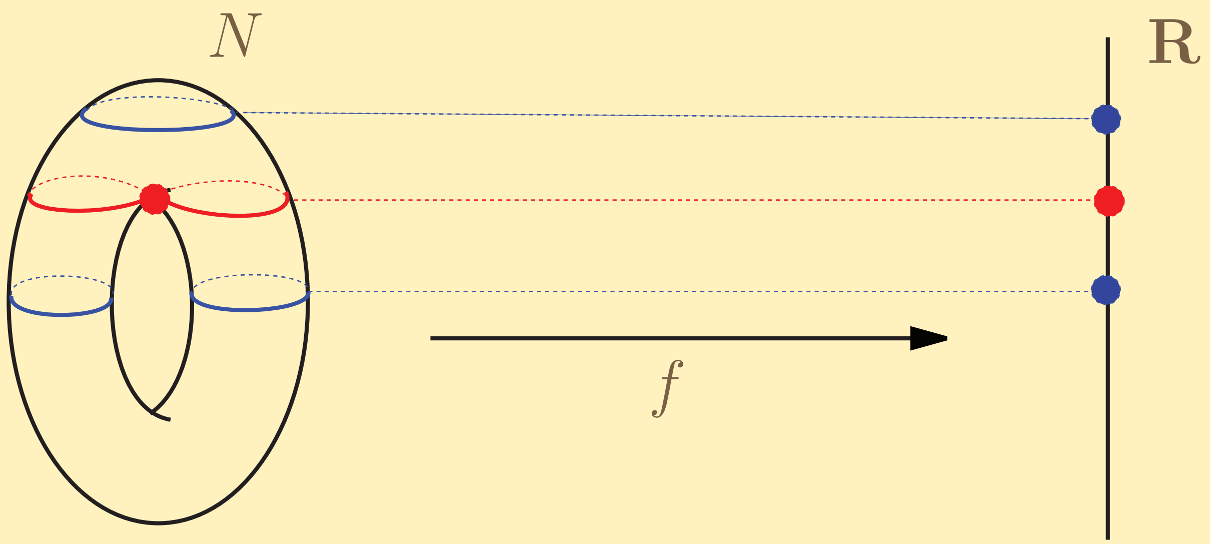

One can grasp the global feature of the data by chasing the level sets. We have some **critical level sets** where **topological transitions of level sets** occur.

Morse lemma

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Theorem 1.3 (Morse lemma) *If f is* **generic** *enough, then around each critical point, f is expressed as*

$$f = \pm x_1^2 \pm x_2^2 \pm \cdots \pm x_n^2 + c$$

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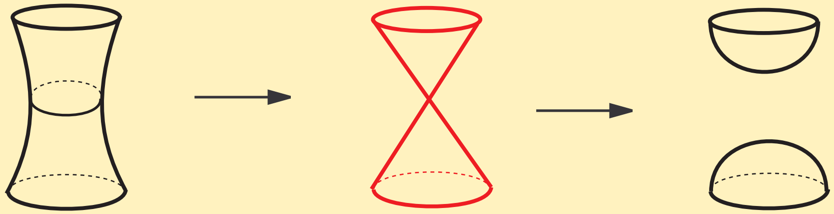
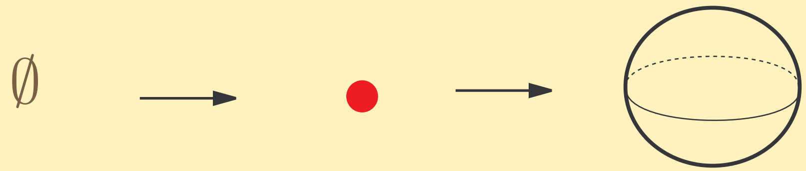
For the study of level-set changes, the Morse lemma is essential !

3-Dimensional example

$f : N^3 \to \mathbf{R} \quad (\dim N^3 = 3)$
Level sets are surfaces "with singularities".





Example of topological transitions of level-surfaces for a 3-dimensional scalar field around **critical level sets**.

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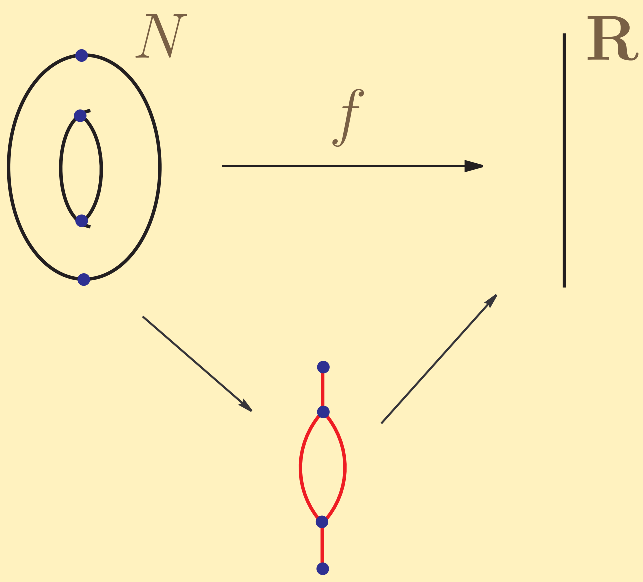

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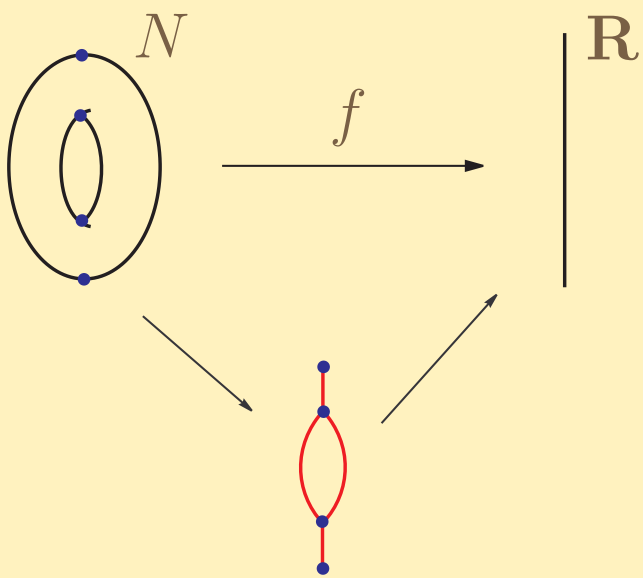
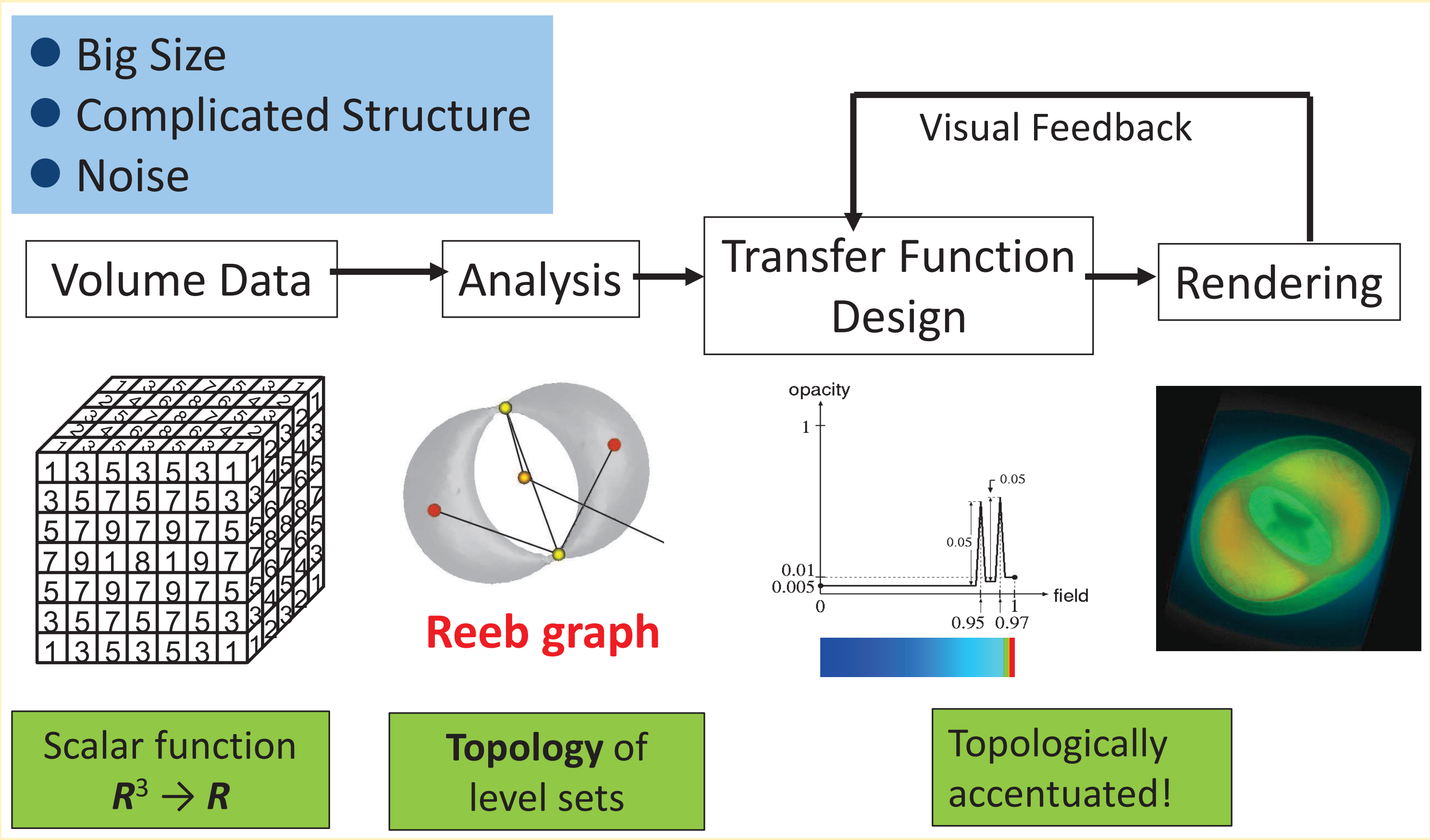


Vertices of a Reeb graph $\Longleftrightarrow$ Critical points of a function
Reeb graph is indispensable for visualizing scalar fields.

An example of an application of Reeb graph:
[Takahashi–Takeshima–Fujishiro, 2004] **Topological Volume Skeletonization and its Application to Transfer Function Design**



§2. Visualizing Multi-field Data

Multivariate data analysis

We study several functions at the same time, rather than a single scalar valued function.

For technical reasons, topological analysis of such **multi-variate data** has just recently begun.

We can attack this problem, using the recently developed "**Joint Contour Net**", a novel technique in Computer Science, on the basis of **Singularity Theory**, a sophisticated discipline in Mathematics.

Fiber

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$f : N^n \to \mathbf{R}^m$ $(m \geq 1)$ differentiable **map** (or **multi-field**)

$$f(x) = (f_1(x), f_2(x), \ldots, f_m(x))$$

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Generically, we have $\dim f^{-1}(\boldsymbol{c}) = n - m$.
Usually, we assume $n \geq m$.

More precisely...

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Mathematically, a fiber is, in fact, NOT just a subset in N^n, but a MAP around a pre-image.

