Lifting Special Generic Maps

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(**I**nstitute of **M**athematics for **I**ndustry, Kyushu University)
（九州大学，**マス・フォア・インダストリ研究所**）

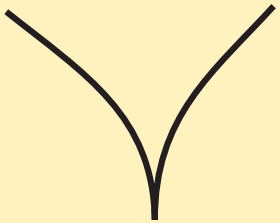
Joint work with **Masamichi Takase** (Seikei University)

May 31, 2011

§1. Lifting Singular Maps

Desingularizing a singular curve

This is a **singular** plane curve.



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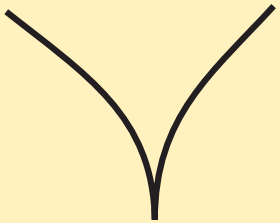
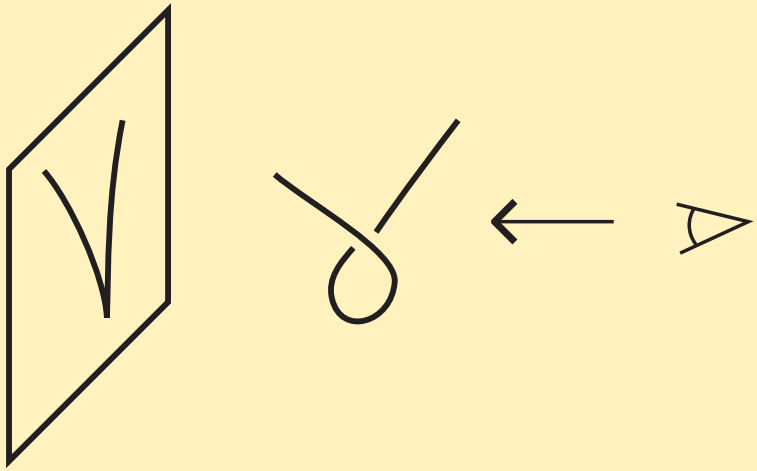


But, this can be the projected image of a **non-singular** space curve.



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M^n: **closed** n-dim. C^∞ manifold,
$f : M^n \to \mathbf{R}^p$ a **generic** C^∞ map $(n \geq p)$.

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η: **immersion** *or* **embedding**

Surface case

Theorem 1.2 (Haefliger, 1960) $f : M^2 \to \mathbf{R}^2$
$\exists$**immersion** $\eta : M^2 \to \mathbf{R}^3$ *s.t.* $f = \pi \circ \eta$
$\Longleftrightarrow$ *For every singular set component* S $(\cong S^1)$ *of* f*:*
if S *has an annulus nbhd,* S *contains an even number of cusps,*
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Equi-dimensional case

Theorem 1.5 (Saito, 1961) M^n: **orientable**
$f : M^n \to \mathbf{R}^n$ *special generic map*
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Theorem 1.6 (Blank–Curley, 1985)
$f : M^n \to N^n$, $\pi : E \to N^n$ *line bundle*
$\exists$**immersion** $\eta : M^n \to E$ *s.t.* $f = \pi \circ \eta$
$\iff \operatorname{rk} df \geq n-1$, *and*
$[\{\textit{cusps}\}]^* + w_1(\nu) + i^* f^* w_1(E) = 0$ *in* $H^1(\{\textit{folds}\}; \mathbf{Z}_2)$,
where ν *is the normal line bundle of* $\{\textit{folds}\}$ *in* M^n *and*
$i : \{\textit{folds}\} \to M^n$ *is the inclusion.*

Special generic maps

Today's topic:
Desingularization of **special generic maps**.
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$$(x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_{p-1}, x_p^2 + x_{p+1}^2 + \cdots + x_n^2)$$

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Definition 1.8 $f : M^n \to N^p$ is a **special generic map** (**SGM**, for short) if it has **only definite fold singularities**.

