Special Generic Maps on Open 4-Manifolds

Osamu Saeki
Kyushu University

December 7, 2009

§1. Introduction

Special generic map

Definition 1.1 M^n, N^p: smooth manifolds $(n \geq p)$
A singularity of a smooth map $M^n \longrightarrow N^p$ that has the normal form

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_{p-1}, x_p^2 + x_{p+1}^2 + \cdots + x_n^2) \quad (1)$$

is called a **definite fold singularity**.
A smooth map $f : M^n \longrightarrow N^p$ is a **special generic map** if it has only definite fold singularities.
Then, $S(f)$, the set of singular points of f, is a submanifold of M^n of dimension $p-1$.

Example 1.2 The map $f : \mathbf{R}^n \longrightarrow \mathbf{R}^p$ defined by (1) is a proper special generic map.
$S(f) = \mathbf{R}^{p-1} \times \{0\}$

Example

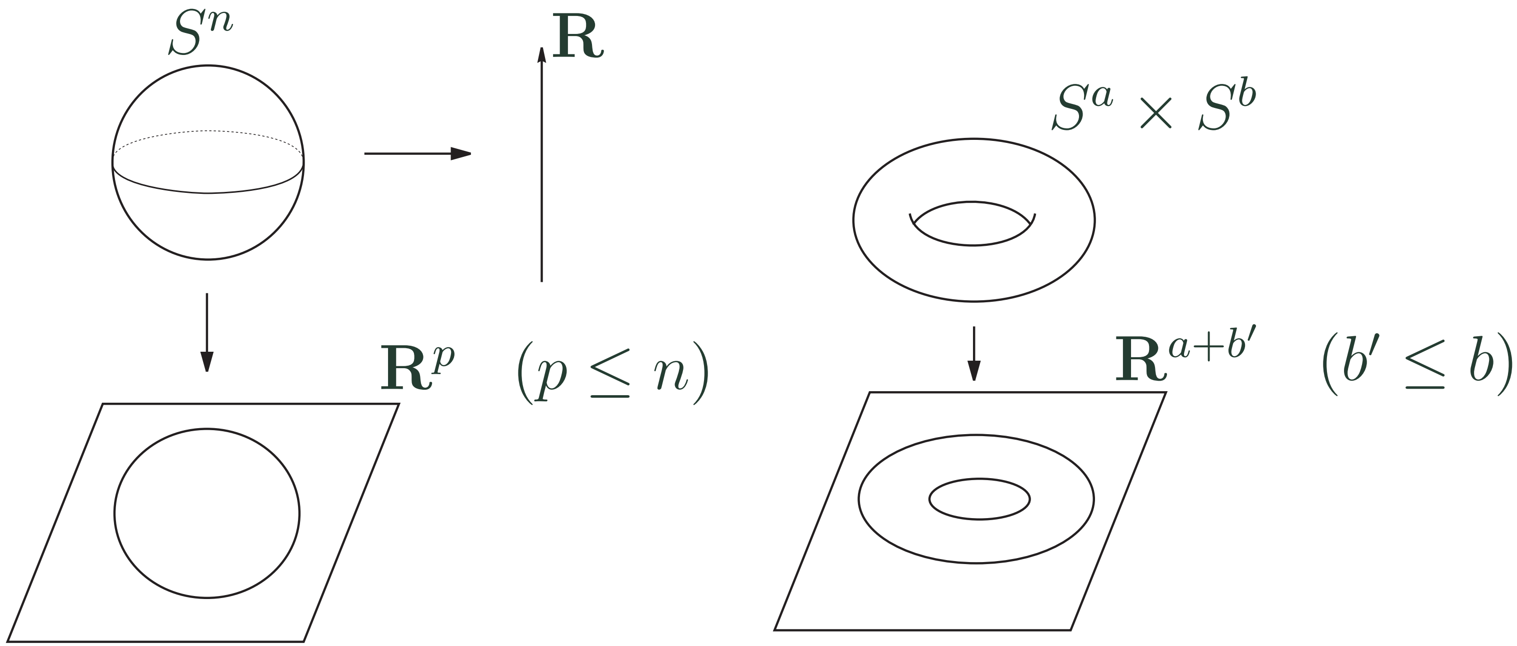


Figure 1: Example of special generic maps

Special generic maps are strongly related to **smooth structures** of manifolds.

