Cobordism Group of Morse Functions on Surfaces with Boundary

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§1. Introduction

Morse function

All manifolds and maps are differentiable of class C^∞.

Let N be a manifold **with boundary**.

A C^∞ function $f : N \to \mathbf{R}$ is a **Morse function** if

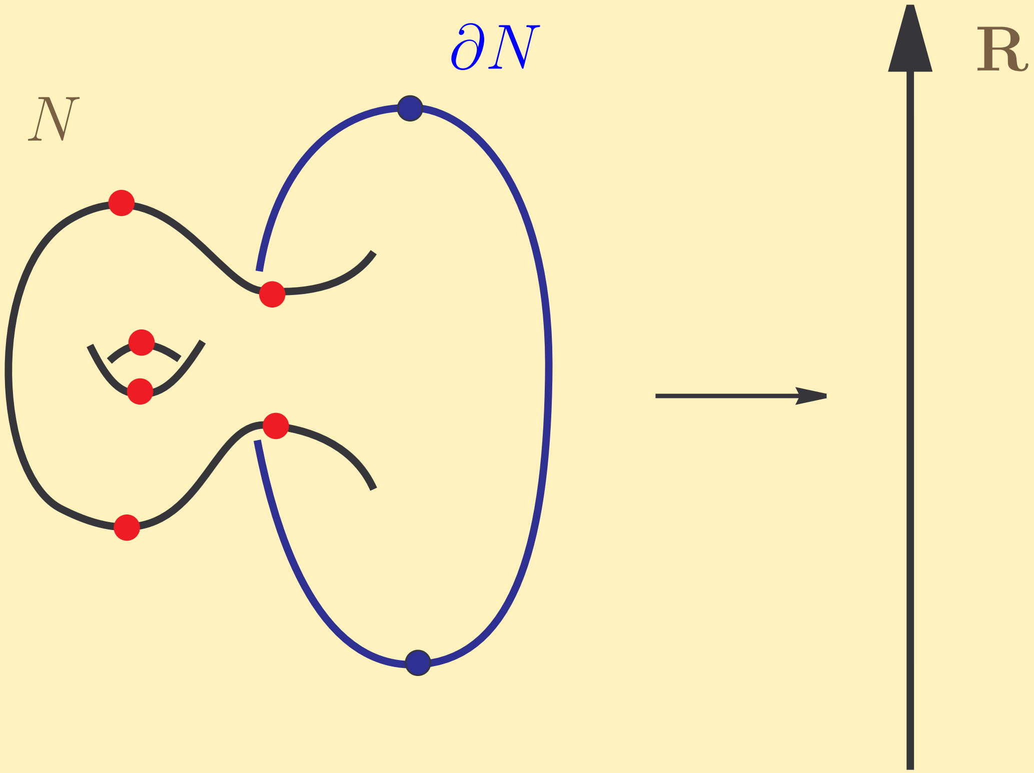
(1) the critical points of f and $f|_{\partial N}$ are all **non-degenerate** and have **distinct values**, and
(2) f is a **submersion** on a neighborhood of ∂N.
($\Longleftrightarrow$ critical points of $f|_{\partial N}$ are all **correct** critical points.)

Fact. f is a Morse function iff it is C^∞ **stable**.
It is also called a **correct function** in the literature.

A smooth map on a manifold with boundary is **admissible** if it is a submersion on a neighborhood of the boundary.

In this sense, every Morse function is admissible.

Example of a Morse function



The critical points of f and $f|_{\partial N}$ are non-degenerate.
The critical values of f and $f|_{\partial N}$ are all distinct.
f is a submersion near the boundary.

Cobordism

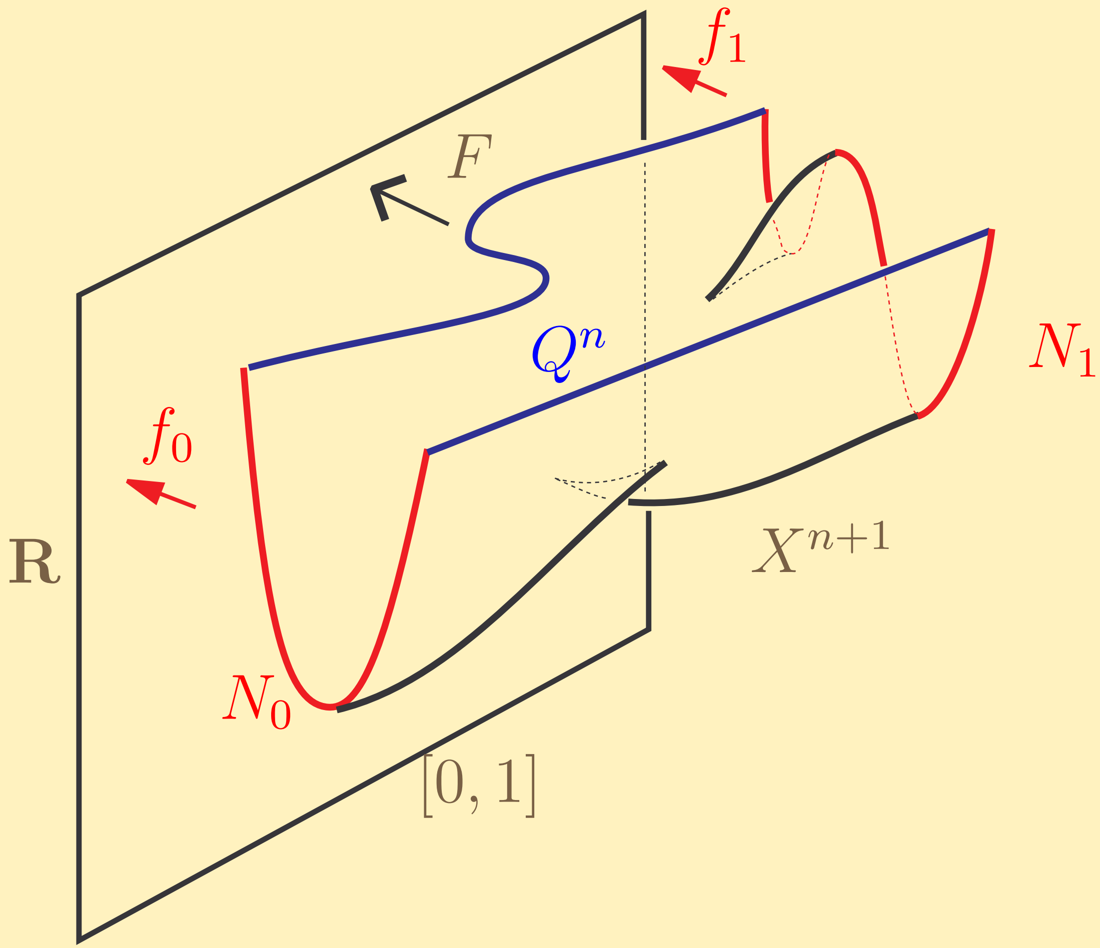
Let N_0 and N_1 be compact n-dim. manifolds with boundary.
Morse functions $f_i\colon\ N_i \to \mathbf{R}$, $i = 0, 1$, are **admissibly cobordant** if