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For points $\boldsymbol{c} \in \mathbf{R}^m$ and $\boldsymbol{d} \in \mathbf{R}^m$, fibers over $\boldsymbol{c}$ and $\boldsymbol{d}$ are **equivalent** (or the points have the same **singular fiber type**) if

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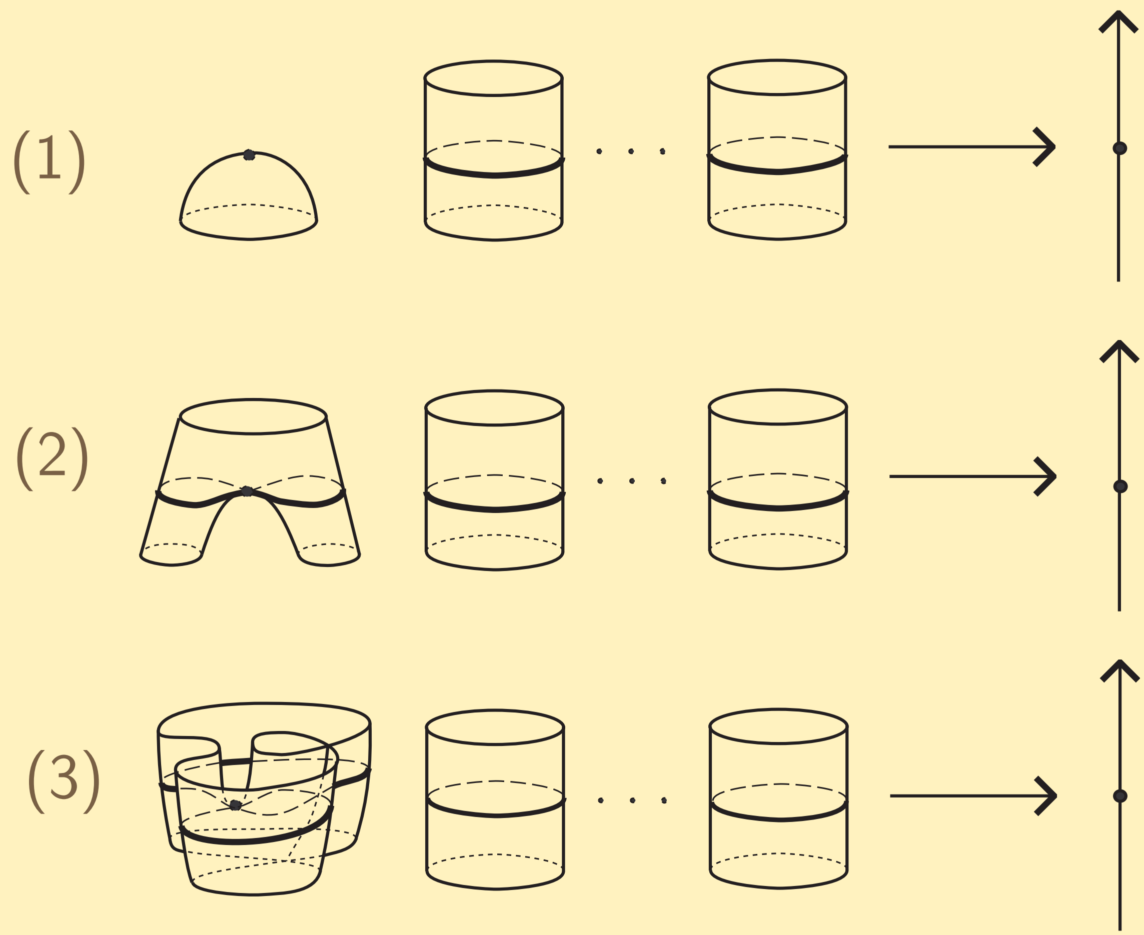
for some neighborhoods $\boldsymbol{c} \in U \subset \mathbf{R}^m$ and $\boldsymbol{d} \in V \subset \mathbf{R}^m$.

Singular fibers for scalar fields



Equivalence classes of singular fibers for Morse functions on surfaces

Example of fibers

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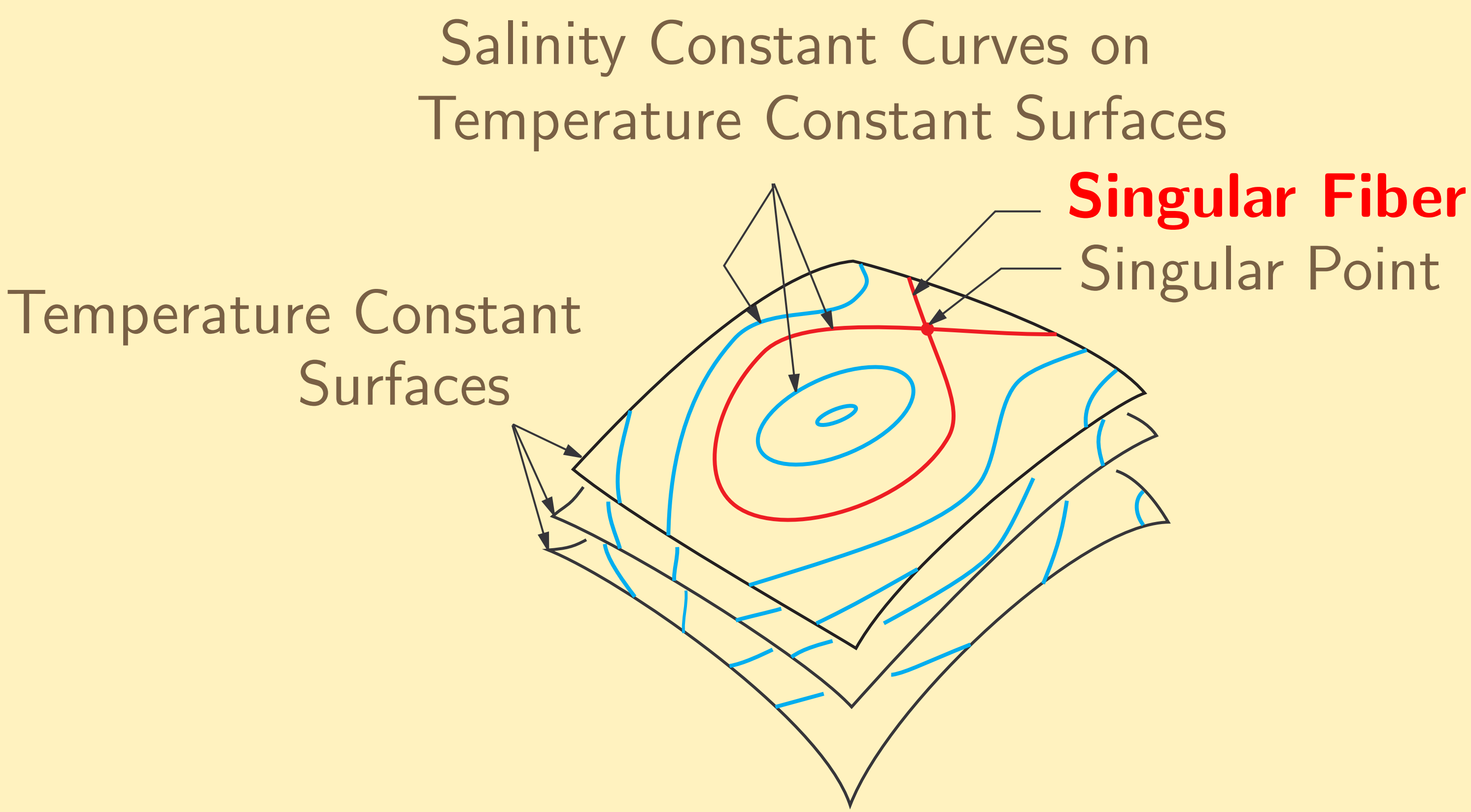

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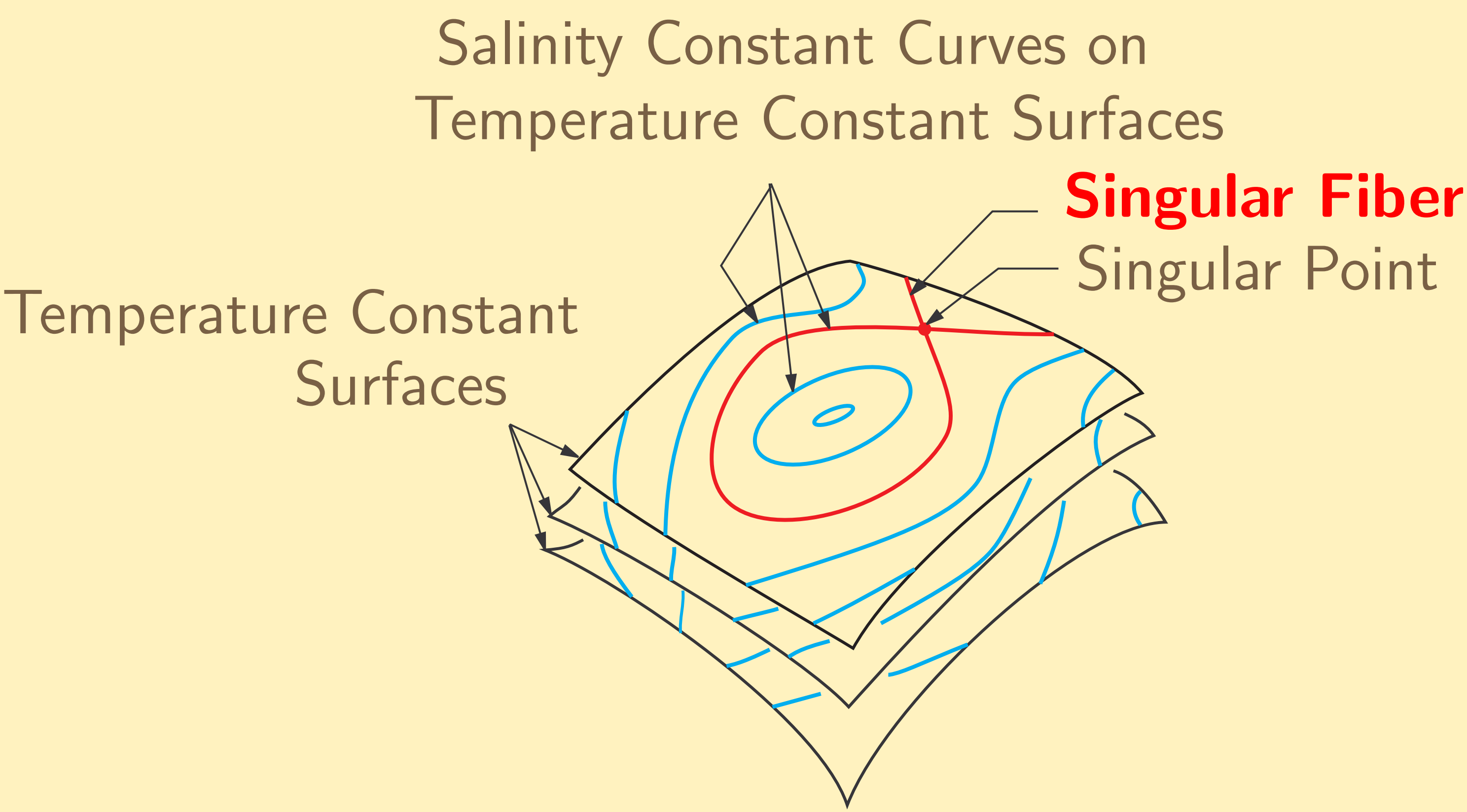

A fiber containing a singular point is called a **singular fiber**.
This is important in grasping the topological feature of the given data !

Singular points and Jacobi set

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Generically, the Jacobi set $J(f)$ is of dimension $m - 1$.

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Morse lemma says that a non-degenerate critical point of a Morse function has the same singularity type as the critical point of a quadratic function $\pm x_1^2 \pm x_2^2 \pm \cdots \pm x_n^2$.