M^n: closed connected manifold of dimension n

Theorem 1.3 (Reeb, Smale) $n \geq 5$

M^n *is a homotopy* n*-sphere (*$\Longleftrightarrow M^n \approx S^n$ *(homeomorphic))*

$\Longleftrightarrow \exists f : M^n \to \mathbf{R}$ *special generic function*

Theorem 1.4 (S, 1993)

$M^n \cong S^n$ *(diffeomorphic)*

$\Longleftrightarrow 1 \leq \forall p \leq n, \exists f : M^n \to \mathbf{R}^p$ *special generic map*

Theorem 1.5 (Sakuma-S, 1990's) $\exists (M_1^4, M_2^4)$ *such that*
$M_1^4 \approx M_2^4$ *(homeomorphic)*
$\exists f_1 : M_1^4 \to \mathbf{R}^3$ *special generic map*
$\not\exists f_2 : M_2^4 \to \mathbf{R}^3$ *special generic map*
In fact, there are infinitely many such pairs.

Theorem 1.6 (S (1993) $+$ 3-dim. Poincaré Conj.)
M^4*: closed* 1*-connected* 4*-dim. manifold*
$\exists f : M^4 \to \mathbf{R}^3$ *special generic map*
$\Longleftrightarrow M^4 \cong \sharp^k(S^2 \times S^2)$ *or* $\sharp^k(S^2 \tilde{\times} S^2)$ *(diffeomorphic)*

Corollary 1.7
$M^4 \approx \sharp^k(S^2 \times S^2)$ *or* $\sharp^k(S^2 \tilde{\times} S^2)$ *(homeomorphic)*
$\exists f : M^4 \to \mathbf{R}^3$ *special generic map*
$\Longleftrightarrow M^4 \cong \sharp^k(S^2 \times S^2)$ *or* $\sharp^k(S^2 \tilde{\times} S^2)$ *(diffeomorphic)*

Remark 1.8 Smooth structures on $\sharp^k(S^2 \times S^2)$ or on $\sharp^k(S^2 \tilde{\times} S^2)$ are not unique. In fact, there are *infinitely many* such structures if k is a sufficiently big odd integer (J. Park, 2003).

Today's topic: How about **non-compact** 4-dim. manifolds?

Note. Usually an open 4-manifold admits *uncountably many* smooth structures.

§2. Main Results

Theorem 2.1

M^4: open 1-connected 4-dim. manifold of "finite type"

$\exists f : M^4 \to N^3$ *proper special generic map*

for some 3-dim. manifold N^3 with $S(f) \neq \emptyset$

$\Longleftrightarrow$ *M^4 is diffeomorphic to the connected sum*

of a finite number of copies of the following manifolds:

$\mathbf{R}^4 (= S^4 \setminus \{\text{point}\})$, $\mathrm{Int}\left(\natural^k (S^2 \times D^2)\right) = S^4 \setminus (\vee^k S^1)$,

$\mathbf{R}^2$*-bundle over* S^2, $S^2 \times S^2$, $S^2 \tilde{\times} S^2$

Remark 2.2

$\tilde{f} : \widetilde{M}^4 \to \mathbf{R}^3$ special generic map on a closed 4-manifold $\widetilde{M}^4$

$C \subsetneqq \mathbf{R}^3$ closed subset

$\Longrightarrow M^4 = \widetilde{M}^4 - \tilde{f}^{-1}(C)$,

$\tilde{f}|_{M^4} : M^4 \to \mathbf{R}^3 \setminus C$ is a proper special generic map.

Corollary 2.3 $M^4 \approx \mathbf{R}^4$ *(homeomorphic)*
$\exists f : M^4 \to \mathbf{R}^p$ *special generic map for some* $1 \leq p \leq 3$
$\iff M^4 \cong \mathbf{R}^4$ *(diffeomorphic)*

Theorem 2.4 L^3 *: closed orientable* 3*-manifold*
$M^4 \approx L^3 \times \mathbf{R}$ *(homeomorphic)*
$\exists f : M^4 \to \mathbf{R}^3$ *special generic map*
$\iff M^4 \cong L^3 \times \mathbf{R}$ *(diffeomorphic) and*
$\exists g : L^3 \to \mathbf{R}^2$ *special generic map*

Remark 2.5 "$\Longleftarrow$" is easy.
Consider $f = g \times \mathrm{id}_{\mathbf{R}} : L^3 \times \mathbf{R} \to \mathbf{R}^2 \times \mathbf{R}$.

