Topology of Manifolds and Global Theory of Singularities

Osamu Saeki (Kyushu University)

One day workshop on
hypersurface singularity and its link manifolds

January 23, 2014

Introduction

This is basically a survey talk.

We will work in the C^∞ category.

Introduction

This is basically a survey talk.

We will work in the C^∞ category.

Contents.

§1. **Special Generic Maps**
Differentiable structures on spheres, $\mathbf{R}^4$, etc.

§2. **Stable Maps**
Cobordism of low dimensional manifolds, Singular fibers, Complexity of maps

§3. **Invariants of Manifolds**
Constructing invariants using stable maps, Problems

§1. Special Generic Maps

Special generic map

Definition 1.1 A singularity of a C^∞ map $M^n \to N^p$, $n \geq p$, that has the normal form

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_{p-1}, \pm x_p^2 \pm x_{p+1}^2 \pm \cdots \pm x_n^2)$$

is called a **fold singularity**.

Special generic map

Definition 1.1 A singularity of a C^∞ map $M^n \to N^p$, $n \geq p$, that has the normal form

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_{p-1}, \pm x_p^2 \pm x_{p+1}^2 \pm \cdots \pm x_n^2)$$

is called a **fold singularity**.

It is a **definite fold singularity** if all the signs are the same.

Special generic map

Definition 1.1 A singularity of a C^∞ map $M^n \to N^p$, $n \geq p$, that has the normal form

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_{p-1}, \pm x_p^2 \pm x_{p+1}^2 \pm \cdots \pm x_n^2)$$

is called a **fold singularity**.

It is a **definite fold singularity** if all the signs are the same.

Definition 1.2 (Burlet–de Rham, 1974; Calabi, 1966)
$f : M^n \to N^p$ is a **special generic map** (**SGM**, for short) if it has **only definite fold singularities**.
This is considered to be a class of maps with mildest singularities.

Special generic map

Definition 1.1 A singularity of a C^∞ map $M^n \to N^p$, $n \geq p$, that has the normal form

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_{p-1}, \pm x_p^2 \pm x_{p+1}^2 \pm \cdots \pm x_n^2)$$

is called a **fold singularity**.

It is a **definite fold singularity** if all the signs are the same.

Definition 1.2 (Burlet–de Rham, 1974; Calabi, 1966)
$f : M^n \to N^p$ is a **special generic map** (**SGM**, for short) if it has **only definite fold singularities**.
This is considered to be a class of maps with mildest singularities.

Example 1.3 A function $f : M^n \to \mathbf{R}$ is a SGM iff it is a Morse function with only critical points of index 0 or n.