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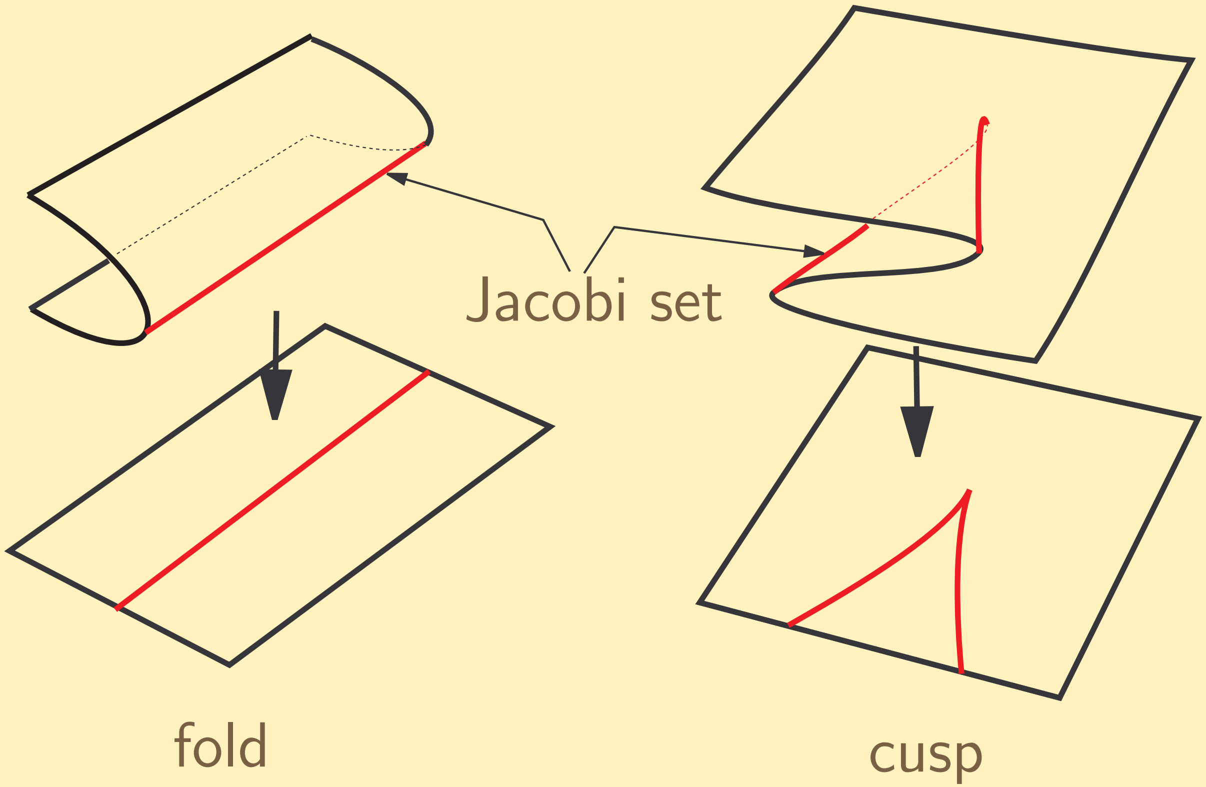

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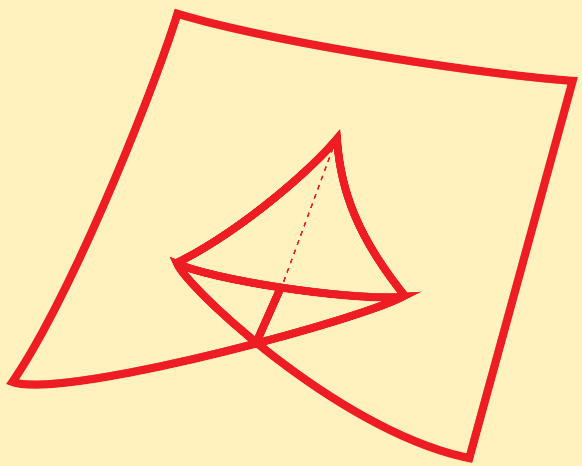

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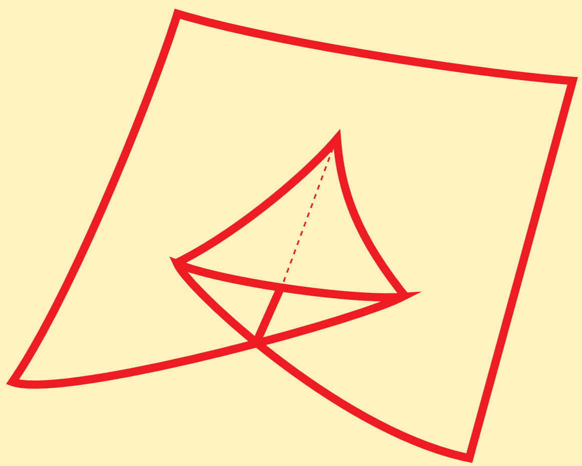

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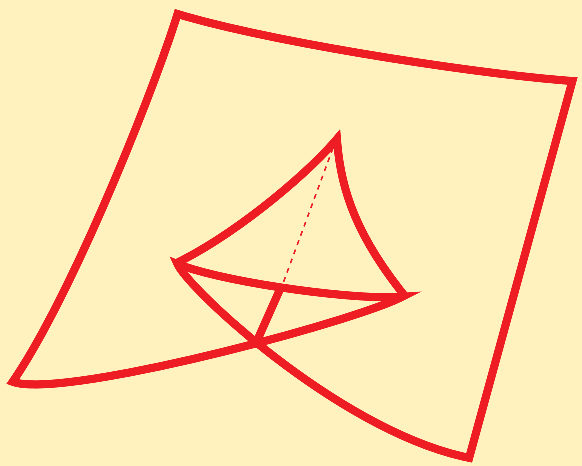

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$\Longrightarrow$ still studied in Singularity Theory as one of its central problems.

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Unfortunately, these studies are apparently not fully based on mathematical theories.

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Singularity theory of differentiable mappings
$\Downarrow$
One can identify the singularity types and the singular fiber types
(to a certain extent...)

Jacobi set image

For a generic $f : N^n \to \mathbf{R}^m$, we have $\dim J(f) = \dim f(J(f)) = m - 1$.
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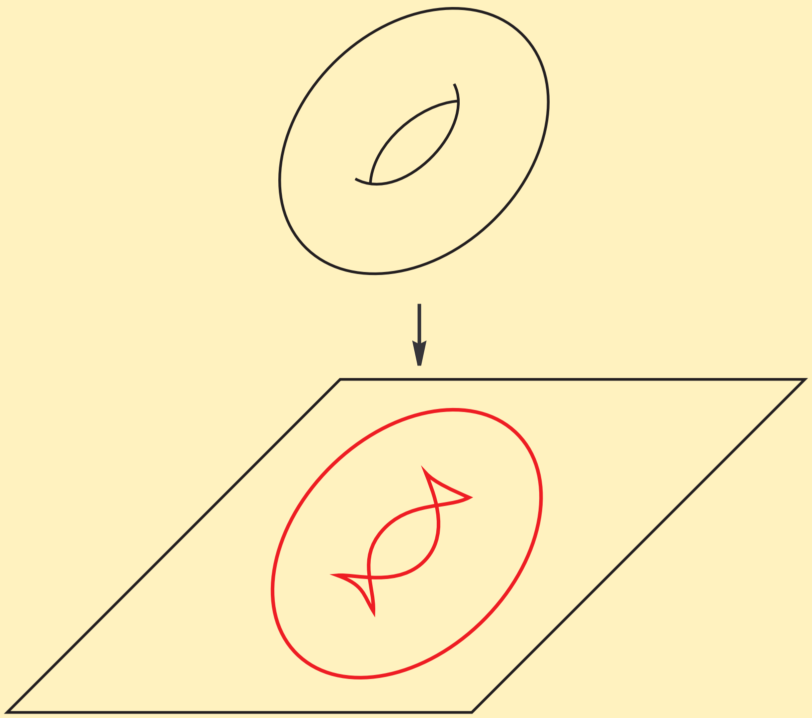

Example of Jacobi set image of a map of a surface into $\mathbf{R}^2$

What is expected

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In fact, when a singularity theorist (like me) analyzes a given multi-field, he/she tries to visualize it by the above method (but, with hand calculation and almost always without success !)
Any way, it is important to identify the **singular fibers** and the **topological transitions of the fibers** near singular fibers.

§3. Visualizing 2-Variate Volume Data

Case of $N^3 \to \mathbf{R}^2$

Let us consider the case of $n = 3$ and $m = 2$: 2-variate volume data.
In the following, $f : N^3 \to \mathbf{R}^2$ will be a multi-field, where N^3 is a bounded region (with boundary) in $\mathbf{R}^3$.

N^3: **spatial domain** (or **domain**)
$\mathbf{R}^2$: **data domain** (or **range**)

We assume that f is differentiable and is sufficiently **generic** (or **non-degenerate**).

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Jacobi set $J(f)$ forms a smooth curve in N^3.

Singular fiber

Let ∂N^3 be the boundary of the spatial domain, which is a compact surface (without boundary).

Set $f_\partial = f|_{\partial N^3} : \partial N^3 \to \mathbf{R}^2$, which is a generic differentiable map.

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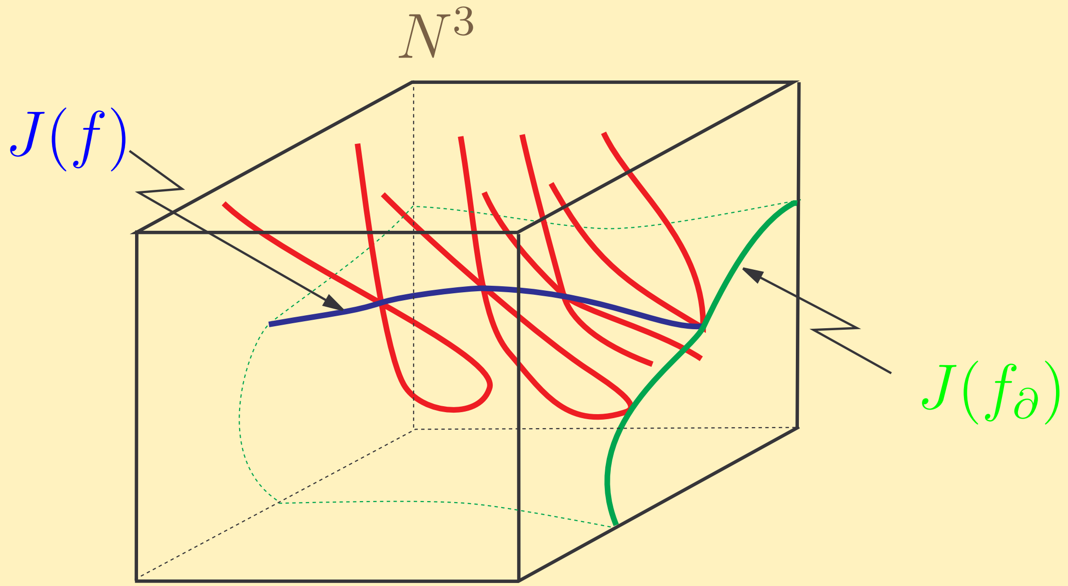
A fiber that passes through $J(f) \cup J(f_\partial)$ is called a **Singular Fiber**.



Joint Contour Net

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Instead of taking a point $\boldsymbol{c} \in \mathbf{R}^2$, we consider a small pixel $P \subset \mathbf{R}^2$.
Instead of a fiber $f^{-1}(\boldsymbol{c})$, we consider a **fat fiber** $f^{-1}(P)$.

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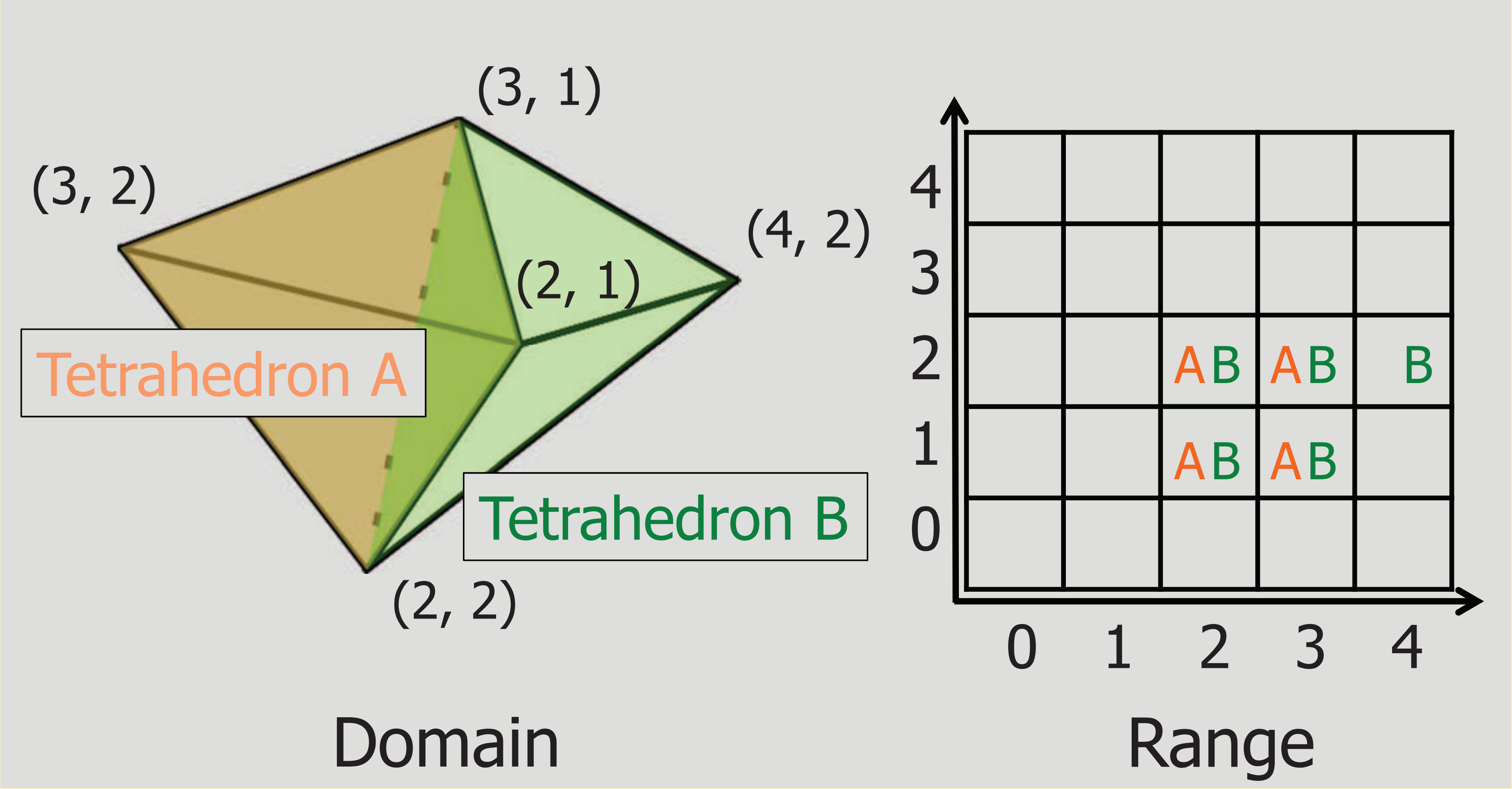
In this way, we can identify singular fibers in a robust way, because fat fibers contain essential information on its central fiber.