Examples

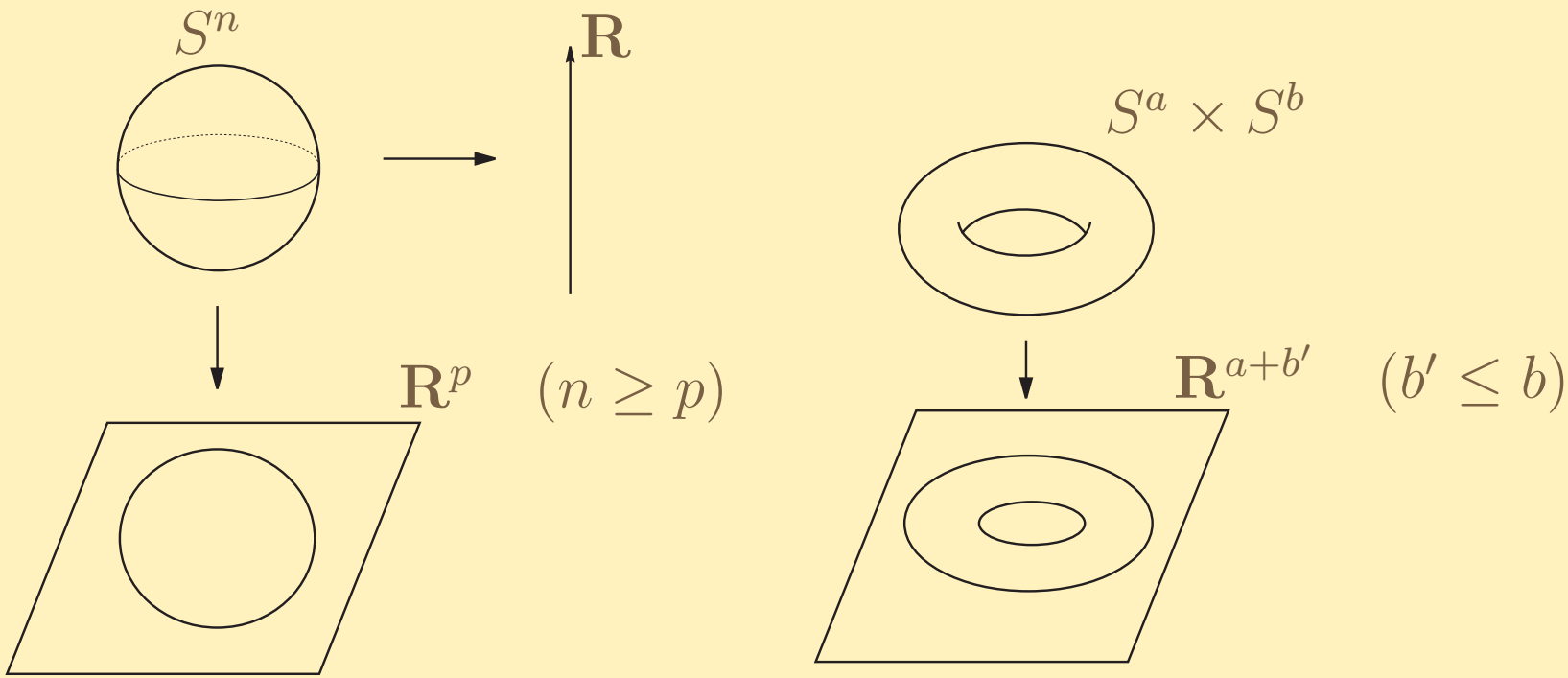

Figure 1: Examples of special generic maps

§2. Lifting Special Generic Functions

Special generic functions

Theorem 2.1 (Reeb, Smale, Cerf et al)
M^n: closed connected n-dim. C^∞ manifold
$\exists$special generic function $M^n \to \mathbf{R}$
$\Longleftrightarrow$
(1) $M^n \approx S^n$ *(homeomorphic)* $(n \neq 4)$
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Generalized Poincaré conjecture is still open in dimension 4 in the C^∞ category.

In the following, M^n will be connected.

Lifting special generic functions

Theorem 2.3 $n \geq 1$
$f : M^n \to \mathbf{R}$ *special generic function*
There always exists an **immersion** $\eta : M^n \to \mathbf{R}^{n+1}$ *s.t.* $f = \pi \circ \eta$.

