Non-trivial real Milnor fibrations

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Joint work with
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§1. Real Milnor Fibrations

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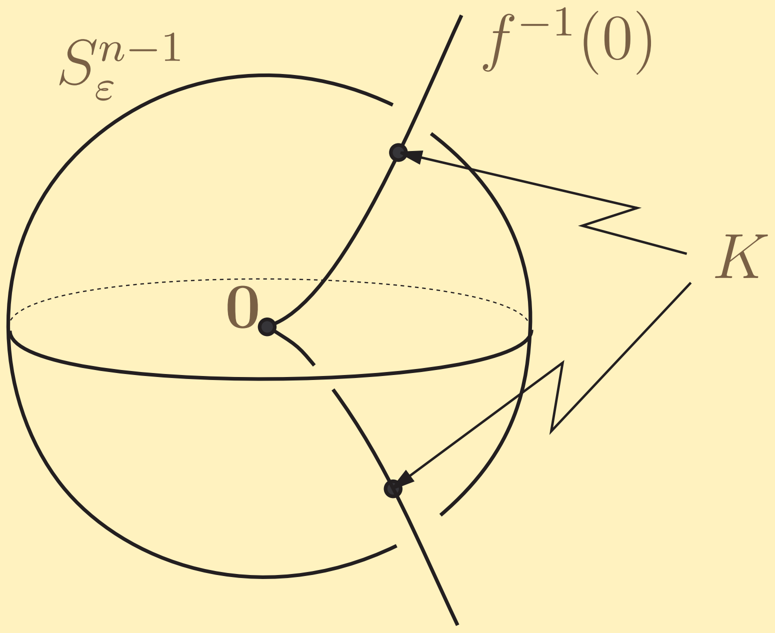
Then, for $0 < \forall\varepsilon << 1$, the **link** $K = f^{-1}(0) \cap S_\varepsilon^{n-1}$ is a codimension p submanifold of S_ε^{n-1}.



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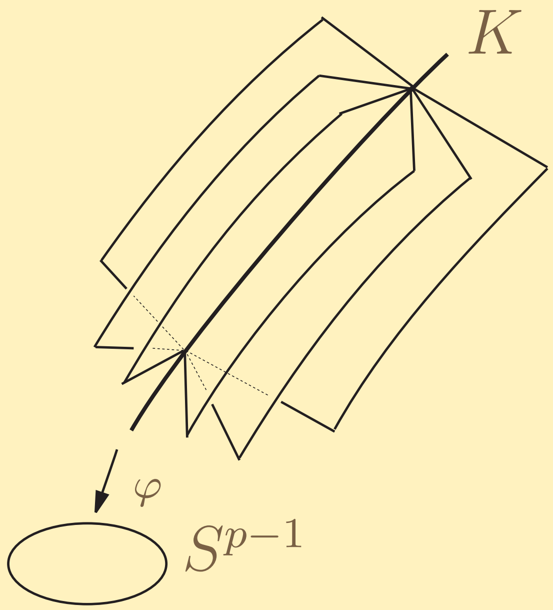
More precisely, there is a trivialization of a tubular nbhd $N(K) = K \times D^p$ s.t. $\varphi|_{N(K)\setminus K}$ coincides with

$$\frac{f}{||f||} : N(K) \setminus K = K \times (D^p \setminus \{0\}) \to D^p \setminus \{0\} \to S^{p-1}.$$



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Problem 1.2 (Milnor, 1968) *For which dimensions $n \geq p \geq 2$ do non-trivial examples exist ?*

Answer

Theorem 1.3 (Church–Lamotke, 1976 + Poincaré Conj.)
(a) *For* $0 \leq n-p \leq 2$, *non-trivial examples occur precisely for* $(n,p) = (2,2), (4,3), (4,2)$.
(b) *For* $n-p \geq 4$, *non-trivial examples occur for all* (n,p).
(c) *For* $n-p = 3$, *non-trivial examples occur precisely for* $(n,p) = (5,2)$, $(8,5)$ *and possibly for* $(6,3)$.

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Theorem 1.4 (R.A. dos Santos, M.A.B. Hohlenwerger, T.O.Souza and O.S., 2014)
There **exist** *polynomial map germs* $(\mathbb{R}^6,0) \to (\mathbb{R}^3,0)$ *with an isolated singularity at* 0 *with* **NON-TRIVIAL** *Milnor fibration.*

§2. Neuwirth–Stallings Pairs

Definition 2.1 (Looijenga, 1971)
$K = K^{n-p-1}$: oriented submanifold of S^{n-1} with trivial normal bundle.
We allow $K = \emptyset$.
Suppose $\exists \psi : S^{n-1} \setminus K \to S^{p-1}$ locally trivial fibration
s.t. for a trivialization $N(K) = K \times D^p$ of a tubular nbhd of K,
$\psi|_{N(K)\setminus K}$ coincides with

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where $\pi(x, y) = y/\|y\|$.

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The closure F of a fiber of ψ is called a **fiber**.