(1) $\exists$compact manifold X^{n+1} with corners
(cobordism between the manifolds N_0 and N_1) s.t.
$\partial X^{n+1} = N_0 \cup Q^n \cup N_1$,
$\partial Q^n = \partial N_0 \cup \partial N_1$ (Q^n is a cobordism between ∂N_0 and ∂N_1),
(2) $\exists F : X^{n+1} \to \mathbf{R} \times [0, 1]$,
(3) $F|_{N_0} = f_0 : N_0 \to \mathbf{R} \times \{0\}$ and $F|_{N_1} = f_1 : N_1 \to \mathbf{R} \times \{1\}$,
(4) $F|_{X^{n+1}\setminus(N_0\cup N_1)} : X^{n+1} \setminus (N_0 \cup N_1) \to \mathbf{R} \times (0, 1)$ is a proper **admissible** C^∞ map which has only **folds** and **cusps** as its singularities.

In this case, we call F an **admissible cobordism** between f_0 and f_1.

fold: $(x_1, x_2, \dots, x_{n+1}) \mapsto (x_1, \pm x_2^2 \pm \cdots \pm x_{n+1}^2)$,
cusp: $(x_1, x_2, \dots, x_{n+1}) \mapsto (x_1, x_2^3 + x_1x_2 \pm x_3^2 \pm \cdots \pm x_{n+1}^2)$

Example



Admissibility is important in the above definition.
If we drop the condition, then any two Morse functions are cobordant!

Cobordism group

If a Morse function is admissibly cobordant to the function on the empty set $\emptyset$, then it is **null-cobordant**.

"Admissible cobordism" defines an equivalence relation on the set of all Morse functions on compact n-dim. manifolds with boundary.
The set of all equivalence classes forms an **additive group** under disjoint union:

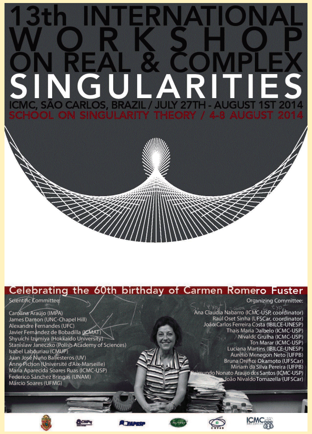
(1) the neutral element is the class of null-cobordant Morse functions,
(2) $-[f : N \to \mathbf{R}] = [-f : N \to \mathbf{R}]$.

Denote by $b\mathfrak{N}_n$ the additive group of all admissible cobordism classes and call it the **n-dim. admissible cobordism group of Morse functions on manifolds with boundary**.

Main theorem

Theorem 1.1 *The 2-dim. admissible cobordism group of Morse functions $b\mathfrak{M}_2$ is cyclic of order two.*

Remark 1.2 *We had previously shown* $\exists$*epimorphism* $b\mathfrak{M}_2 \to \mathbf{Z}_2$*, using cohomology of the* **universal complex of singular fibers***.*
This was presented in 13th International Workshop on $\mathbf{R}$ *&* $\mathbf{C}$ *Singularities, São Carlos, in 2014.*
Prof. **Terry Gaffney** *asked if it is an isomorphism.*
The above theorem affirmatively answers his question!



Morse functions on closed surfaces

For Morse functions on manifolds **without boundary**, the **fold cobordism groups** have been studied.
Two Morse functions on closed n-dim. manifolds are **fold cobordant** if there exists a cobordism $F : X^{n+1} \to \mathbf{R} \times [0,1]$ between them which has only **fold** points as its singularities. (No cusp is allowed.)

Theorem 1.3 (Ikegami–Saeki 2003, Ikegami 2004)
The fold cobordism group of Morse functions on closed surfaces is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}$.
The fold cobordism group for oriented closed surfaces is isomorphic to $\mathbf{Z}$.

The idea of our proof of the main theorem is based on [Ikegami–Saeki 2003].

§2. Reeb Graph and Reeb Space

Definition 2.1 $f : N \to P$ smooth map
For $x, x' \in N$, define $x \sim_f x'$ if
(i) $f(x) = f(x')(= y)$, and
(ii) x and x' belong to the **same connected component of $f^{-1}(y)$**.

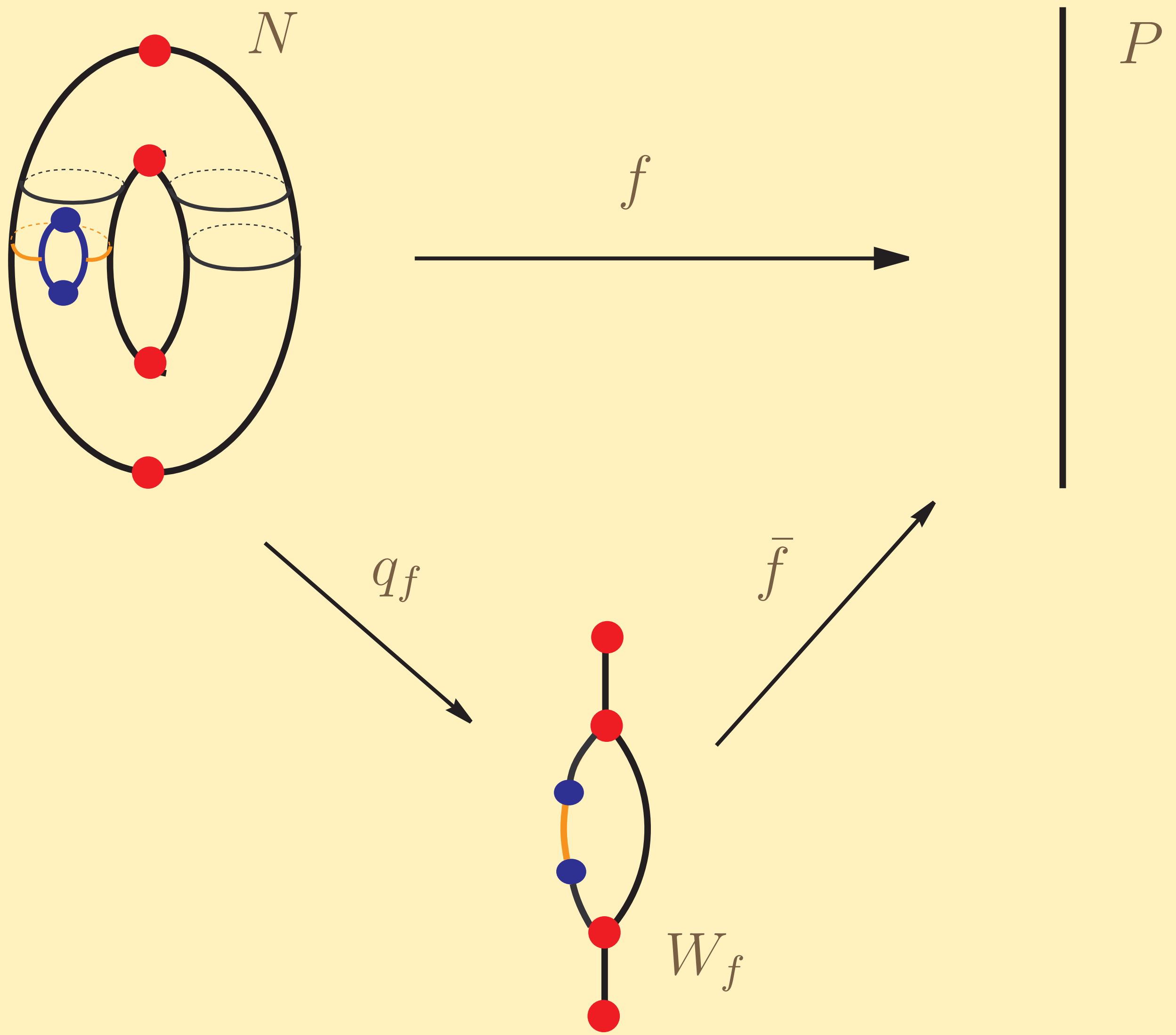
$W_f = N/\sim_f$ quotient space
$q_f : N \to W_f$ quotient map