Examples of SGMs

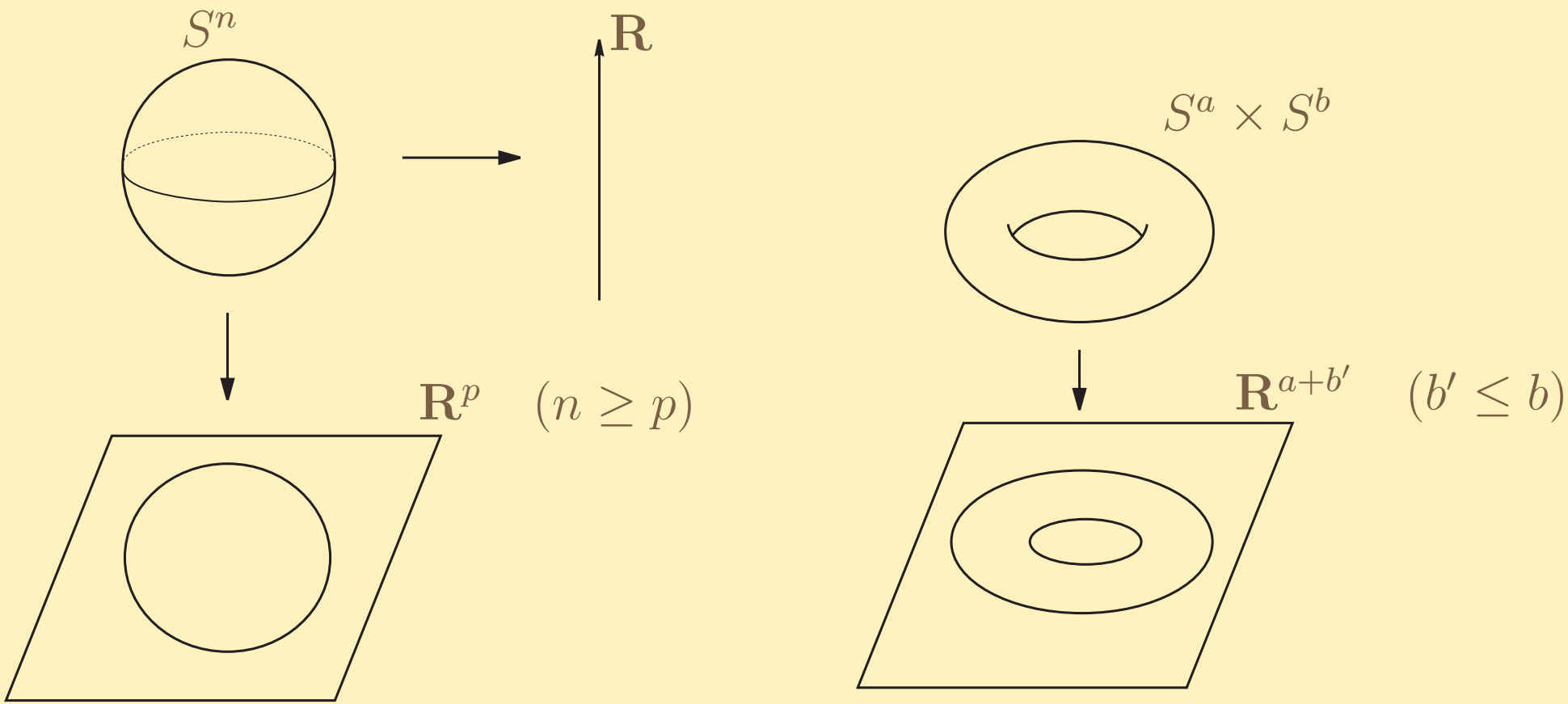


Figure 1: Examples of special generic maps

An invariant

Definition 1.4 M^n: closed n-dimensional C^∞ manifold

$$\mathcal{S}(M^n) = \{p \in \mathbf{Z} \,|\, 1 \leq p \leq n,\ \exists f : M^n \to \mathbf{R}^p \ \ \textbf{SGM}\}$$

An invariant

Definition 1.4 M^n: closed n-dimensional C^∞ manifold

$$\mathcal{S}(M^n) = \{p \in \mathbf{Z} \,|\, 1 \leq p \leq n,\ \exists f : M^n \to \mathbf{R}^p \ \ \textbf{SGM}\}$$

This is a diffeomorphism invariant of M^n.

$$M_0 \cong M_1 \quad \text{(diffeomorphic)} \Longrightarrow \mathcal{S}(M_0) = \mathcal{S}(M_1)$$

An invariant

Definition 1.4 M^n: closed n-dimensional C^∞ manifold

$$\mathcal{S}(M^n) = \{p \in \mathbf{Z} \,|\, 1 \leq p \leq n,\ \exists f : M^n \to \mathbf{R}^p \ \ \textbf{SGM}\}$$

This is a diffeomorphism invariant of M^n.

$$M_0 \cong M_1 \quad \text{(diffeomorphic)} \Longrightarrow \mathcal{S}(M_0) = \mathcal{S}(M_1)$$

Example 1.5

(1) $\mathcal{S}(S^n) = \{1, 2, \ldots, n\}$

An invariant

Definition 1.4 M^n: closed n-dimensional C^∞ manifold

$$\mathcal{S}(M^n) = \{p \in \mathbf{Z} \,|\, 1 \leq p \leq n,\ \exists f : M^n \to \mathbf{R}^p \ \ \textbf{SGM}\}$$

This is a diffeomorphism invariant of M^n.

$$M_0 \cong M_1 \quad \text{(diffeomorphic)} \Longrightarrow \mathcal{S}(M_0) = \mathcal{S}(M_1)$$

Example 1.5

(1) $\mathcal{S}(S^n) = \{1, 2, \ldots, n\}$
(2) $\mathcal{S}(S^a \times S^b) = \{a+1, a+2, \ldots, a+b\} \quad (a \leq b)$

Characterization of the sphere

Theorem 1.6 (Calabi, 1966; S., 1993)
M^n: closed n-dimensional C^∞ manifold

$$\mathcal{S}(M^n) = \{1, 2, \ldots, n\} \iff M^n \cong S^n \quad \text{(diffeomorphic)}$$

Characterization of the sphere

Theorem 1.6 (Calabi, 1966; S., 1993)
M^n: closed n-dimensional C^∞ manifold

$$\mathcal{S}(M^n) = \{1, 2, \ldots, n\} \iff M^n \cong S^n \quad \text{(diffeomorphic)}$$

There exist lots of **exotic spheres**.