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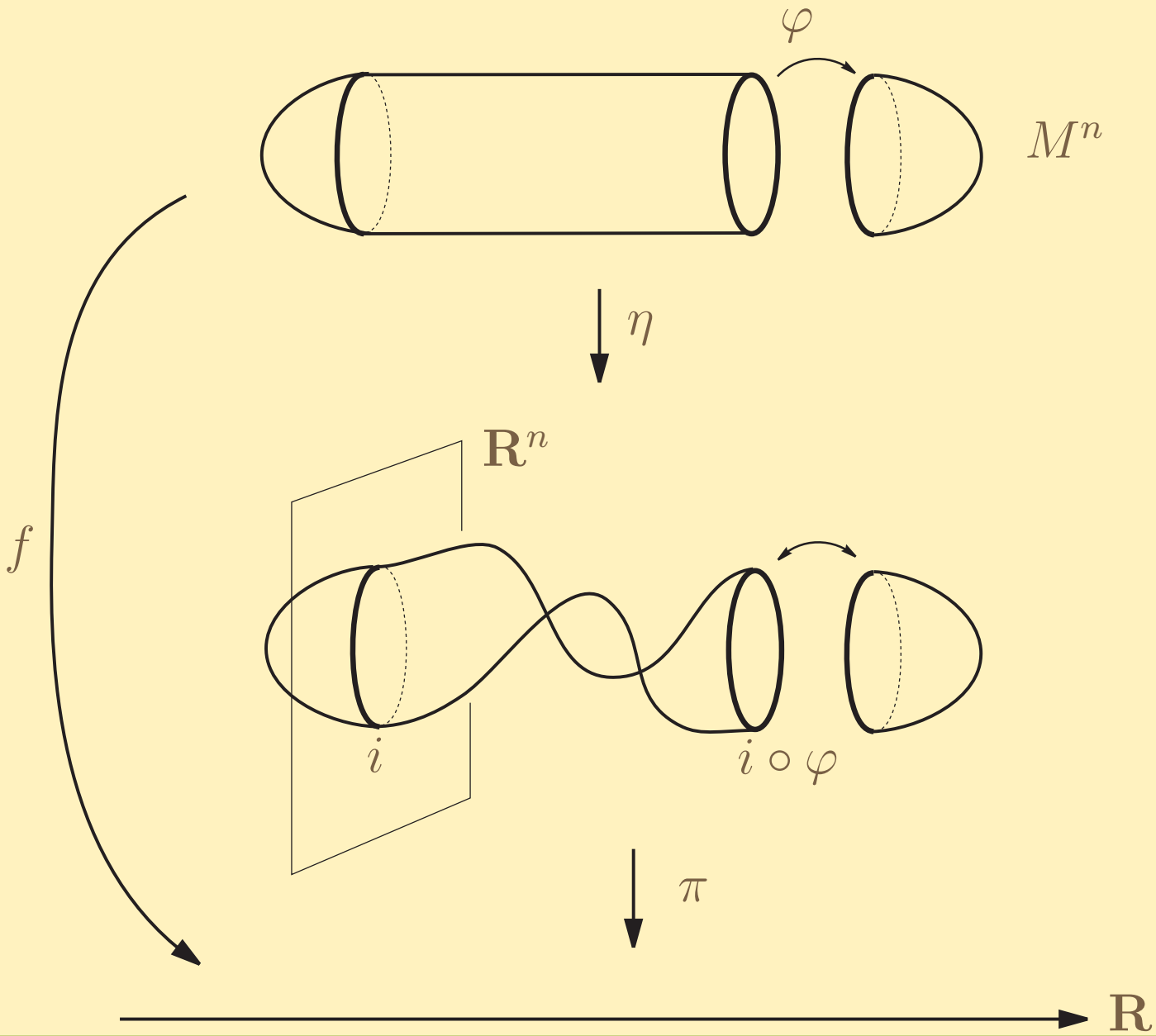
Lemma 2.4 (Kaiser, 1988)
Let $i : S^{n-1} \to \mathbf{R}^n$ be the standard embedding.
For $\forall$diffeomorphism $\varphi : S^{n-1} \to S^{n-1}$ that preserves the orientation, the immersions i and $i \circ \varphi$ are regularly homotopic.