$\exists! \bar{f} : W_f \to P$ that makes the diagram commutative:

$$\begin{array}{ccccc} N & & \xrightarrow{f} & & P \\ & q_f \searrow & & \nearrow \bar{f} & \\ & & W_f & & \end{array}$$

The above diagram is called the **Stein factorization** of f.
W_f is called the **Reeb space** and $\bar{f}$ the **Reeb map**.

Example



$f : N \to \mathbf{R}$ Morse function on a compact **surface** with boundary

Lemma 2.2 *Reeb space* W_f *is a finite* **graph** *whose vertices are the* q_f*-images of the critical points of* f *and* $f|_{\partial N}$*.*

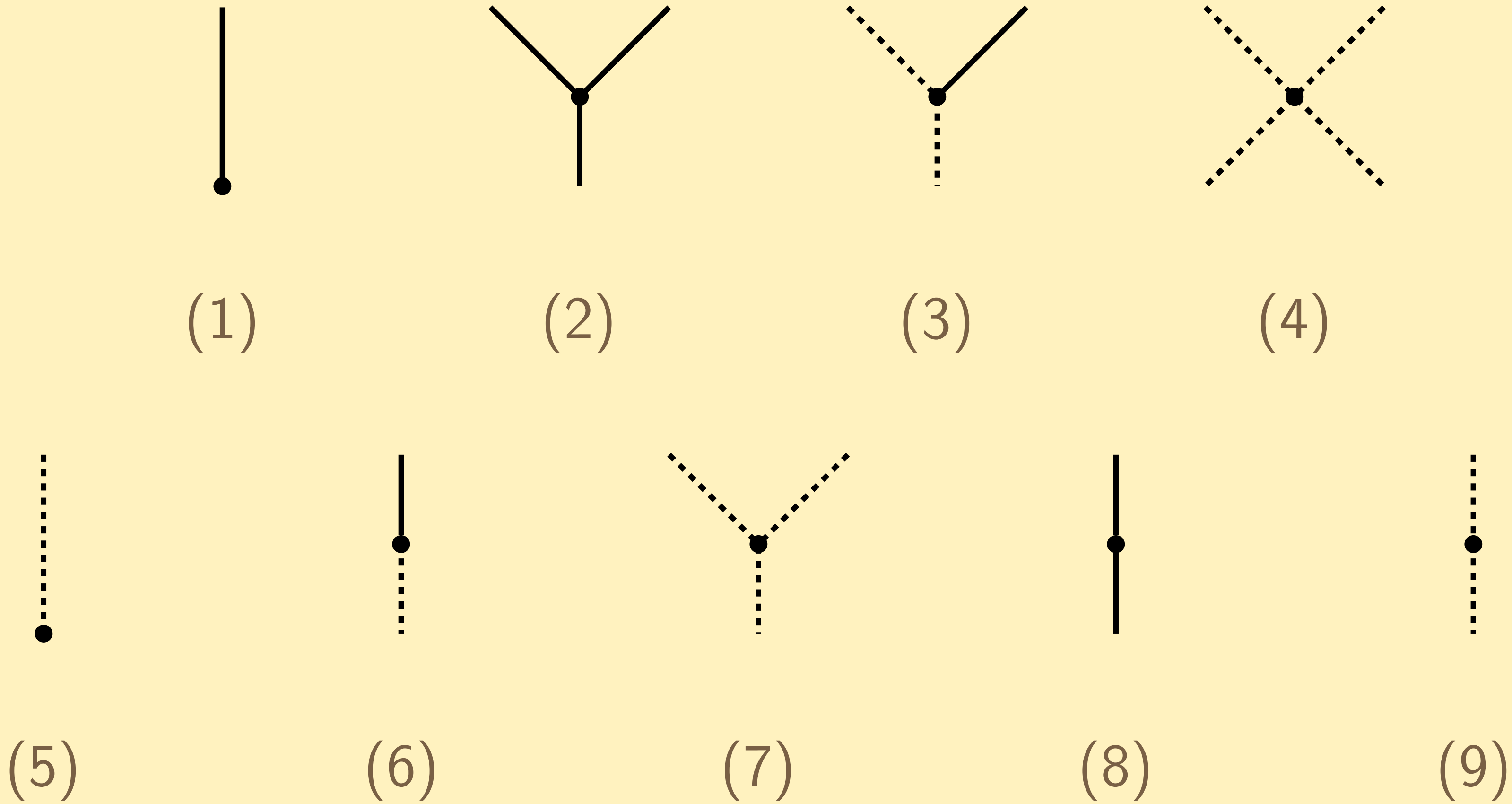
W_f is also called a **Reeb graph**, and the continuous map $\overline{f} : W_f \to \mathbf{R}$ a **Reeb function**.

Each edge corresponds to a **circle regular fiber** or an **arc regular fiber**.
We label each edge by 0 or 1, where 0 (resp. 1) means that it corresponds to a **circle** regular fiber (resp. an **arc** regular fiber).
Such a Reeb graph is called a **labeled Reeb graph**.

Edge with label 0 ——— thick line : circle fiber
Edge with label 1 $\cdots\cdots$ dotted line : arc fiber

Local forms of Reeb function

Around each vertex, $\overline{f} : W_f \to \mathbf{R}$ is locally equivalent to one of the height functions below (or their negatives).
(The map $\overline{f}$ is an embedding on each edge.)