Constructing JCN (1)

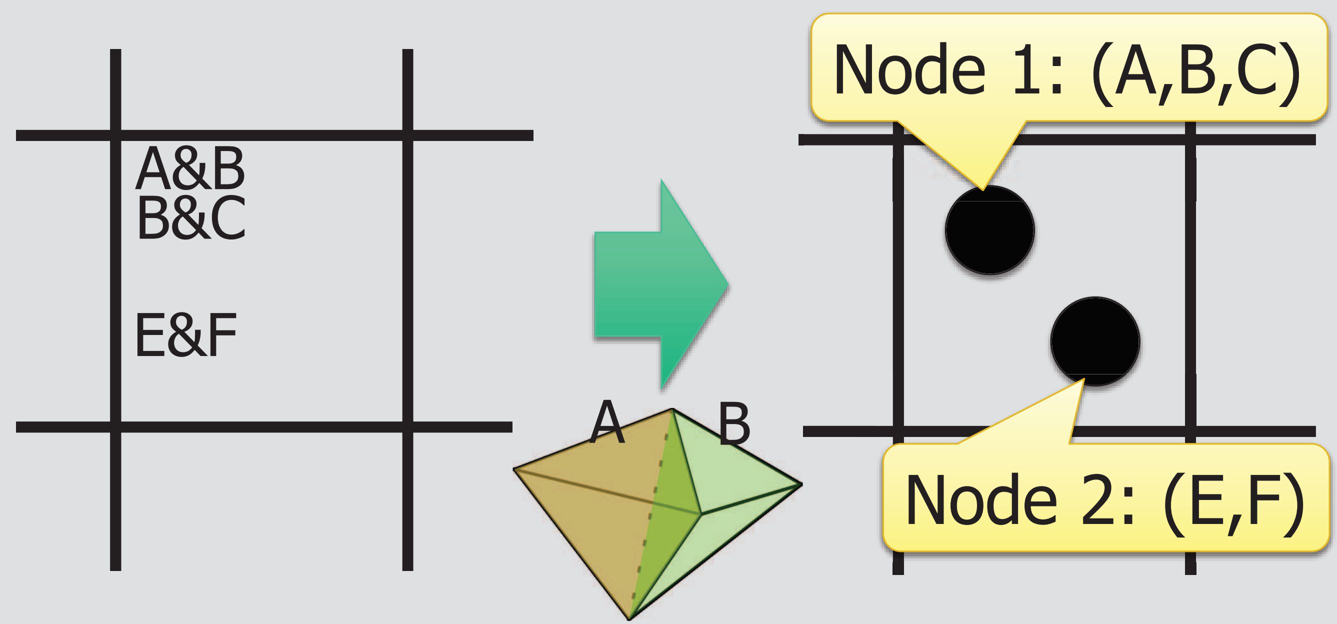
Domain (3D): Tetrahedral mesh / Range (2D): Rectangular mesh
Tetrahedra in the domain are decomposed into smaller pieces according to their (quantized) values.



Constructing JCN (2)

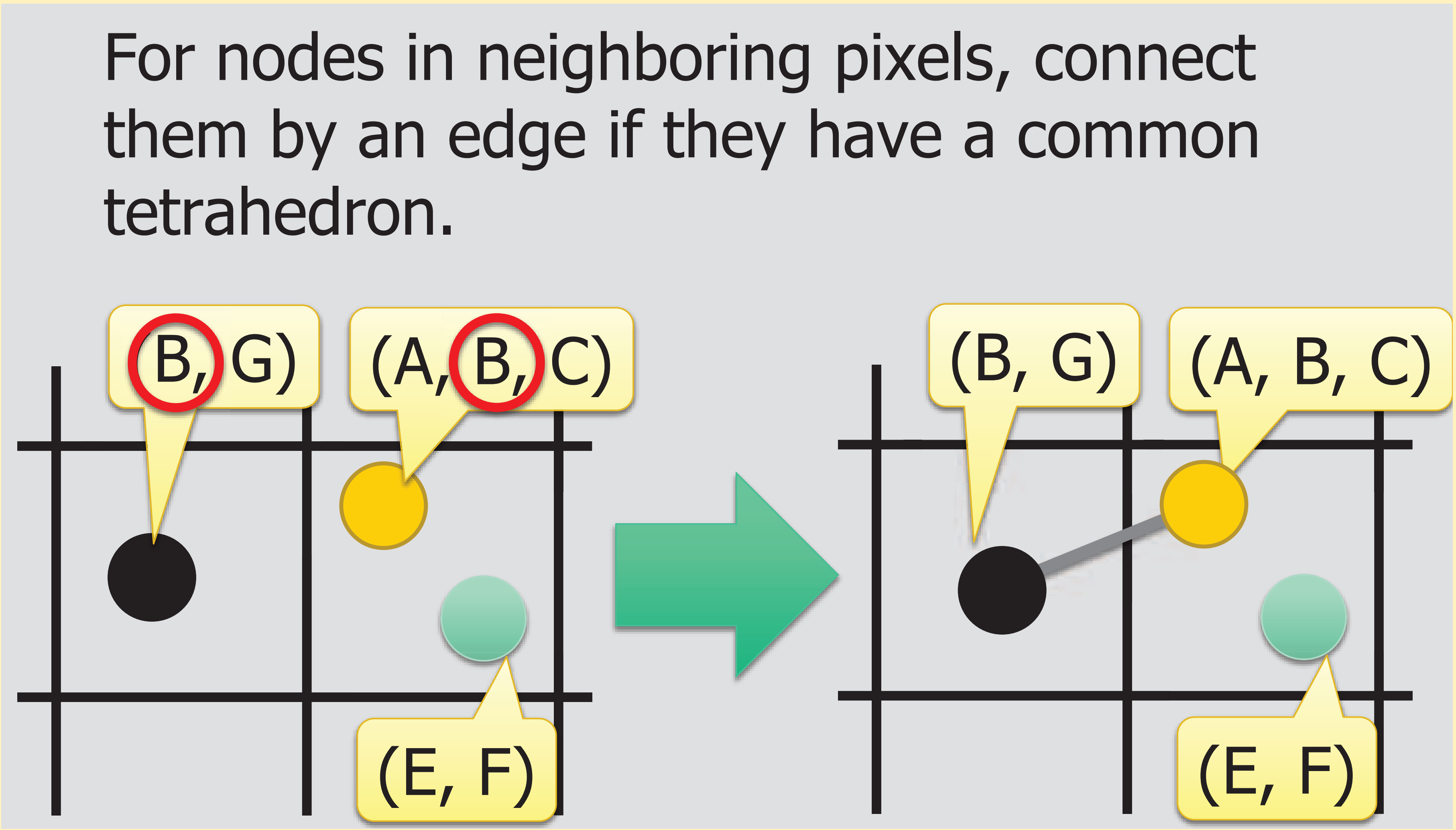
Unite neighboring pieces that have the same value.
They correspond to the connected components of the inverse image of a pixel — a **fat fiber**.

At each pixel (square), put a node for each set of neighboring tetrahedra.



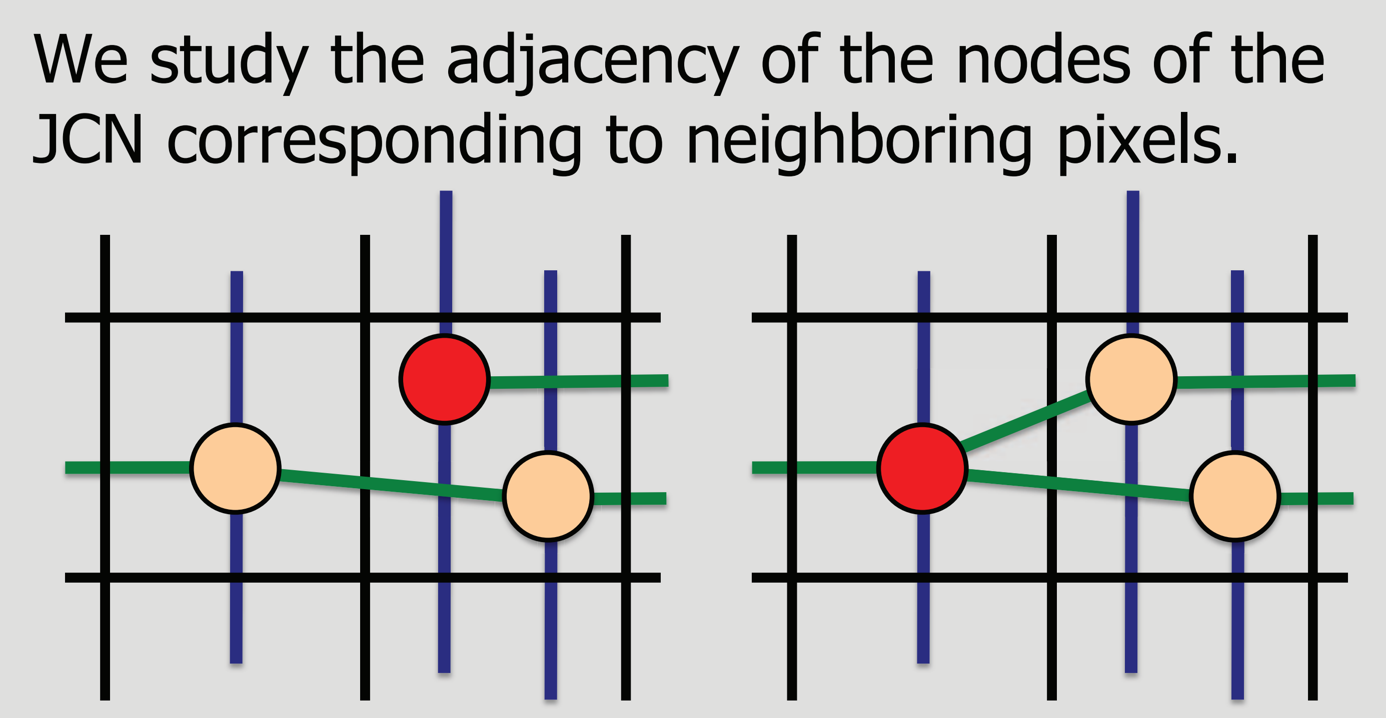
Constructing JCN (3)

Encode the adjacency information of the fat fibers by edges.



Constructing JCN (4)

In this way, we get a graph called **Joint Contour Net**, describing the adjacency relations among the connected components of fat fibers. This, in turn, can be used to detect birth-death or merge-splitting of fibers.



Birth-Death of fibers   Merge-Splitting of fbers

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In other words, we have the decomposition

$$\begin{array}{ccc} N^n & \xrightarrow{\quad f \quad} & \mathbf{R}^m \\ & {}_{q_f}\searrow \qquad \nearrow_{\overline{f}} & \\ & W_f, & \end{array}$$

for some map $\overline{f}$, where q_f is the natural quotient map.
This is called the **Stein factorization** of f (in singularity theory).