Proof of Theorem 2.3

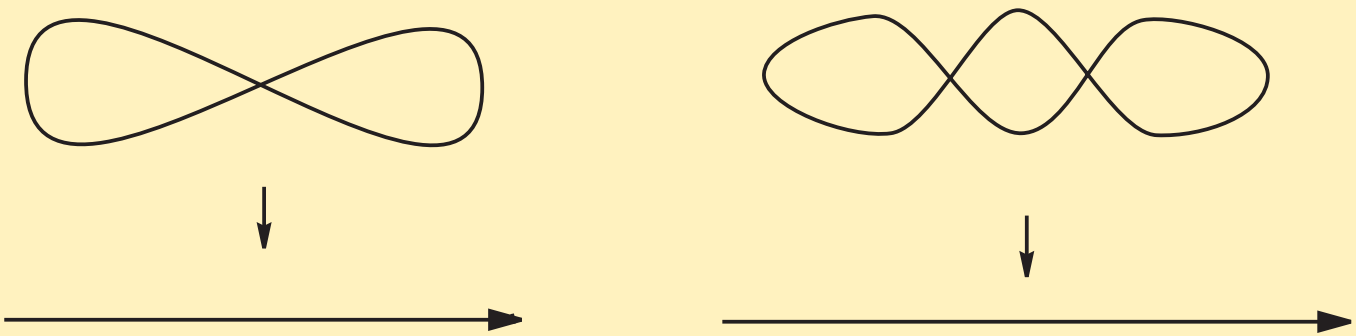


Characterization of immersion lifts

Theorem 2.5 $n \geq 2$, $f : M^n \to \mathbf{R}$ *special generic function*
$\eta_0 : M^n \to \mathbf{R}^{n+1}$ *immersion*
$\exists$*immersion* $\eta : M^n \to \mathbf{R}^{n+1}$ *regularly homotopic to* η_0 *s.t.* $f = \pi \circ \eta$
$\Longleftrightarrow$ *normal degree of* η_0 *is equal to*

$$\begin{cases} \pm 1, & n \neq 3, 7 \\ \pm 1 \text{ or } 0, & n = 3, 7. \end{cases}$$

S^3 or S^7



Embedding lift

Theorem 2.6 $n \geq 2$, $n \neq 5$
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Problem 2.7 *How about* $n = 1$ *or* $n = 5$*?*

§3. Lifting Special Generic Maps into R^2

Theorem 3.1 (Burlet–de Rham, 1974; Porto–Furuya, 1990; S, 1993)

M^n: closed connected **orientable** $(n \geq 2)$
$\exists$special generic map $f : M^n \to \mathbf{R}^2$
$\Longleftrightarrow$ M^n is diffeomorphic to

$$\Sigma^n \sharp \left(\sharp_{i=1}^{r} (\Sigma_i^{n-1} \times S^1) \right)$$

for some homotopy spheres Σ^n and Σ_i^{n-1}
(for $n \leq 6$, they are standard spheres).

Lifting SGM's into R^2

Theorem 3.2 M^n: **orientable**, $2 \leq n \leq 7$ *or* $n = 4m$.
$f : M^n \to \mathbf{R}^2$ *special generic map*
There always exists an **immersion** $\eta : M^n \to \mathbf{R}^{n+1}$ *s.t.* $f = \pi \circ \eta$.

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Θ_n: the group of homotopy n-spheres

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$\Longrightarrow$ A **"canonical" homotopy n-sphere** $\Sigma(f) \in \Theta_n$ can be defined in such a way that

$$M^n \cong \Sigma(f) \sharp \left(\sharp_{i=1}^{r} (\Sigma_i^{n-1} \times S^1) \right) \quad \text{(diffeomorphic)}$$

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On the other hand, we have the homomorphism
$\mathrm{SH} : \Theta_n \to \mathbf{Z}_2$ called the "**Smale–Hirsch map**" (Budney, 2004).