§3. Ends of Open Manifolds

Definition 3.1 (Siebenmann, 1965) X: Hausdorff space
ε: collection of subsets of X such that

(i) Each $G \in \varepsilon$ is a connected open non-empty set with compact frontier $\overline{G} - G$,

(ii) $G, G' \in \varepsilon \Longrightarrow \exists G'' \in \varepsilon$ with $G'' \subset G \cap G'$,

(iii) $\bigcap_{G \in \varepsilon} \overline{G} = \emptyset$.

A maximal such collection ε is called an **end** of X.
A **neighborhood** of an end ε is any set $N \subset X$ that contains some member of ε.

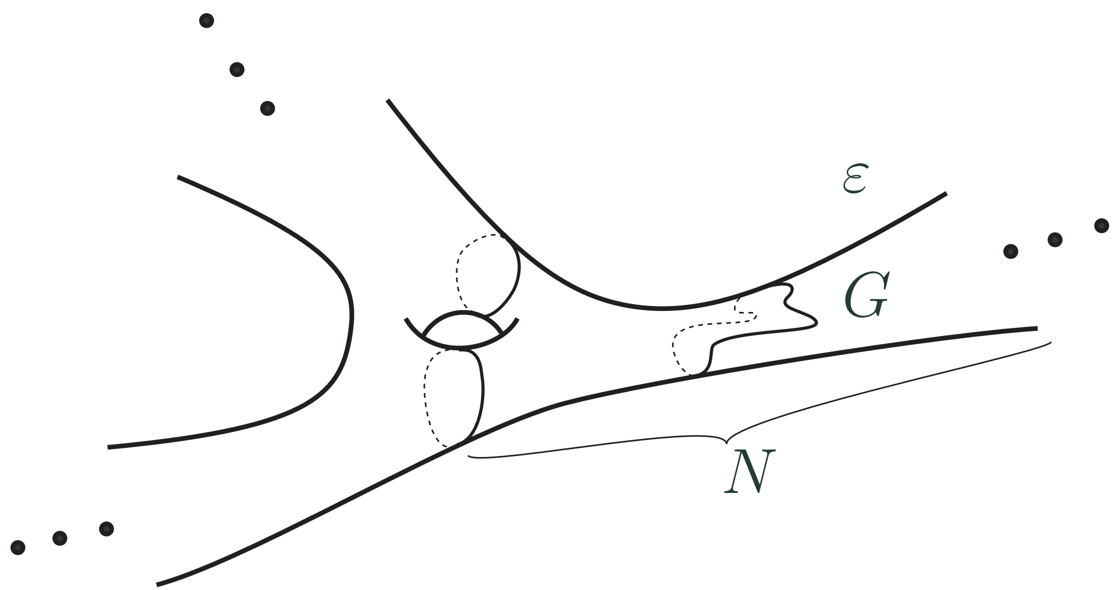


Figure 2: Ends of a manifold

Definition 3.2 ε: an end of a topological manifold X

π_1 is **stable** at ε

$\Longleftrightarrow$

$\exists X_1 \supset X_2 \supset \cdots$ a sequence of path connected neighborhoods of ε such that $\bigcap \overline{X}_i = \emptyset$ and the sequence

$$\mathcal{G}: \qquad \pi_1(X_1) \xleftarrow{f_1} \pi_1(X_2) \xleftarrow{f_2} \cdots$$

induced by the inclusions induces isomorphisms

$$\mathrm{Im}(f_1) \xleftarrow{\cong} \mathrm{Im}(f_2) \xleftarrow{\cong} \cdots .$$

Definition 3.3 Suppose π_1 is stable at an end ε.
Define $\boldsymbol{\pi_1}(\varepsilon)$ to be the projective limit $\varprojlim \mathcal{G}$ for some $\mathcal{G}$ as above.

According to Siebenmann, $\pi_1(\varepsilon)$ is well defined up to isomorphism.

Definition 3.4 An open manifold M is of **finite type** if

(i) M has finitely many ends,

(ii) for each end ε, $\pi_1(\varepsilon)$ is finitely presentable,

(iii) $H_*(M; \mathbf{Z}_2)$ is finitely generated.