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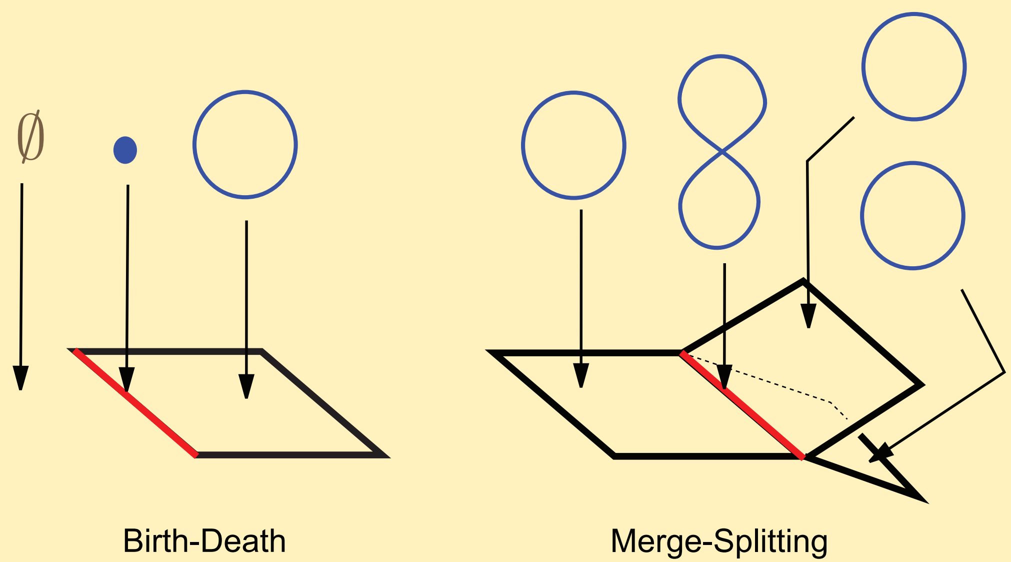

Singular fiber types

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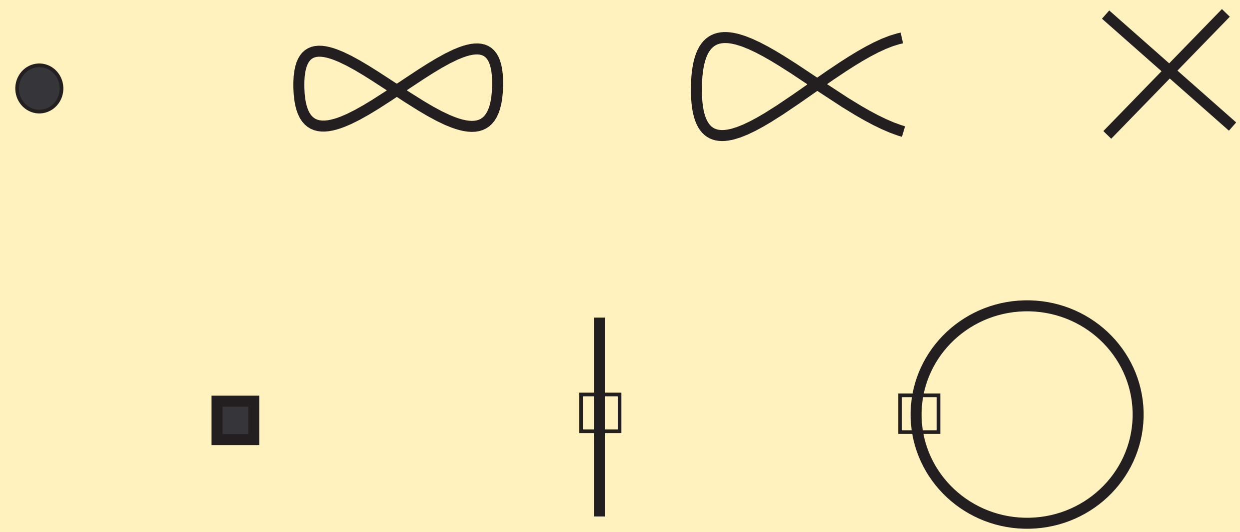
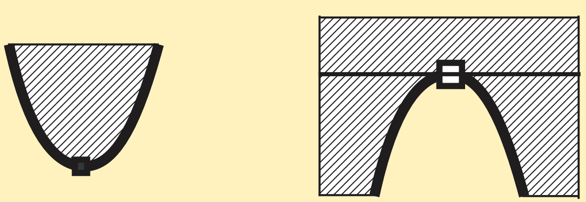
Combining JCN together with the classification theorem, we can identify the singular fiber types !

Singular fibers of $\kappa = 1$

The following 7 singular fibers appear along curves in the range.

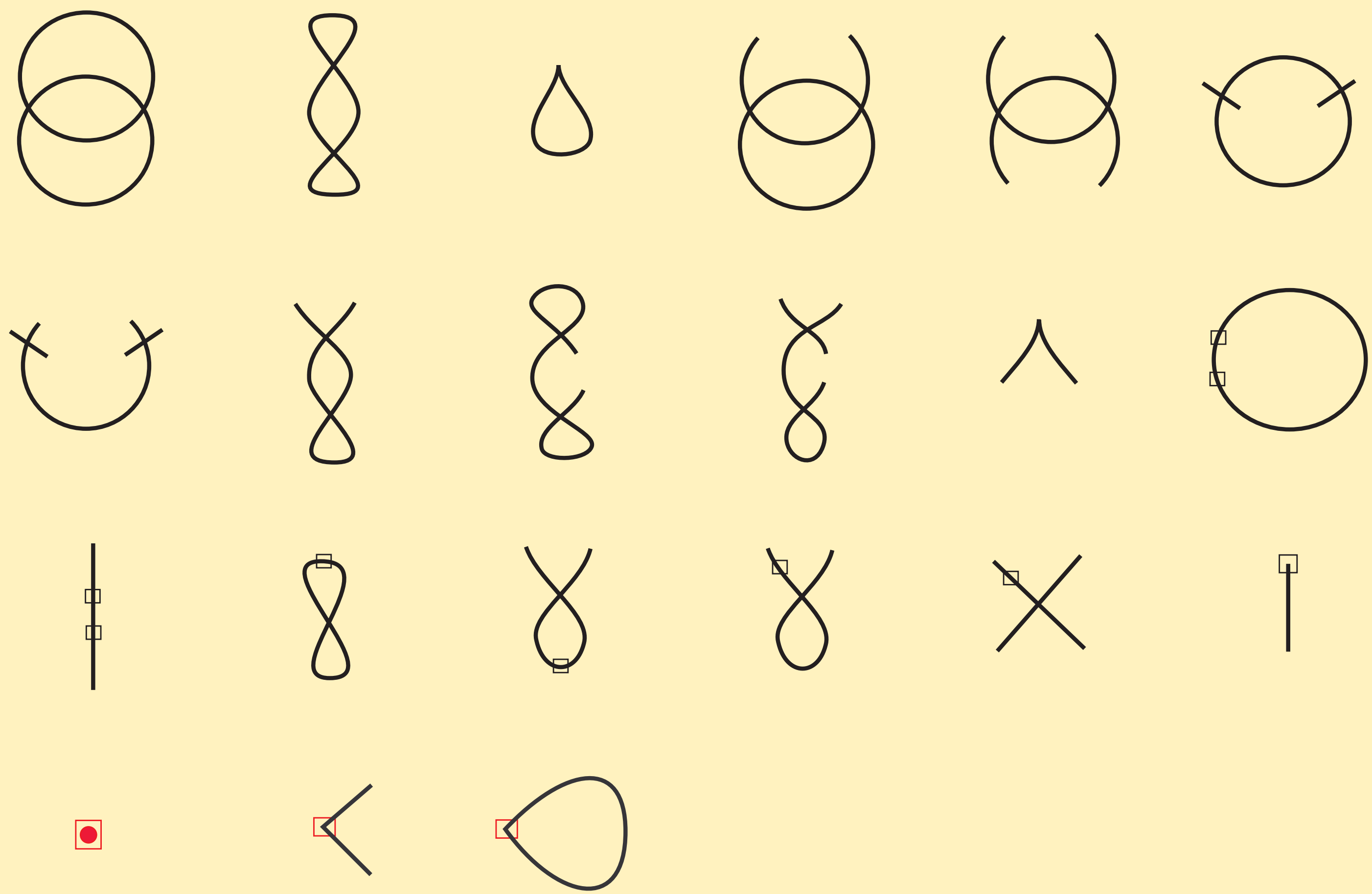


In the list, squares correspond to boundary tangency points in ∂N^3 as in the figures on the right.



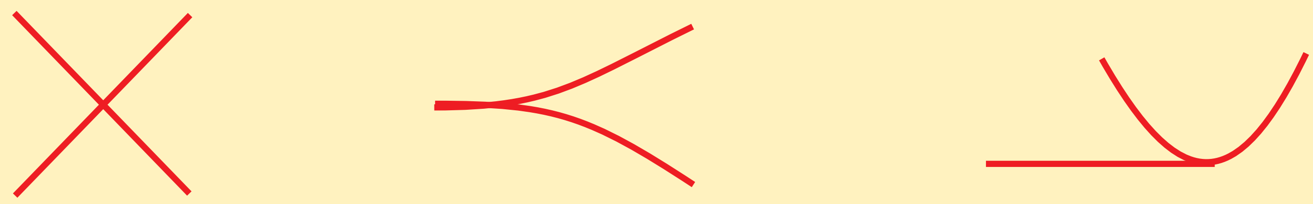
Singular fibers of $\kappa = 2$

The following 21 singular fibers appear discretely.
Red points correspond to $J(f) \cap \partial N^3 = J(f) \cap J(f_\partial)$.



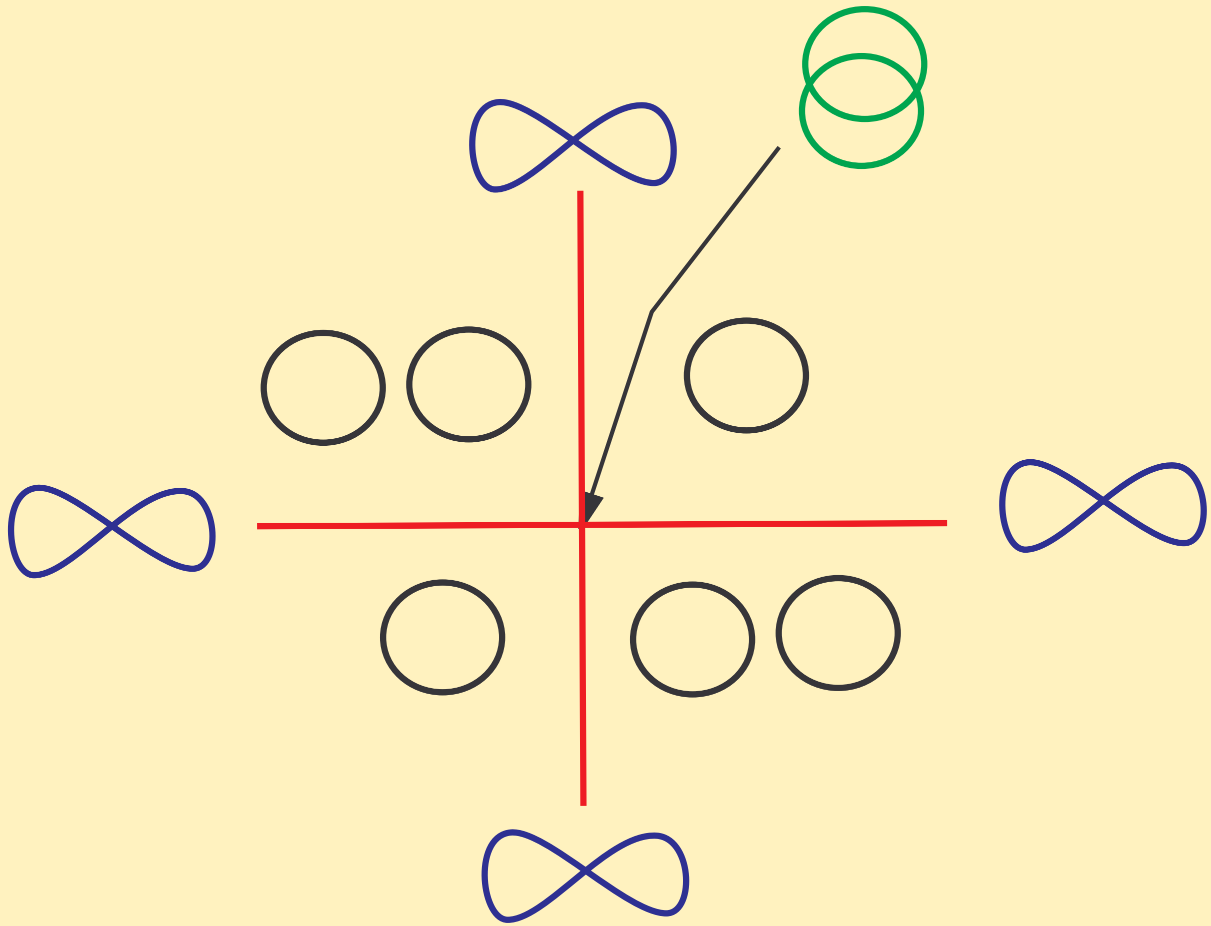
Jacobi set image

The curve of the Jacobi set image $f(J(f)) \cup f_\partial(J(f_\partial)) \subset \mathbf{R}^2$ has 3 types of singularities.