SGMs can detect the standard differentiable structure on a sphere!

Characterization of the sphere

Theorem 1.6 (Calabi, 1966; S., 1993)
M^n: closed n-dimensional C^∞ manifold

$$\mathcal{S}(M^n) = \{1, 2, \ldots, n\} \iff M^n \cong S^n \quad \text{(diffeomorphic)}$$

There exist lots of **exotic spheres**.
SGMs can detect the standard differentiable structure on a sphere!

Example 1.7
Σ^7 : Milnor's exotic 7-sphere
$\{1, 2, 7\} \subset \mathcal{S}(\Sigma^7) \subset \{1, 2, 3, 7\}$

Characterization of the sphere

Theorem 1.6 (Calabi, 1966; S., 1993)
M^n: closed n-dimensional C^∞ manifold

$$\mathcal{S}(M^n) = \{1, 2, \ldots, n\} \iff M^n \cong S^n \quad \text{(diffeomorphic)}$$

There exist lots of **exotic spheres**.
SGMs can detect the standard differentiable structure on a sphere!

Example 1.7
Σ^7 : Milnor's exotic 7-sphere
$\{1, 2, 7\} \subset \mathcal{S}(\Sigma^7) \subset \{1, 2, 3, 7\}$

How to prove Theorem 1.6 ?

Definition 1.8 $f : M \to N$ smooth map

For $x, x' \in M$, define $x \sim_f x'$ if

(i) $f(x) = f(x')(= y)$, and

(ii) x and x' belong to the same **connected component** of $f^{-1}(y)$.

Stein factorization

Definition 1.8 $f : M \to N$ smooth map
For $x, x' \in M$, define $x \sim_f x'$ if
(i) $f(x) = f(x')(= y)$, and
(ii) x and x' belong to the same **connected component** of $f^{-1}(y)$.

$W_f = M/\sim_f$ **quotient space**
$q_f : M \to W_f$ quotient map

Stein factorization

Definition 1.8 $f : M \to N$ smooth map

For $x, x' \in M$, define $x \sim_f x'$ if

(i) $f(x) = f(x')(= y)$, and

(ii) x and x' belong to the same **connected component** of $f^{-1}(y)$.

$W_f = M/\sim_f$ **quotient space**

$q_f : M \to W_f$ quotient map

$\exists! \bar{f} : W_f \to N$ that makes the following diagram commutative:

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "q_f"'] & & N \\ & W_f \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Stein factorization

Definition 1.8 $f : M \to N$ smooth map
For $x, x' \in M$, define $x \sim_f x'$ if
(i) $f(x) = f(x')(= y)$, and
(ii) x and x' belong to the same **connected component** of $f^{-1}(y)$.

$W_f = M/\sim_f$ **quotient space**
$q_f : M \to W_f$ quotient map

$\exists! \bar{f} : W_f \to N$ that makes the following diagram commutative:

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "q_f"'] & & N \\ & W_f \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

The above diagram is called the **Stein factorization** of f.