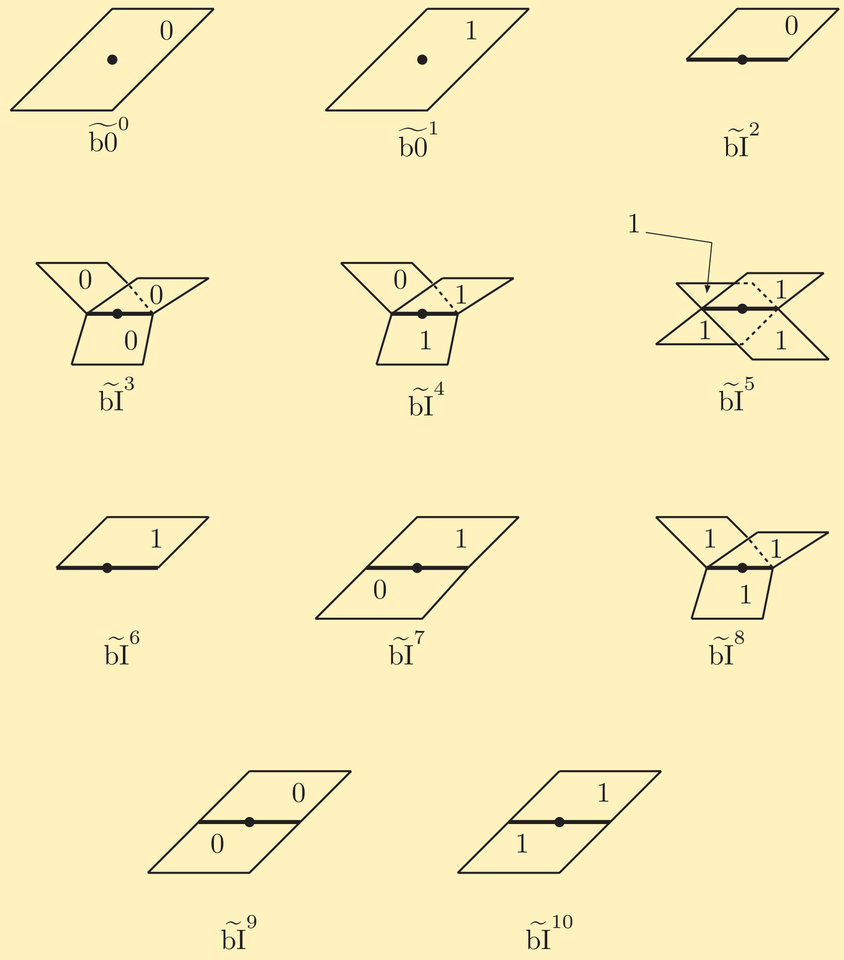


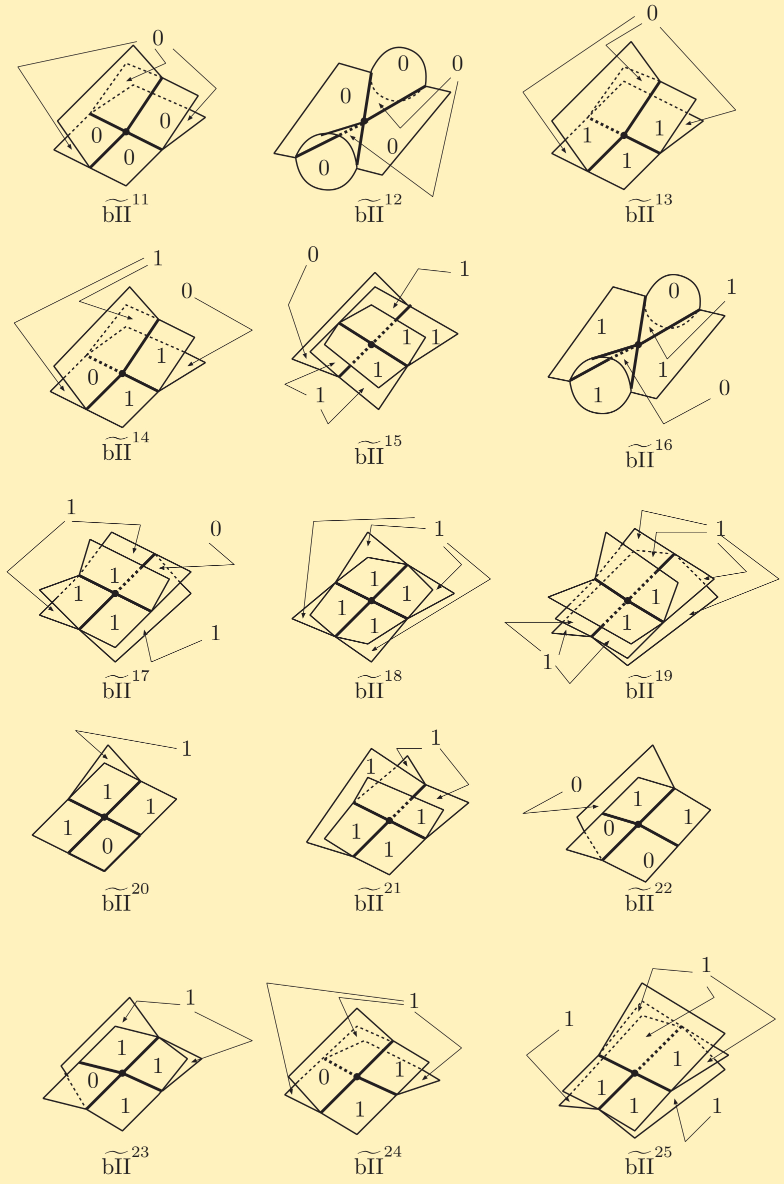
Reeb space of a stable map

Lemma 2.3 *Let $f : X^3 \to P^2$ be an* **admissible** C^∞ *stable map of a compact 3-dim. manifold with boundary into a surface without boundary. Then, the Reeb space W_f is a compact 2-dim.* **polyhedron**, *which is* **labeled**: *each component of $W_f \setminus q_f(S(f))$ is labeled with 0 or 1, where $S(f)$ is the set of singular points of f.*