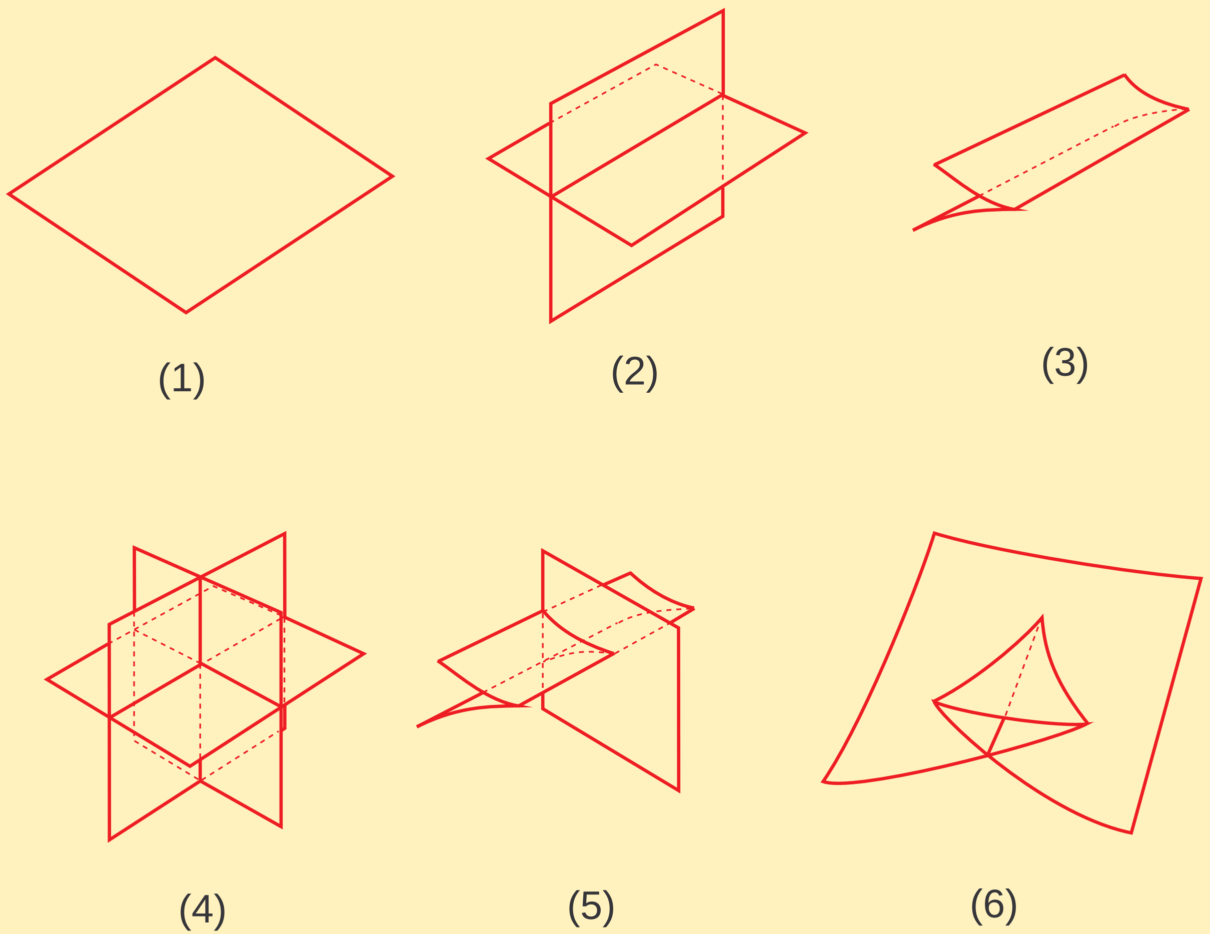
With the help of JCN, we can also identify the curve of the Jacobi set image in the range.

Example of transition of fibers



Fiber over each part of $f(J(f))$ and $\mathbf{R}^2 \setminus f(J(f))$

When $n = 4$, $m = 3$



Local configurations of the Jacobi set image for maps $f : N^4 \to \mathbf{R}^3$

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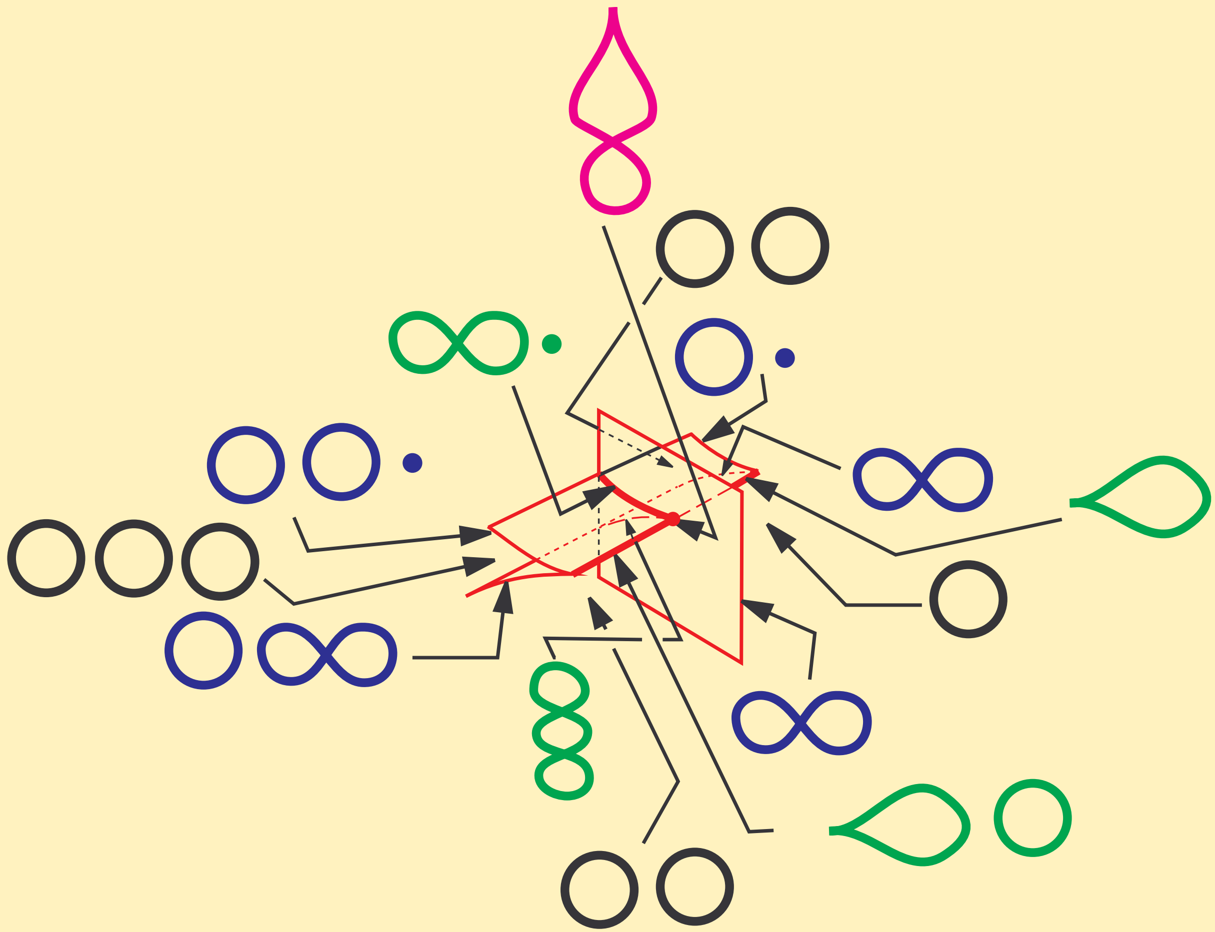
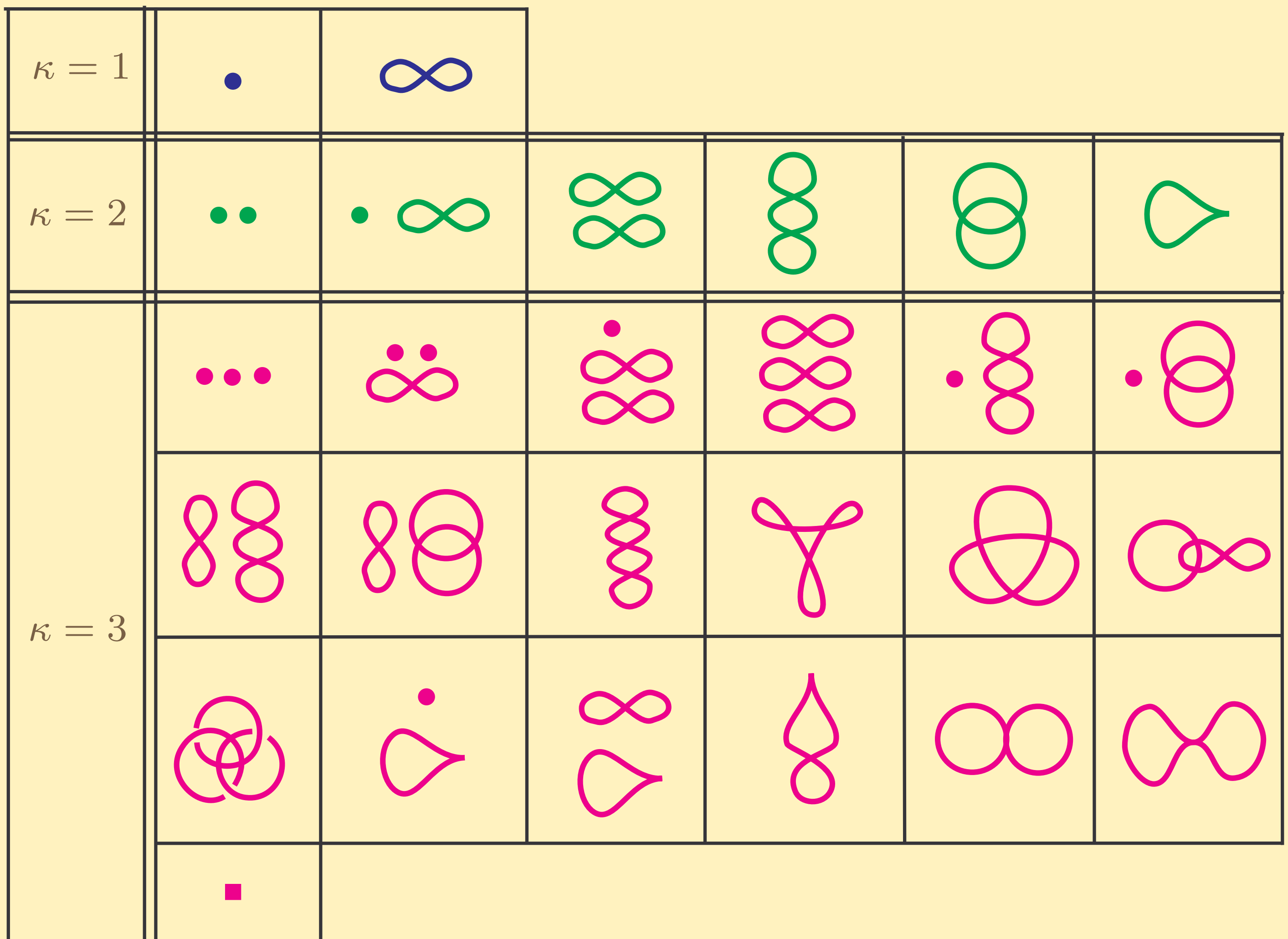


An example of topological transition of fibers for a map $f : N^4 \to \mathbf{R}^3$

List of singular fibers

For $n = 4$, $m = 3$, we have the following classification list.