Example

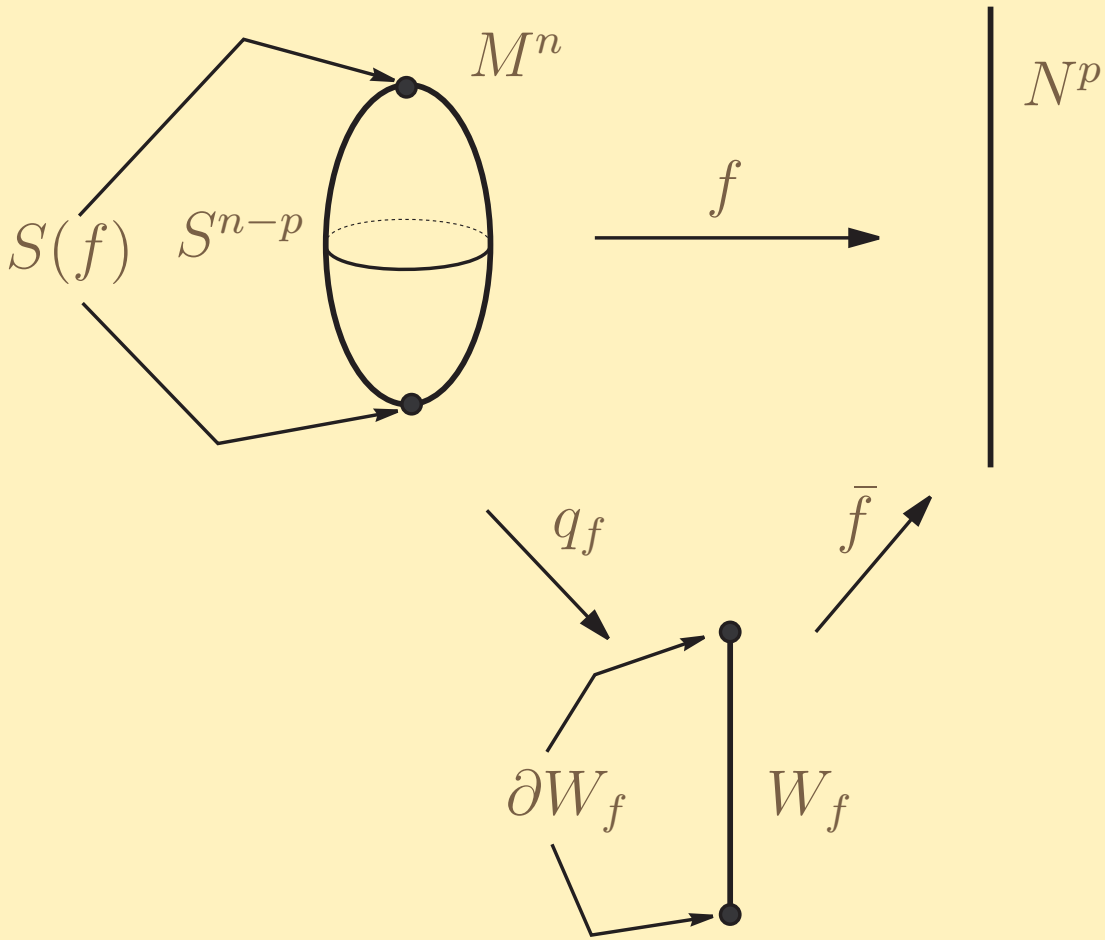


Figure 2: Stein factorization of a SGM

Disk bundle theorem

If f is a special generic map, then W_f has the structure of a smooth p-dimensional manifold possibly with boundary.

Disk bundle theorem

If f is a special generic map, then W_f has the structure of a smooth p-dimensional manifold possibly with boundary.

Theorem 1.9 (S., 1993)
$f : M^n \to N^p$ *proper special generic map with* $n - p = 1, 2, 3$ *s.t.* $S(f) \neq \emptyset$
$\Longrightarrow$
M^n *is diffeomorphic to the boundary of a* D^{n-p+1}**-bundle** *over* W_f.

Disk bundle theorem

If f is a special generic map, then W_f has the structure of a smooth p-dimensional manifold possibly with boundary.

Theorem 1.9 (S., 1993)
$f : M^n \to N^p$ *proper special generic map with* $n - p = 1, 2, 3$ *s.t.* $S(f) \neq \emptyset$
$\Longrightarrow$
M^n *is diffeomorphic to the boundary of a* D^{n-p+1}**-bundle** *over* W_f.

Proof of Theorem 1.6:
$1 \in \mathcal{S}(M^n) \Longrightarrow M^n$ is a homotopy sphere (Reeb).
$n - 1 \in \mathcal{S}(M^n) \Rightarrow \exists f : M^n \to \mathbf{R}^{n-1}$ SGM $\Rightarrow W_f$ is contractible
& M^n is the boundary of a D^2-bundle over $W_f \Rightarrow M^n \cong S^n$. $\square$

Disk bundle theorem

If f is a special generic map, then W_f has the structure of a smooth p-dimensional manifold possibly with boundary.

Theorem 1.9 (S., 1993)
$f : M^n \to N^p$ *proper special generic map with* $n - p = 1, 2, 3$ *s.t.* $S(f) \neq \emptyset$
$\Longrightarrow$
M^n *is diffeomorphic to the boundary of a* D^{n-p+1}**-bundle** *over* W_f.

Proof of Theorem 1.6:
$1 \in \mathcal{S}(M^n) \Longrightarrow M^n$ is a homotopy sphere (Reeb).
$n-1 \in \mathcal{S}(M^n) \Rightarrow \exists f : M^n \to \mathbf{R}^{n-1}$ SGM $\Rightarrow W_f$ is contractible & M^n is the boundary of a D^2-bundle over $W_f \Rightarrow M^n \cong S^n$. $\square$

Observation: W_f is the core (or spine) of a "good manifold" whose boundary is the given manifold M^n.

A characterization of R^4

It is known that the differentiable structure on $\mathbf{R}^n$, $n \neq 4$, is **unique**.