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Theorem 3.4 $f : M^n \to \mathbf{R}^2$ *special generic map*, $n \geq 5$
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That is, does there exist a SGM $f : M^n \to \mathbf{R}^2$ that cannot be lifted to an immersion into $\mathbf{R}^{n+1}$?

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That is, does there exist a SGM $f : M^n \to \mathbf{R}^2$ that cannot be lifted to an immersion into $\mathbf{R}^{n+1}$?
(2) *Is $\Sigma(f)$ determined only by the source manifold M^n ?*

§4. Further Results

Stein factorization

Definition 4.1 $f : M^n \to \mathbf{R}^p$ C^∞ map $(n > p)$
For $x, x' \in M^n$, define $x \sim_f x'$ if
(i) $f(x) = f(x')(= y)$, and
(ii) x and x' belong to the same connected component of $f^{-1}(y)$.

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$W_f = M^n/\sim_f$ quotient space, $q_f : M^n \to W_f$ quotient map

$\exists! \bar{f} : W_f \to \mathbf{R}^p$ that makes the diagram commutative:

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The above diagram is called the **Stein factorization** of f.

Example

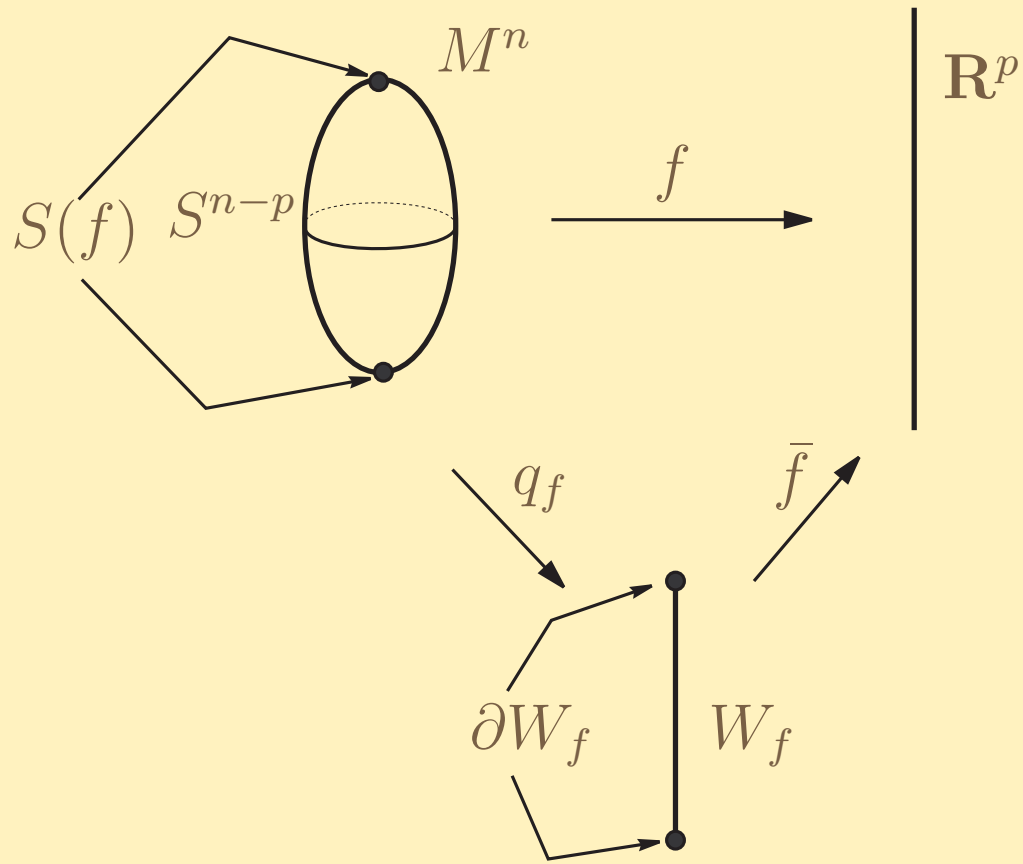

Figure 2: Stein factorization of a SGM

Fundamental properties

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(1) *The singular point set $S(f)$ is a regular submanifold of M^n of dimension $p-1$,*