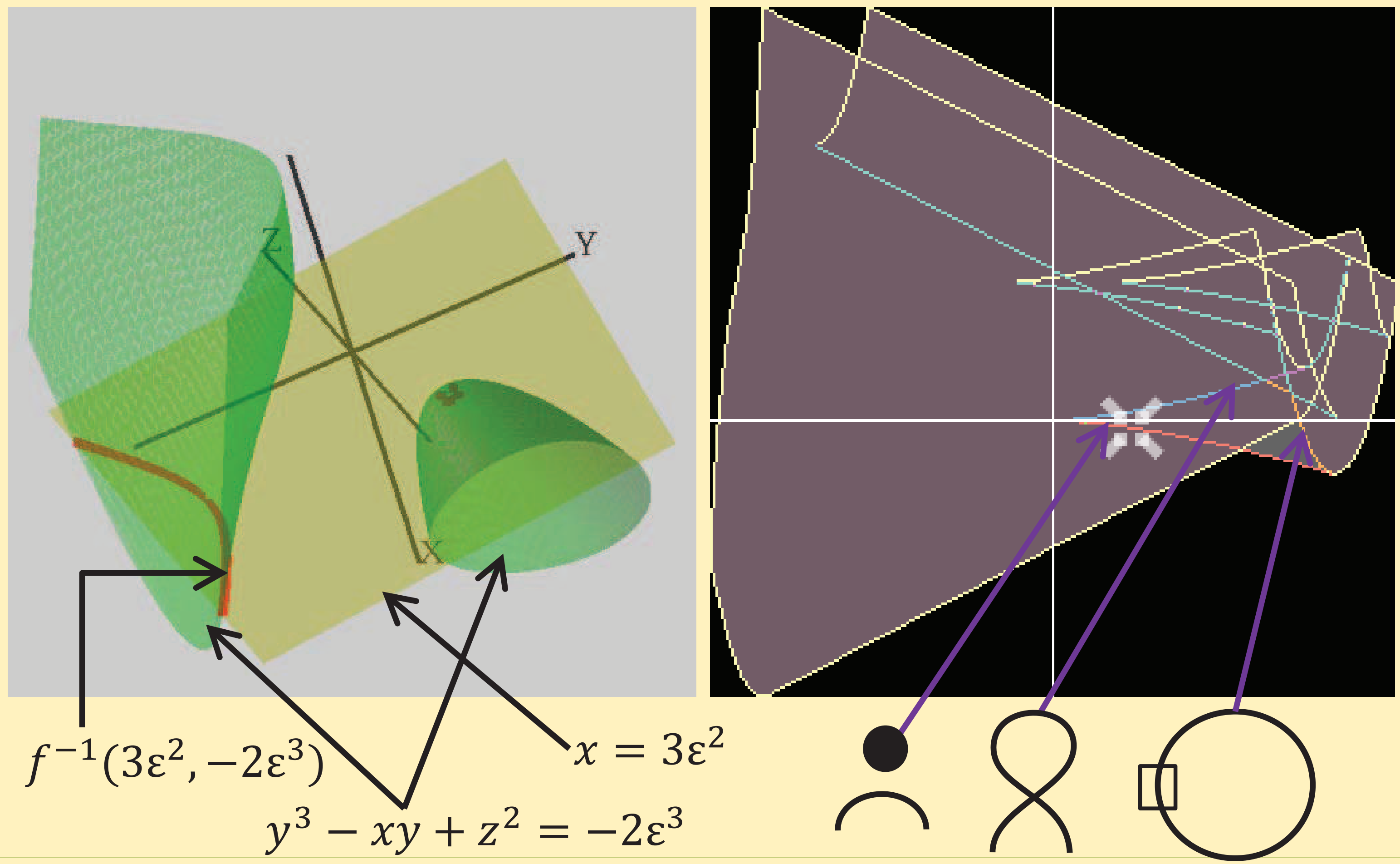


§4. Examples of Visualization

Sample: Analytic multi-field (1)

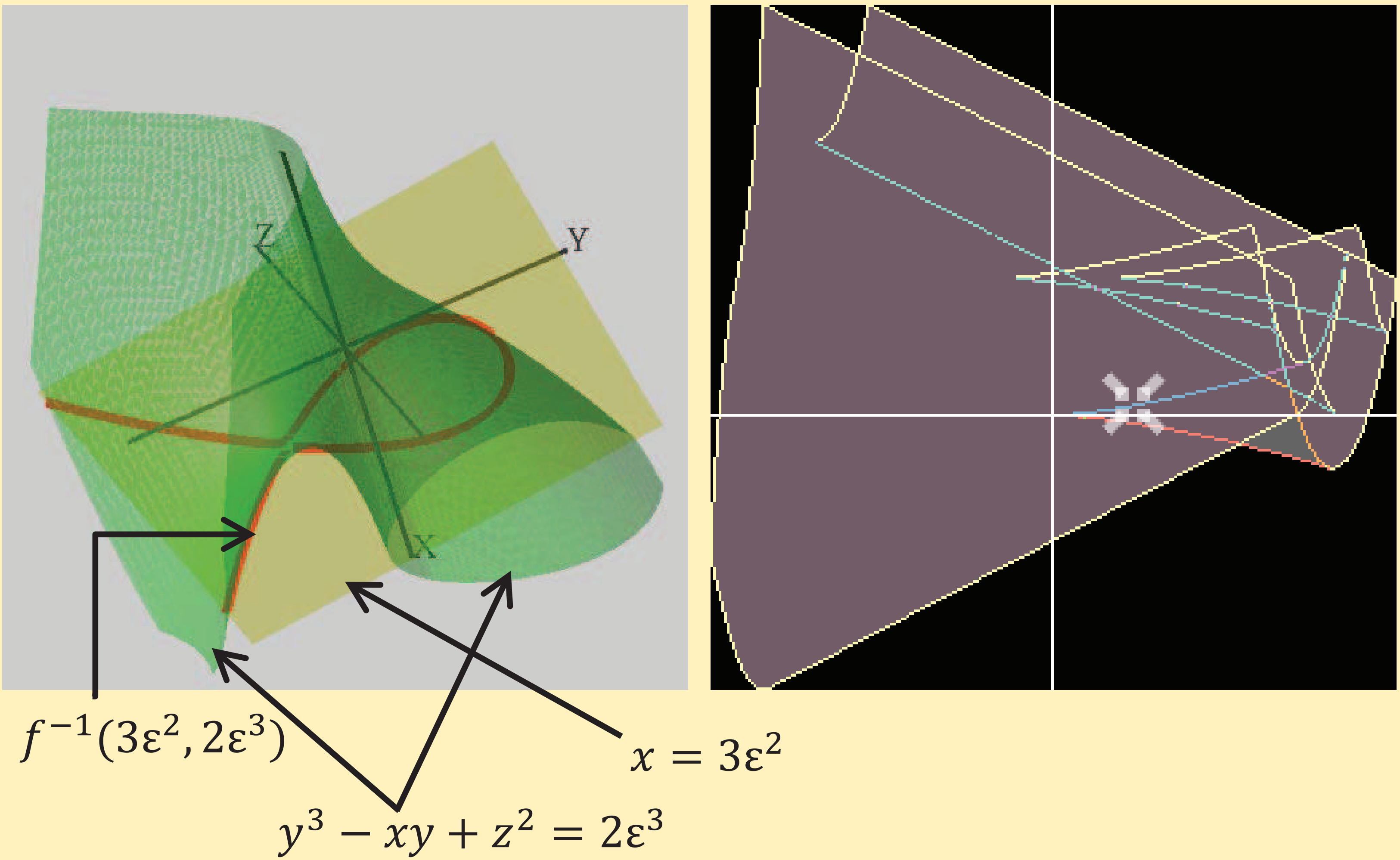
Analytic map $f(x, y, z) = (x, y^3 - xy + z^2)$.
A birth-death of fibers can be observed.

Domain **Range**



Sample: Analytic multi-field (2)

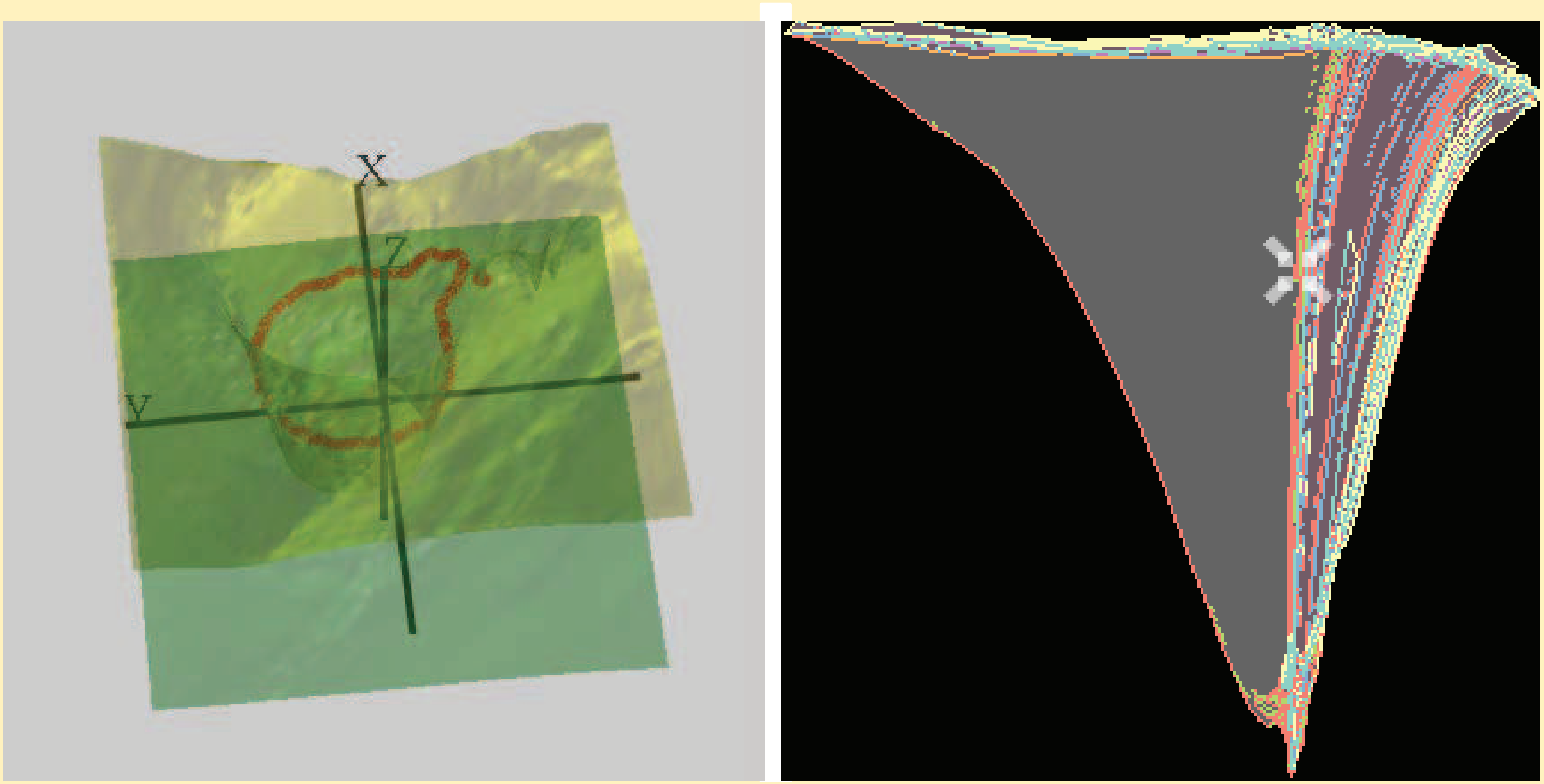
Same analytic map $f(x, y, z) = (x, y^3 - xy + z^2)$.
A merge-splitting of fibers can be observed.