Looijenga construction

Theorem 2.2 (Looijenga, 1971)

(S^{n-1}, K^{n-p-1}): an NS-pair with fiber F s.t. $K^{n-p-1} \neq \emptyset$
$\Longrightarrow$ $\exists$polynomial map germ $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with an isolated singularity at 0
s.t. the associated NS-pair (Milnor fibration) is isomorphic to the connected sum

$$(S^{n-1}, K^{n-p-1}) \sharp ((-1)^n S^{n-1}, (-1)^{n-p} K^{n-p-1}),$$

with fiber the boundary connected sum $F \natural (-1)^{n-p} F$.

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This is purely a differential topology problem !

Strategy

1°. We **classify** smooth locally trivial fibrations

$$F^3 \hookrightarrow E^5 \to S^2$$

with fiber F^3 a compact (simply connected) 3-manifold with $\partial F^3 = K^2$ s.t. the boundary fibration $\partial F^3 \hookrightarrow \partial E^5 \to S^2$ is trivial.

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2°. We then **characterize** those fibrations with $\widetilde{E^5} \cong S^5$.
Such $(\widetilde{E^5}, K^2 \times \{0\})$, $0 \in D^3$, gives an NS-pair with fiber F^3.
If $F^3 \not\cong D^3$, we are done.

Let (S^5, K^2) be an NS-pair.
Since S^5 never fibers over S^2, we have $K^2 \neq \emptyset$.
We have a fibration $\psi : S^5 \setminus \operatorname{Int} N(K^2) \to S^2$ with fiber F.

By the homotopy exact sequence

$$\pi_2(S^5 \setminus \operatorname{Int} N(K^2)) \to \pi_2(S^2) \to \pi_1(F) \to \pi_1(S^5 \setminus \operatorname{Int} N(K^2)),$$

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$$S^3_{(k+1)} \hookrightarrow E^5 \to S^2.$$

Classification

Such fibrations are classified by $\pi_1(\mathrm{Diff}(S^3_{(k+1)}, \partial S^3_{(k+1)}))$, where $\mathrm{Diff}(S^3_{(k+1)}, \partial S^3_{(k+1)})$ is the group of diffeomorphisms which fix the boundary pointwise.

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Consider a disjoint union $\cup^{k+1} B^3$ "standardly" embedded in S^3.
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Lemma 2.3 (Cerf, 1968) *The canonical map*

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Thus, our fiber bundles are classified by $\pi_1(\mathrm{Diff}(S^3, \cup^{k+1} B^3))$.

Cerf-Palais Theorem

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We can also prove

Lemma 2.5 $\mathrm{Emb}(\cup^{k+1}B^3, S^3) \simeq \mathbb{F}_{k+1}(S^3) \times O(3)^{k+1}$, *where*

$$\mathbb{F}_{k+1}(S^3) = \{(x_1, x_2, \ldots, x_{k+1}) \,|\, x_i \in S^3,\ x_j \neq x_\ell, j \neq \ell\},$$

which is called the **configuration space**.

Classification result

By the homotopy exact sequence of the bundle

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together with the above lemma, we see that

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Lemma 2.6 (Fadell–Husseini 2001) $\pi_2(\mathbb{F}_{k+1}(S^3)) \cong \mathbb{Z}^{k(k-1)/2}$.

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Thus, our bundles $S^3_{(k+1)} \hookrightarrow E^5 \to S^2$ are in one-to-one correspondence with the elements in $\mathbb{Z}^{k(k-1)/2}$.
This corresponds to a $k \times k$ **skew-symmetric integer matrix** as follows.

Linking matrix

Among the $k+1$ boundary components of E^5, we choose k of them and attach k copies of the trivial fibration $D^3 \times S^2 \to S^2$ to get

$$S^3_{(1)} \hookrightarrow Y \stackrel{\widetilde{\psi}}{\to} S^2,$$

where $S^3_{(1)} = S^3_{(k+1)} \cup (\cup^k D^3) \cong D^3$ and $Y = E^5 \cup (\cup^k D^3 \times S^2)$.

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Then, the k disjoint sections give k disjoint (oriented) 2-spheres S^2_i, $i = 1, 2, \ldots, k$, embedded in S^5.

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Thus, the above fibration is identified with $Y \cong D^3 \times S^2 \to S^2$.
Furthermore, the k copies of $D^3 \times S^2$ attached to E^5 give rise to k disjoint embedded 2-spheres $\{0\} \times S^2$.
We can attach $S^2 \times D^3$ to $Y = D^3 \times S^2$ to get S^5.

Then, the k disjoint sections give k disjoint (oriented) 2-spheres S^2_i, $i = 1, 2, \ldots, k$, embedded in S^5.
Thus, we have the **linking matrix** $L = (\mathrm{lk}(S^2_i, S^2_j))_{1 \leq i,j \leq k}$.

Classification by linking matrix

We adopt the convention that the diagonal entries $\mathrm{lk}(S_i^2, S_i^2)$ are all zero.