Lemma 3.5 (Husch–Price, 1970)

W^3: *open orientable* 3*-manifold of finite type*

$\Longrightarrow \exists \widetilde{W}^3$ *compact orientable* 3*-manifold and*

$\exists h : W^3 \to \widetilde{W}^3$ *embedding*

such that $h(\operatorname{Int} W^3) = \operatorname{Int} \widetilde{W}^3$.

§4. Stein Factorization

Definition 4.1 $f : M \longrightarrow N$ smooth map

For $x, x' \in M$, define $x \sim_f x'$ if

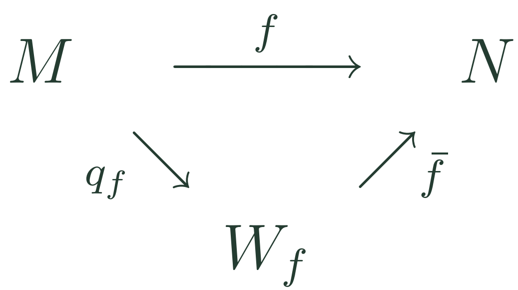
(i) $f(x) = f(x')(= y)$, and

(ii) x and x' belong to the same connected component of $f^{-1}(y)$.

$W_f = M/\sim_f$ quotient space

$q_f : M \longrightarrow W_f$ quotient map

$\exists ! \bar{f} : W_f \longrightarrow N$ that makes the diagram commutative:



The above diagram is called the **Stein factorization** of f.

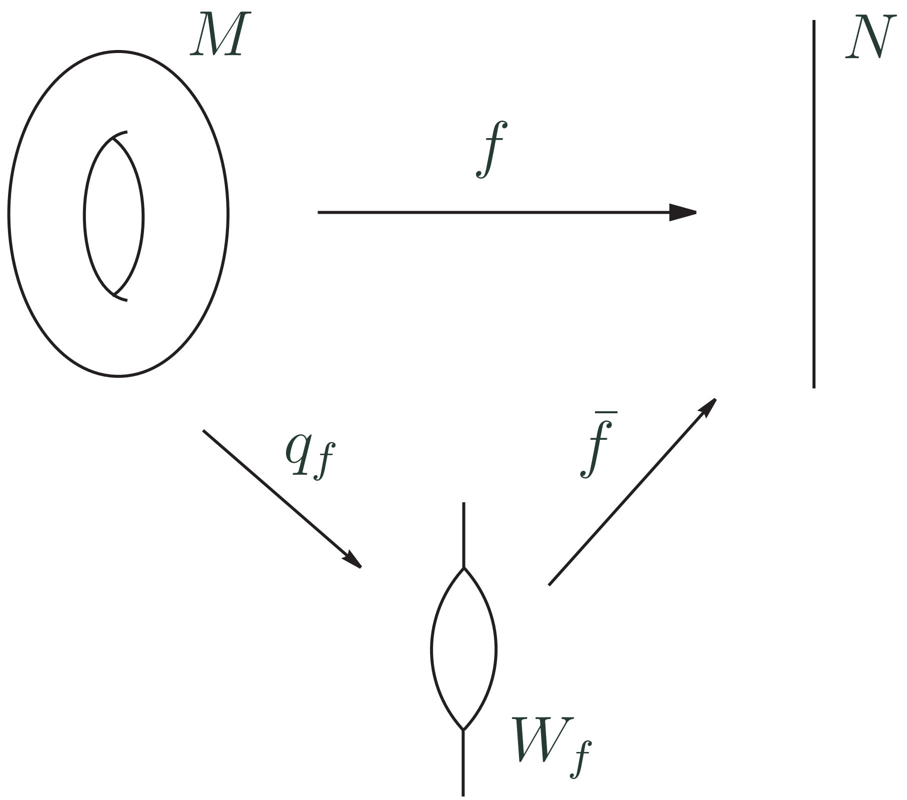


Figure 3: Stein factorization

Proposition 4.2 *Let $f : M^n \to N^p$ be a proper special generic map with $n > p$. Then, we have the following.*

(1) *The singular point set $S(f)$ is a regular submanifold of M^n of dimension $p - 1$,*

(2) *W_f has the structure of a smooth p-dim. manifold possibly with boundary such that $\bar{f} : W_f \to N^p$ is an immersion.*