(2) *W_f has the structure of a smooth p-dim. manifold possibly with boundary such that $\bar{f} : W_f \to \mathbf{R}^p$ is an immersion.*

(3) *$q_f|_{S(f)} : S(f) \to \partial W_f$ is a diffeomorphism.*

(4) *$q_f|_{M^n \setminus S(f)} : M^n \setminus S(f) \to \operatorname{Int} W_f$ is a smooth S^{n-p}-bundle.*

Immersion lift

Theorem 4.3 *M^n: orientable, $(n,p) = (5,3), (6,3), (6,4)$ or $(7,4)$*
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*$\exists$**immersion** $\eta : M^n \to \mathbf{R}^{n+1}$ s.t. $f = \pi \circ \eta$*
$\Longleftrightarrow$ M^n is spin, i.e. $w_2(M^n) = 0$.

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The Stein factorization induces a smooth S^{n-p}-bundle

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If $w_2(M^n) = 0$, then we can show that this is a trivial bundle.

Codimension -1 case

$f : M^n \to \mathbf{R}^p$ special generic map $(n > p)$
Orient $\mathbf{R}^p$. Then the quotient space W_f has the induced orientation.
Then $\partial W_f \cong S(f)$ also have the induced orientations.

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*$\exists$**immersion** $\eta : M^n \to \mathbf{R}^{n+1}$ s.t. $f = \pi \circ \eta$*
$\iff [S(f)] = 0$ in $H_{n-2}(M^n; \mathbf{Z})$.

Key to the proof:
The Stein factorization induces a smooth S^{n-p}-bundle

$$M^n \setminus S(f) \to \operatorname{Int} W_f.$$

Codimension -1 case

$f : M^n \to \mathbf{R}^p$ special generic map $(n > p)$
Orient $\mathbf{R}^p$. Then the quotient space W_f has the induced orientation.
Then $\partial W_f \cong S(f)$ also have the induced orientations.

Theorem 4.4 *M^n: orientable, $f : M^n \to \mathbf{R}^{n-1}$ special generic*
*$\exists$**immersion** $\eta : M^n \to \mathbf{R}^{n+1}$ s.t. $f = \pi \circ \eta$*
$\iff [S(f)] = 0$ in $H_{n-2}(M^n; \mathbf{Z})$.

Key to the proof:
The Stein factorization induces a smooth S^{n-p}-bundle

$$M^n \setminus S(f) \to \operatorname{Int} W_f.$$

If $[S(f)] = 0$, then we can show that this is a trivial bundle.

Embedding results

Theorem 4.5 *M^n: orientable, $f : M^n \to \mathbf{R}^p$ special generic map $(n, p) = (2, 1), (3, 2), (4, 3), (5, 3), (6, 3), (6, 4)$ or $(7, 4)$ $\Longrightarrow \exists$regular homotopy of* **immersions** *$\eta_t : M^n \to \mathbf{R}^{n+1}$, $t \in [0, 1]$, with $f = \pi \circ \eta_0$ s.t. $f_t = \pi \circ \eta_t$ is a special generic map, $t \in [0, 1]$, and η_1 is an* **embedding**.

Embedding results

Theorem 4.5 *M^n: orientable, $f : M^n \to \mathbf{R}^p$ special generic map $(n,p) = (2,1),(3,2),(4,3),(5,3),(6,3),(6,4)$ or $(7,4)$ $\Longrightarrow$ $\exists$regular homotopy of* **immersions** *$\eta_t : M^n \to \mathbf{R}^{n+1}$, $t \in [0,1]$, with $f = \pi \circ \eta_0$ s.t. $f_t = \pi \circ \eta_t$ is a special generic map, $t \in [0,1]$, and η_1 is an* **embedding**.

Theorem 4.6 *M^4: orientable, $\exists f : M^4 \to \mathbf{R}^3$ special generic map M^4 can be embedded into $\mathbf{R}^5$ $\Longleftrightarrow$ M^4 is spin, i.e. $w_2(M^4) = 0$.*

Thank you!