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All these arguments imply that we have the following one-to-one correspondence by the **linking matrix**:

$$\{\text{isomorphism classes of our fiber bundles } S^3_{(k+1)} \hookrightarrow E^5 \to S^2\}$$
$$\updownarrow$$
$$\{k \times k \text{ skew-symmetric integer matrices } L = (\mathrm{lk}(S_i^2, S_j^2))_{1 \leq i,j \leq k}\}$$

Characterization theorem

We see easily that $\widetilde{E}^5$ is simply connected.
It is known that a smooth **homotopy 5-sphere** is always **diffeomorphic to S^5**.

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Therefore, by a standard (but tedious) computation based on Mayer-Vietoris exact sequence for **homology**, we get the following.

Theorem 2.7
We have $\widetilde{E}^5 \cong S^5$, i.e., our fiber bundle comes from an NS pair iff $\det L = \pm 1$.

Main theorem

It is easy to construct $k \times k$ skew-symmetric matrix of determinant ± 1 as long as k is even.
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Corollary 2.8
For $\forall k = 0, 2, 4, \ldots$, $\exists$*NS-pair* (S^5, L_{k+1}) *with* $L_{k+1} \cong \cup^{k+1} S^2$.
Consequently, $\exists$*polynomial map germ* $(\mathbb{R}^6, 0) \to (\mathbb{R}^3, 0)$ *with an isolated singularity at* 0 *s.t. associated NS-pair is isomorphic to* $(S^5, L_{k+1} \sharp (-L_{k+1}))$.
In particular, $L_{k+1} \sharp (-L_{k+1})$ *consists of* $2k + 1$ *components.*

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This answers **Milnor's non-triviality question** for $(n, p) = (6, 3)$.

§3. Topology of Milnor Fibers

Bouquet theorem

If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is a **complex** polynomial map germ with an isolated singularity at 0, then the Milnor fiber is homotopy equivalent to the **bouquet of spheres**

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In the **real** case, it is easy to construct an example of a polynomial map germ with an isolated singularity whose Milnor fiber is NOT homotopy equivalent to the bouquet of spheres.

However, we have the following.

Proposition 3.1 *Let* $f : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^n, 0)$ *be a polynomial map germ with an isolated singularity at the origin,* $n \geq 2$*.*

Then, the Milnor fiber F_f *has the homotopy type of* $\bigvee^{\mu} S^{n-1}$*, where it means a point when* $\mu = 0$*.*

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Proposition 3.2 *Let* $f : (\mathbb{R}^{2n+1}, 0) \to (\mathbb{R}^n, 0)$ *be a polynomial map germ with an isolated singularity at the origin,* $n \geq 3$.

Then, $H_{n-1}(F_f)$ *is torsion free if and only if* F_f *has the homotopy type of* $\left(\bigvee^{\mu} S^{n-1}\right) \vee \left(\bigvee^{\mu} S^{n}\right)$, *where it means a point when* $\mu = 0$.

Non-trivial examples

Non-trivial examples due to Church–Lamotke (1976) have **contractible Milnor fibers** (but with non-simply connected links).

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Using our techniques for $(n, p) = (6, 3)$, we get the following.

Proposition 3.3 *For each pair of dimensions* $(2n, p)$, $2 \leq p \leq n$, $\exists$*polynomial map germ* $(\mathbb{R}^{2n}, 0) \to (\mathbb{R}^p, 0)$ *with an isolated singularity at* 0 *s.t. the Milnor fiber is homotopy equivalent to*

$$\bigvee^{\mu} S^{n-1}, \quad \mu > 0.$$

Non-trivial examples (II)

Proposition 3.4

For each pair of dimensions $(2n+1, p)$, $2 \leq p \leq n$, ∃polynomial map germ $(\mathbb{R}^{2n+1}, 0) \to (\mathbb{R}^p, 0)$ with an isolated singularity at 0 s.t. the Milnor fiber is homotopy equivalent to

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These provide NEW NON-TRIVIAL examples !

Idea of construction

We first construct a fiber bundle

$$S^n_{(k+1)} \hookrightarrow E^{2n-1} \to S^{n-1}$$

such that it is trivial on the boundary, using a $k \times k$ integer matrix L which is $(-1)^n$-symmetric whose diagonal entries all vanish.

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This may not be diffeomorphic to S^{2n-1}.
However, $\widetilde{E}^{2n-1} \sharp (-\widetilde{E}^{2n-1})$ **is diffeomorphic to** S^{2n-1}.

Idea of construction (II)

Then, the **Looijenga construction** leads to a polynomial map germ $f : (\mathbb{R}^{2n}, 0) \to (\mathbb{R}^n, 0)$ with an isolated singularity at 0 whose Milnor fibration is non-trivial.

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For $(\mathbb{R}^{2n+1}, \mathbb{R}^p)$, we first apply the **spinning construction** to the non-trivial NS-pair (S^{2n-1}, K^{n-1}) constructed above, to get a non-trivial NS-pair $(S^{2n}, \widetilde{K}^n)$.

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Then, consider the **composition with a canonical projection** $\mathbb{R}^n \to \mathbb{R}^p$.

Thank you for your attention !