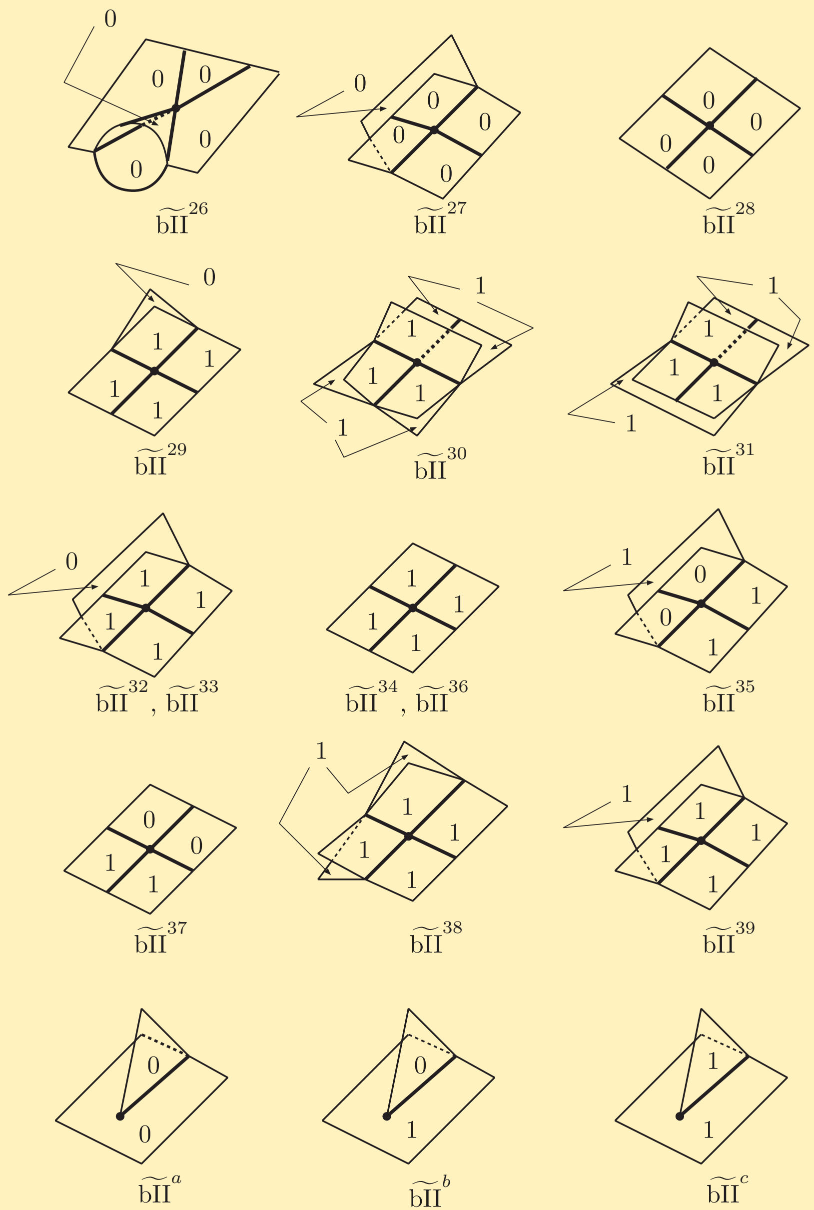
Furthermore, around each point of W_f, the Reeb map $\overline{f} : W_f \to P^2$ is **locally equivalent** *to one of the maps as depicted below, where the relevant map is the vertical projection to a plane.*

Local forms of Reeb maps (I)





Local forms of Reeb maps (II)



§3. Proof

Reeb-like function

Definition 3.1
Let G be a finite graph whose edges are **labeled** by 0 or 1.
We assume that around each vertex of G, it is **locally homeomorphic** to one of the 9 local labeled Reeb graphs for Morse functions.
Then, we call G a **labeled Reeb-like graph**.

Let $r : G \to \mathbf{R}$ be a continuous function such that

1. around each vertex of G, r is **locally equivalent** to one of the local Reeb functions of a Morse function, and
2. r is an embedding on each edge.

Then, we call $r : G \to \mathbf{R}$ a **Reeb-like function**.

These are **abstract generalizations of labeled Reeb graphs and Reeb functions** for Morse functions on compact surfaces with boundary.

Definition 3.2 Two Reeb-like functions $r_i : G_i \to \mathbf{R}$, $i = 0, 1$, on **labeled** Reeb-like graphs are **cobordant** if
$\exists$compact 2-dim. polyhedron W,
$\exists$1-dim. subpolyhedron $\Sigma(W)$, and
$\exists$continuous map $R : W \to \mathbf{R} \times [0, 1]$ s.t.

1. connected components of $W \setminus \Sigma(W)$ are **labeled** with 0 or 1,
2. G_i are identified with "labeled" subcomplexes of W with regular neighborhoods of the forms $G_i \times [0, \varepsilon]$,
3. $r_0 = R|_{G_0} : G_0 \to \mathbf{R} \times \{0\}$ and $r_1 = R|_{G_1} : G_1 \to \mathbf{R} \times \{1\}$,
4. around each point of $W \setminus (G_0 \cup G_1)$, R is **locally equivalent** to the Reeb map $\overline{f} : W_f \to \mathbf{R} \times [0, 1]$ of a proper admissible C^∞ stable map f of a 3-dim. manifold with boundary into a surface.

Cobordism group

This is an **abstract generalization of the Reeb map of an admissible cobordism between two Morse functions** on compact surfaces with boundary.

The above relation defines an equivalence relation for Reeb-like functions. Furthermore, the set of all cobordism classes forms an **additive group** under the disjoint union.

We denote by $b\mathfrak{R}$ the additive group of all cobordism classes of Reeb-like functions on the labeled Reeb-like graphs and call it the **cobordism group of Reeb-like functions**.

We have the natural map $\rho\colon b\mathfrak{N}_2 \to b\mathfrak{R}$, which associates to each admissible cobordism class of a Morse function f on a compact surface with boundary the cobordism class of the Reeb function $\overline{f}\colon W_f \to \mathbf{R}$.
It is straightforward to see that this defines a homomorphism of additive groups.

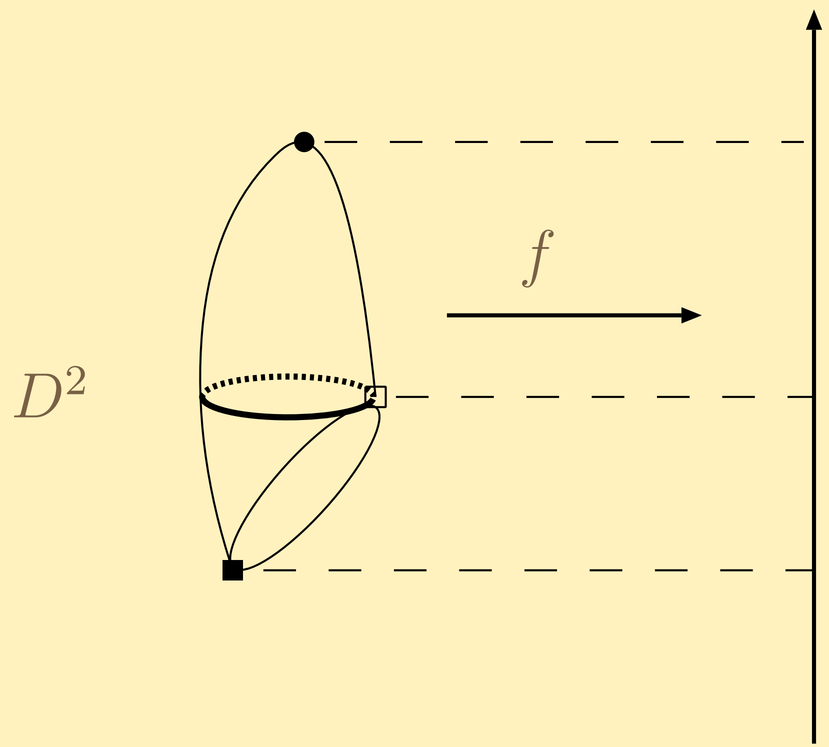
Proposition 3.3 *The homomorphism $\rho\colon b\mathfrak{N}_2 \to b\mathfrak{R}$ is an isomorphism.*

Surjectivity: Given a labeled Reeb-like graph, one can construct an associated Morse function on a compact surface with boundary.