Hurricane Isabel Data (1)

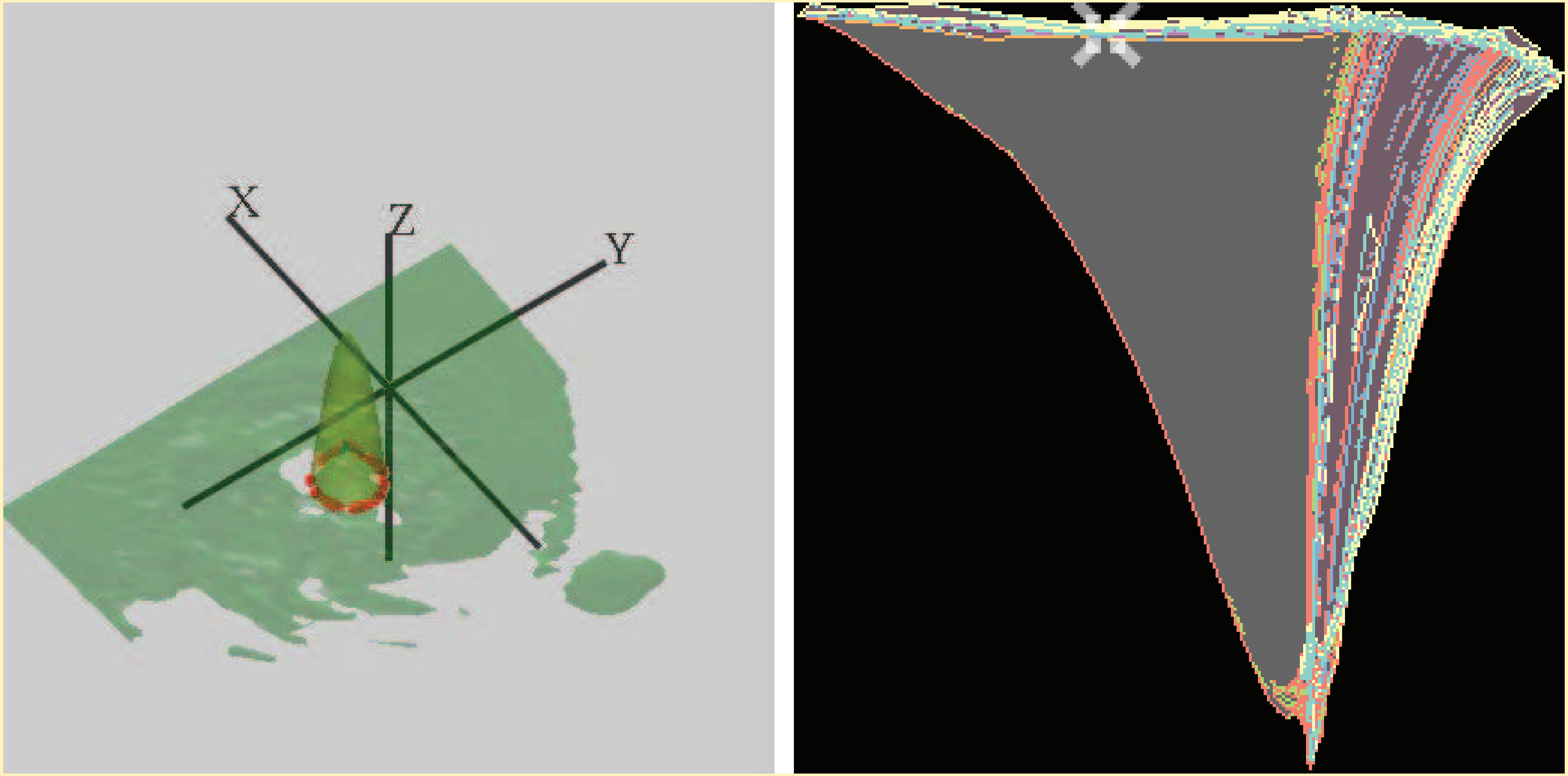
Volume data for the Hurricane Isabel

f = (Pressure, Temperature)



The singular fiber in the left corresponds to the crossing in the right.

Hurricane Isabel Data (2)



(∗) The "Hurricane Isabel" data set was produced by the Weather Research and Forecast (WRF) model, courtesy of NCAR and the U.S. National Science Foundation (NSF).

Future problems

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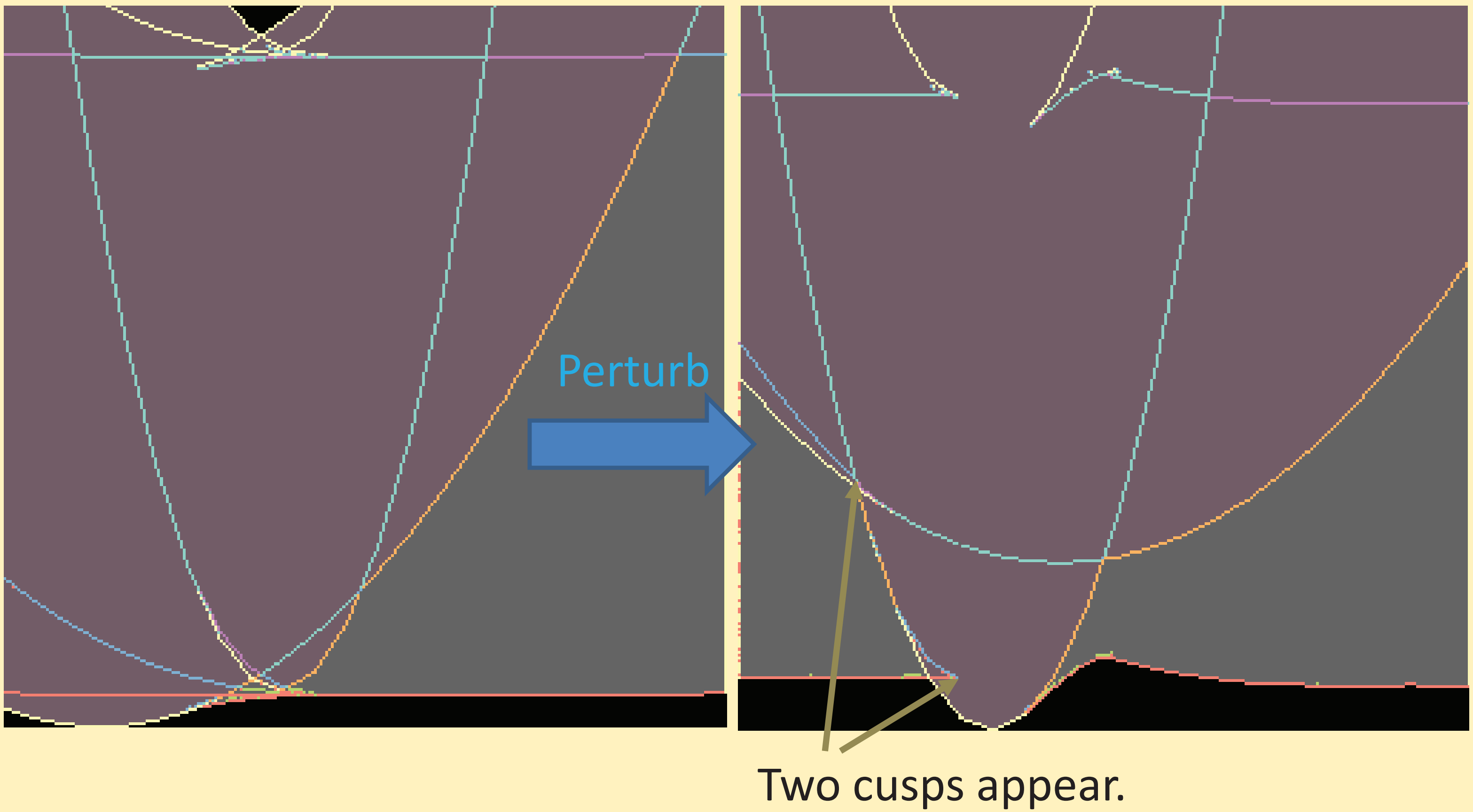
because many important analytic maps are waiting for us to analyze their structures visually.

On the other hand, our technique should be improved for visualizing general scientific data.
It works relatively well for **simulation data**, but sometimes we have serious problems with noise or sparsity of **real data**.

Impact on Mathematics (1)



This supports a theoretical result: the map on the left is degenerated: however, after a perturbation, two or more cusps appear.
This was predicted by a theorem [Ikegami & Saeki, 2009] in singularity theory: now it has been visually verified.

Impact on Mathematics (2)

For $a, b \in \mathbf{R}_+$, set

$$f_{a,b}(z, w) = z^3 + w^2 + a\bar{z} + b\bar{w}, \quad (z, w) \in \mathbf{C}^2$$

How does the family $\{f_{a,b}\}$ bifurcate if $(a, b) \in \mathbf{R}_+^2$ varies?

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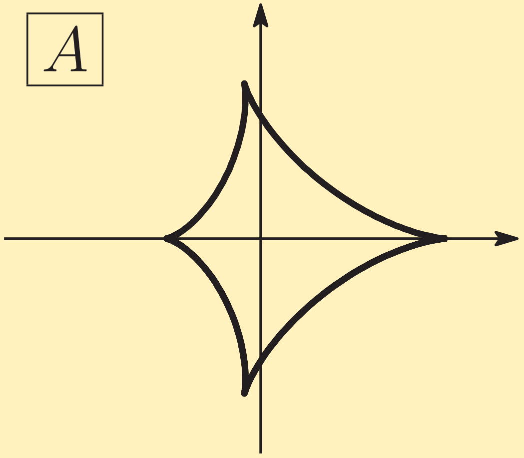
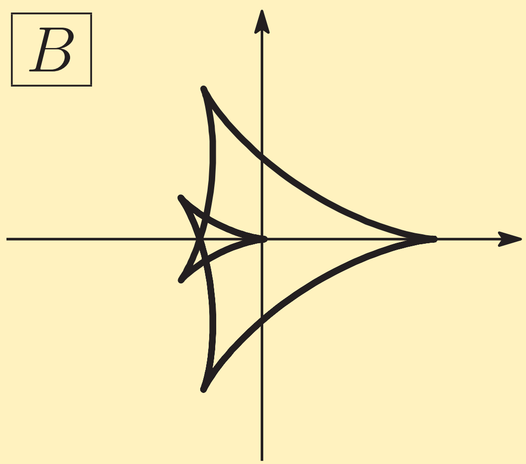
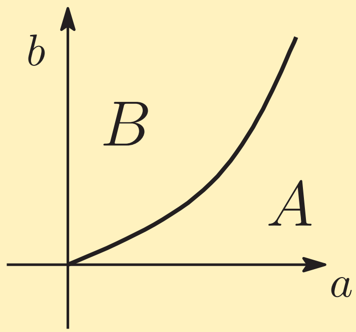
$\mathbf{R}_+^2$ is divided into two regions A and B.
The left 2 figures below show the Jacobi set images of $f_{a,b} : \mathbf{C}^2 = \mathbf{R}^4 \to \mathbf{R}^2 = \mathbf{C}$ for $(a, b) \in A$ and $(a, b) \in B$, respectively.







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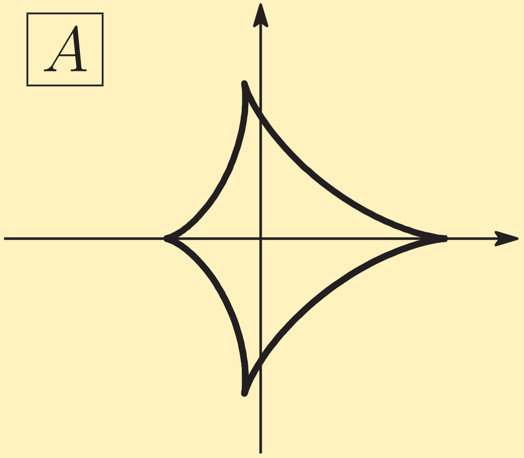

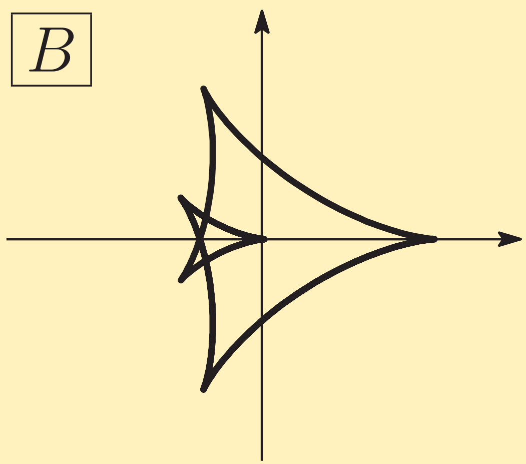

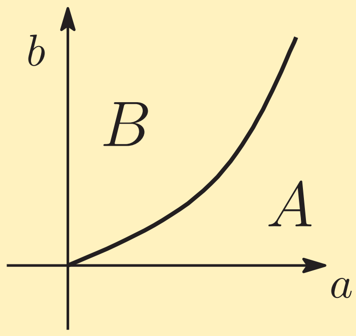

We would be very happy if we can visualize the singular fibers for $f_{a,b}$.

Conclusion

By using the **singularity theory** of differentiable mappings,

- We can **list up** **singularity types** and **singular fiber types** that appear generically;
- Accordingly, we can **identify** the **singularities** and **singular fibers** together with their types.

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I am sure !

Thank you for your attention !