(3) *$q_f|_{S(f)} : S(f) \to \partial W_f$ is a diffeomorphism.*

(4) *$q_f|_{M \setminus S(f)} : M \setminus S(f) \to \operatorname{Int} W_f$ is a smooth S^{n-p}-bundle.*

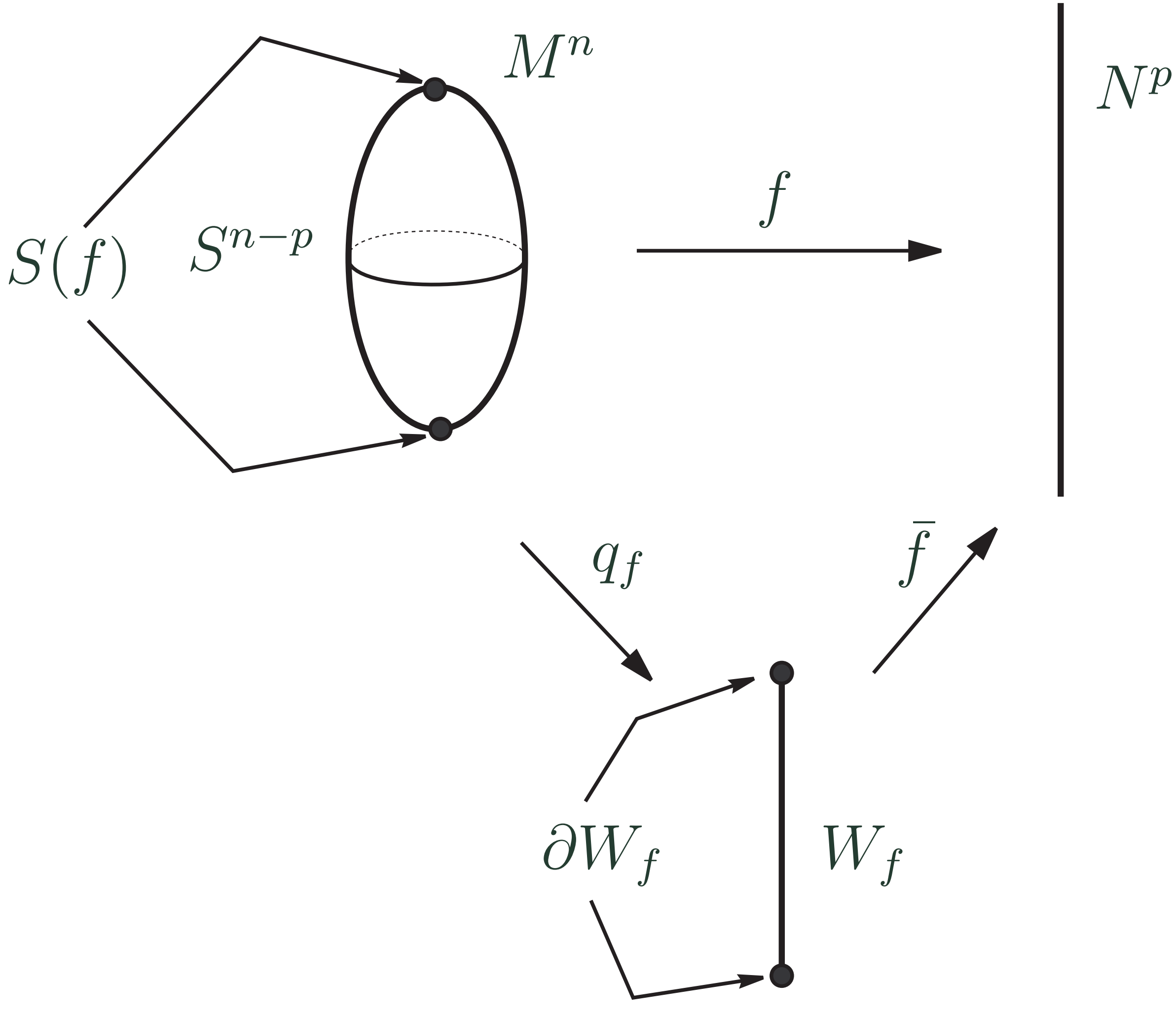


Figure 4: Proposition 4.2

Theorem 4.3 (S, 1993)

$f : M^n \to N^p$ *special generic map with* $n - p = 1, 2, 3$

$\Longrightarrow$

M^n *is diffeomorphic to the boundary of a* D^{n-p+1}*-bundle over* W_f.