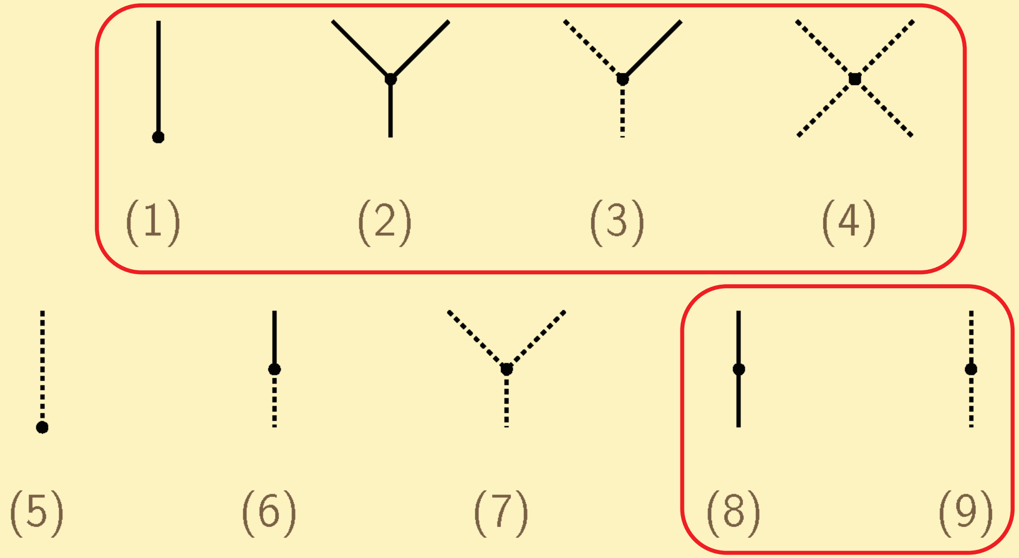
Injectivity: Given an abstract cobordism W between two labeled Reeb graphs R_{f_i} for Morse functions, we first modify the cobordism in such a way that it is a regular neighborhood of $\Sigma(W) \cup R_{f_0} \cup R_{f_1}$.
Then, one can construct an associated admissible cobordism between the Morse functions.

It suffices to prove the following.

Proposition 3.4 *The cobordism group* $b\mathfrak{R}$ *of Reeb-like functions is a* **cyclic group of order two** *generated by the cobordism class of the Reeb function of the Morse function as depicted below.*



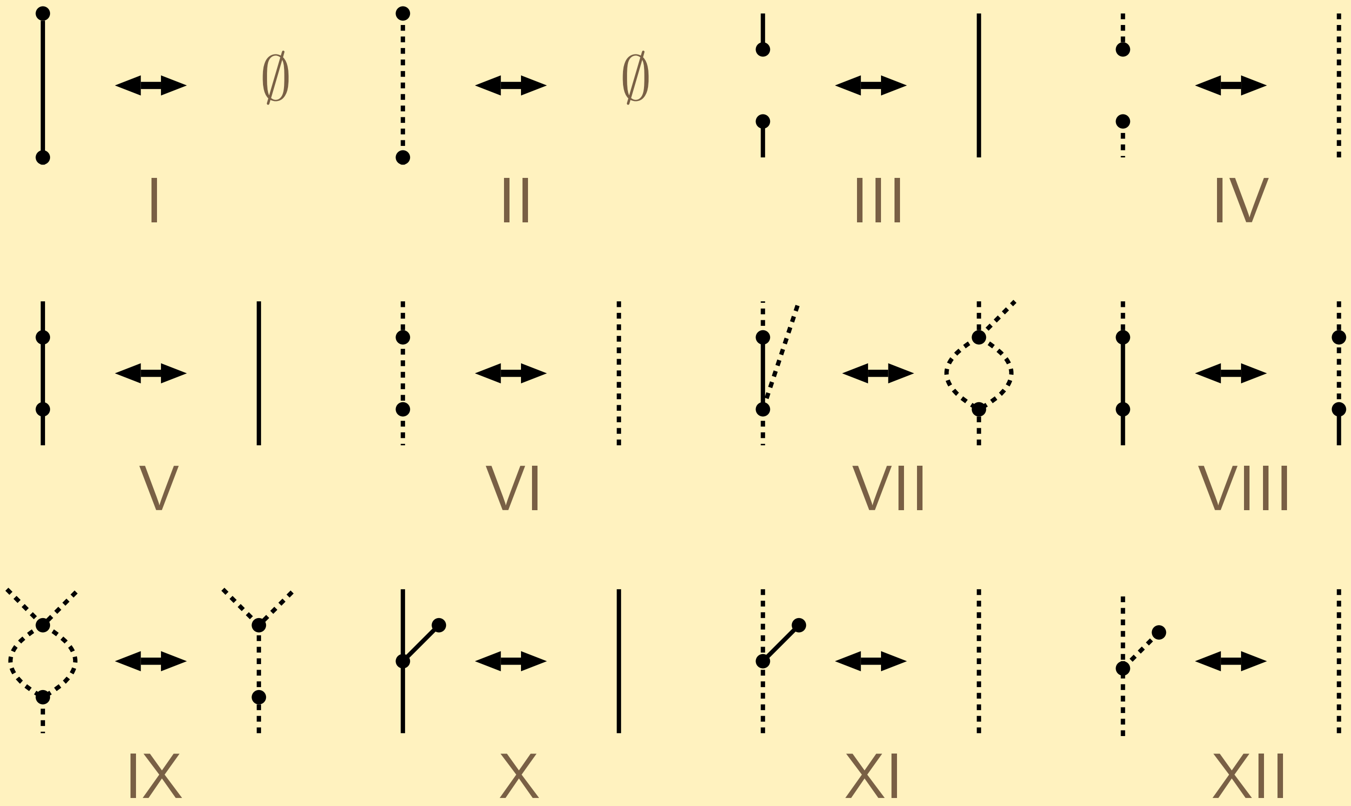
Define $\sigma : b\mathfrak{R} \to \mathbf{Z}_2$ by setting $\sigma([r : G \to \mathbf{R}])$ to be the modulo two number of vertices of type (1), (2), (3), (4), (8) and (9).



σ is a well-defined homomorphism of abelian groups.
In fact, the homomorphism corresponds to a certain cohomology class of the **universal complex of singular fibers**.
The well-definedness of σ is a direct consequence of the fact that the representative of the cohomology class is a **cocycle**.