§5. Proofs of Theorems

Let us prove the following.

Theorem 2.1:
M^4: *open* 1*-connected* 4*-dim. manifold of "finite type"*
$\exists f : M^4 \to N^3$ *proper special generic map*
for some 3*-dim. manifold* N^3 *with* $S(f) \neq \emptyset$
$\Longleftrightarrow$ M^4 *is diffeomorphic to the connected sum*
of a finite number of copies of the following manifolds:
$\mathbf{R}^4$, $\mathrm{Int}\left(\natural^k (S^2 \times D^2)\right)$, $\mathbf{R}^2$*-bundle over* S^2,
$S^2 \times S^2$, $S^2 \tilde{\times} S^2$

Proof of Theorem 2.1:
M^4: open 4-manifold of finite type
N^3: orientable 3-manifold
$f : M^4 \to N^3$ proper special generic map
$\Longrightarrow$
W_f is an open 3-manifold of finite type
$\pi_1(M^4) = 1 \Rightarrow \pi_1(W_f) = 1$
By the solution to the Poincaré Conjecture $+$ Husch–Price Lemma,
$W_f \cong D^3 \setminus F$ or $\natural^k(S^2 \times [0,1]) \setminus F$, where F is a compact surface (possibly with boundary) contained in the boundary.
On the other hand, M^4 is diffeomorphic to the boundary of a D^2-bundle over W_f (by the Disk bundle theorem).
Then we easily get the desired conclusion.

Remark 5.1 Every 4-dim. manifold as in Theorem 2.1 admits infinitely many (or uncountably many) distinct smooth structures.
Theorem 2.1 implies that among them there is exactly one structure that allows the existence of a proper special generic map into an orientable 3-manifold.

Let us now prove the following.

Theorem 2.4:
L^3 : *closed orientable* 3*-manifold*
$M^4 \approx L^3 \times \mathbf{R}$ *(homeomorphic)*
$\exists f : M^4 \to \mathbf{R}^3$ *special generic map*
$\Longleftrightarrow M^4 \cong L^3 \times \mathbf{R}$ *(diffeomorphic) and*
$\exists g : L^3 \to \mathbf{R}^2$ *special generic map*

Proof of Theorem 2.4:

$M^4 \approx L^3 \times \mathbf{R}$, $f : M^4 \to N^3$ special generic map

$\Longrightarrow$

W_f is of "finite type" and has exactly two ends $F_i \times [0, \infty)$, $i = 1, 2$

$F_i \times \{0\} \hookrightarrow W_f$ induce isomorphisms of fundamental groups

$W_f \cong (F_1 \times \mathbf{R}) \sharp (\sharp^k D^3)$

Since $M^4 \approx L^3 \times \mathbf{R}$, we see $W_f \cong F_1 \times \mathbf{R}$.

$\Longrightarrow M^4 \cong L' \times \mathbf{R}$ for some 3-manifold L'

$\pi_1(L') \cong \pi_1(L^3)$ is free

$L' \cong L^3 \cong \sharp^\ell (S^1 \times S^2)$, and hence

$\exists g : L^3 \to \mathbf{R}^2$ special generic map (Burlet–de Rham, 1974)

Conjecture 5.2

M^4: *topological* 4*-manifold*

$\Longrightarrow$ *There exists at most one smooth structure on* M^4 *that allows the existence of a proper special generic map into* $\mathbf{R}^3$*.*

Remark 5.3 In the above conjecture, the **properness** of the special generic map is essential.

$f : M^4 \to N^3$ special generic map of an open 4-manifold

$M' \approx M^4$ (homeomorphic)

$\Longrightarrow \exists$ "formal solution" over M' on the jet level for the open differential relation corresponding to special generic maps.

M' admits a special generic map by the Gromov h-principle for open manifolds.

Final remark

Note: Even if f is proper, the resulting special generic map on M' may not be proper.

Compare this with the following:

$$M^4 \approx \mathbf{R}^4 \Longrightarrow \exists g : M^4 \to \mathbf{R}^4 \quad \text{proper special generic map}$$

In the **equidimensional case**, the C^0 dense h-principle holds and the properness can be preserved.

Thank you!