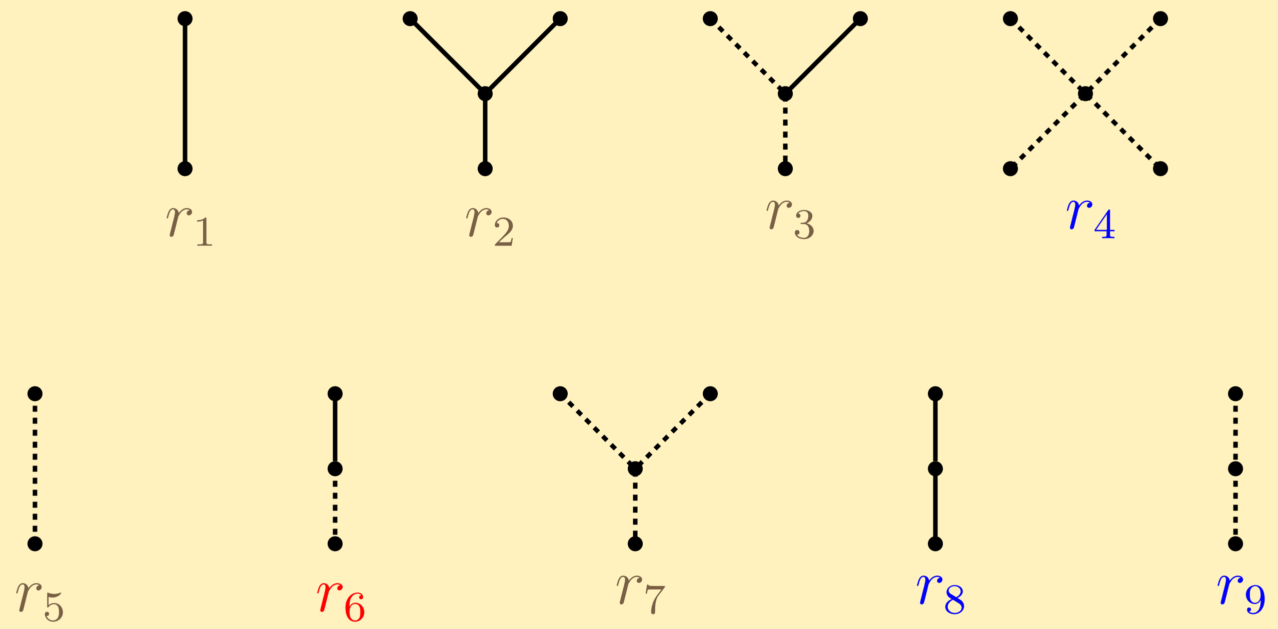
Moves for Reeb-like functions

Lemma 3.5 *Let $r_i : G_i \to \mathbf{R}$, $i = 0, 1$, be Reeb-like functions on labeled Reeb-like graphs. If r_1 is obtained from r_0 by the* **local moves** *as depicted below or their negatives, then r_0 and r_1 are cobordant.*



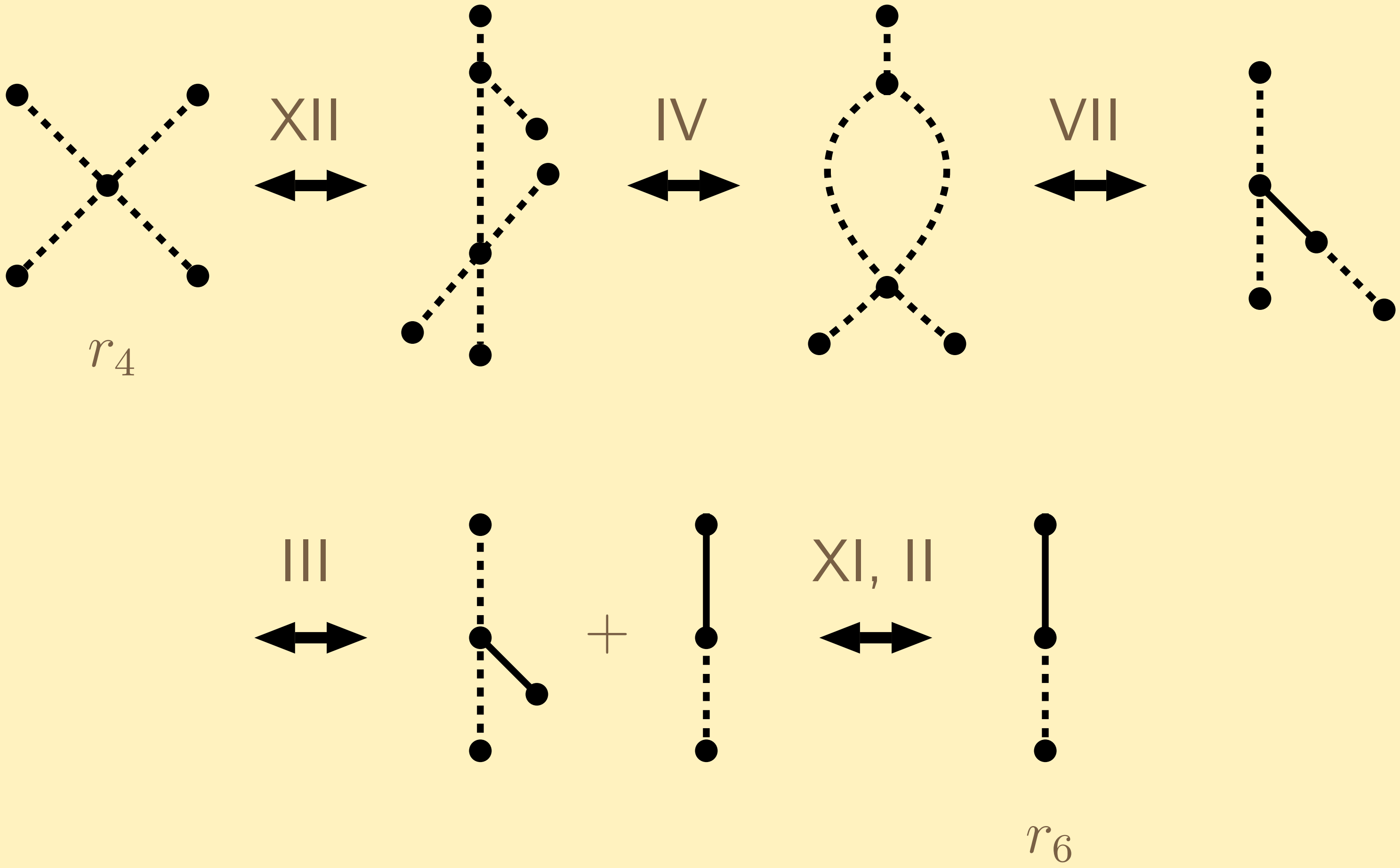
In fact, there are many more; we use only the above ones in the proof.

Let $r : G \to \mathbf{R}$ be an arbitrary Reeb-like function.
We show $[r] = 0$ or $[r_6]$, where r_6 is the Reeb function of the Morse function mentioned above (see below).
We first **cut** the edges of G by moves III and IV to get a disjoint union of **elementary Reeb-like functions** (or their negatives) as follows.

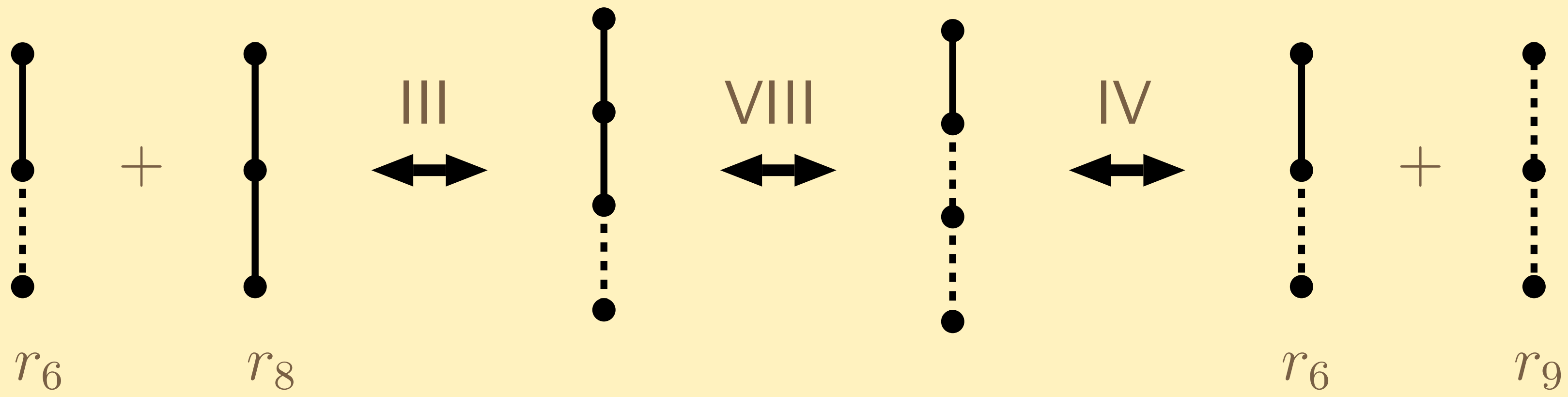


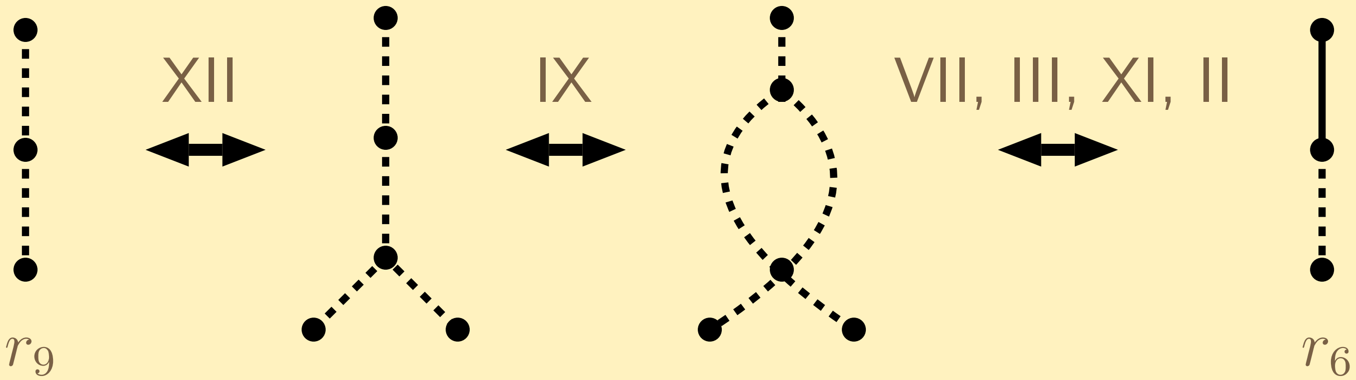
We see easily $[r_1] = [r_5] = 0$ and $[r_2] = [r_3] = [r_7] = 0$.

$[r_4] = [r_6]$

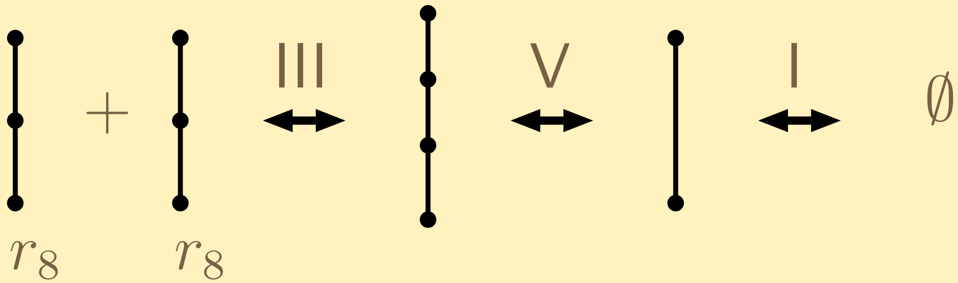


$[r_6] + [r_8] = [r_6] + [r_9]$





Thus, we have shown that $[r_4] = [r_6] = [r_8] = [r_9]$ generates $b\mathfrak{N}_2$.
On the other hand, we have $\sigma([r_8]) = 1 \in \mathbf{Z}_2$ and $[r_8] + [r_8] = 0$ as below.



Hence, σ is an isomorphism. $\square$

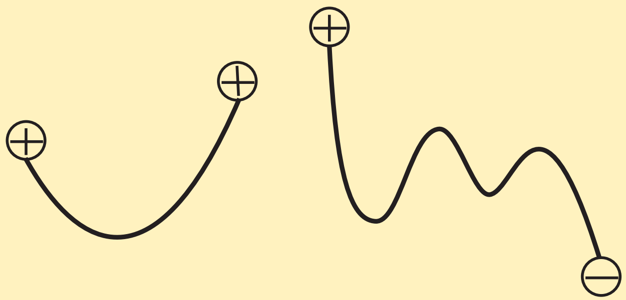
§4. Low Dimensional Cases

$b\mathfrak{N}_0$ and $b\mathfrak{N}_1$

Proposition 4.1 *The 0-dimensional admissible cobordism group of Morse functions $b\mathfrak{N}_0$* **is trivial**.

Proposition 4.2 *The 1-dimensional admissible cobordism group of Morse functions $b\mathfrak{N}_1$* **is an infinite cyclic group** *generated by the admissible cobordism class of $f_0\colon [-1,2] \to \mathbf{R}$ given by $f_0(x) = x^2$, $x \in [-1,2]$.*

In fact, $(\sharp(\text{positive end points}) - \sharp(\text{negative end points}))/2$ gives an isomorphism.



Problems

Problem 4.3 *Study the group structure of* $b\mathfrak{N}_n$, $n \geq 3$.

Problem 4.4 *Study the group structure of the* **oriented version** $b\Omega_n$.

Problem 4.5 *Study the group structures of the* **admissible fold cobordism group** $b\mathfrak{F}_n$ *and its oriented version.*

Thank you for your attention!