A characterization of R^4

It is known that the differentiable structure on $\mathbf{R}^n$, $n \neq 4$, is **unique**.

However, for $n = 4$, it has been known that $\mathbf{R}^4$ admits more than one differentiable structures (Freedman, Donaldson, Kirby, $\sim$1982).
In fact, there exist uncountably many **exotic $\mathbf{R}^4$'s** (Taubes, 1987).

A characterization of R^4

It is known that the differentiable structure on $\mathbf{R}^n$, $n \neq 4$, is **unique**.

However, for $n = 4$, it has been known that $\mathbf{R}^4$ admits more than one differentiable structures (Freedman, Donaldson, Kirby, $\sim$1982).
In fact, there exist uncountably many **exotic R^4's** (Taubes, 1987).

Theorem 1.10 $M^4 \approx \mathbf{R}^4$ *(homeomorphic)*
$\exists f : M^4 \to \mathbf{R}^p$ *proper SGM for* $1 \leq \exists p \leq 3$
$\Longleftrightarrow M^4 \cong \mathbf{R}^4$ *(diffeomorphic)*

SGMs on complex surfaces

We can also prove the following, using the "Stein factorization techniques".

Theorem 1.11 (Sakuma–S., 1999)
Let M^4 be the underlying smooth manifold of a compact complex analytic surface.
Then, $\exists f : M^4 \to \mathbf{R}^3$ SGM
$\Longleftrightarrow$ M^4 is a ruled surface or a Hopf surface diffeomorphic to $S^1 \times S^3$.

§2. Stable Maps

Stable maps

Let $f : M^n \to N^{n-1}$ be a C^∞ stable map of codimension -1.
In this case, **regular fibers** are disjoint unions of S^1.
$\Longrightarrow$ Each of their component bounds D^2.

Stable maps

Let $f : M^n \to N^{n-1}$ be a C^∞ stable map of codimension -1.
In this case, **regular fibers** are disjoint unions of S^1.
$\Longrightarrow$ Each of their component bounds D^2.

It would be nice to have a "disk bundle" over the quotient space W_f whose boundary coincides with M^n.
$\Longleftarrow$ Observation from the study of SGMs

Stable maps

Let $f : M^n \to N^{n-1}$ be a C^∞ stable map of codimension -1.
In this case, **regular fibers** are disjoint unions of S^1.
$\Longrightarrow$ Each of their component bounds D^2.

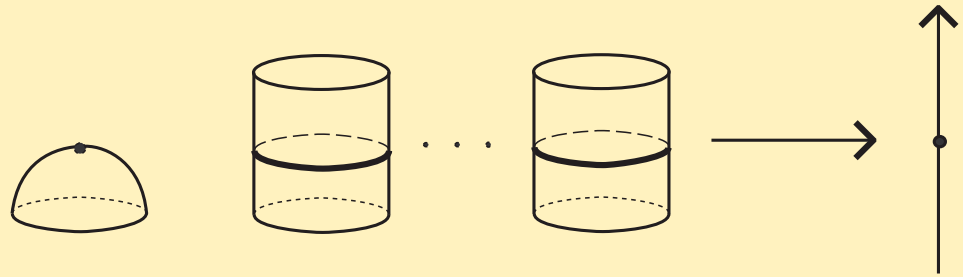
It would be nice to have a "disk bundle" over the quotient space W_f whose boundary coincides with M^n.
$\Longleftarrow$ Observation from the study of SGMs

Obstructions to constructing such a "D^2-bundle" over W_f are concentrated around the **singular fibers**.

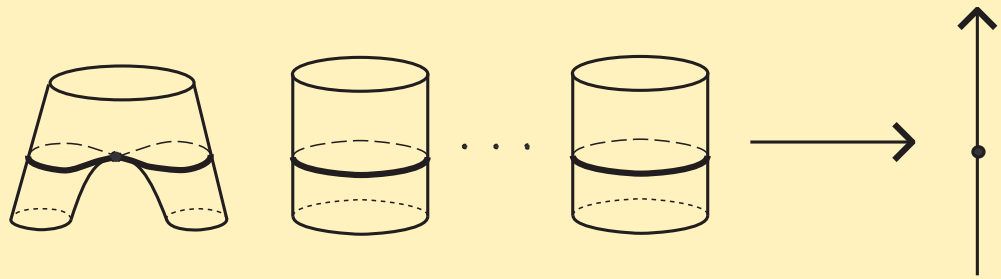
Surface case

Let M^2 be a closed oriented surface, and $f : M^2 \to \mathbf{R}$ a Morse function. In this case, singular fibers are classified as follows:

(1)

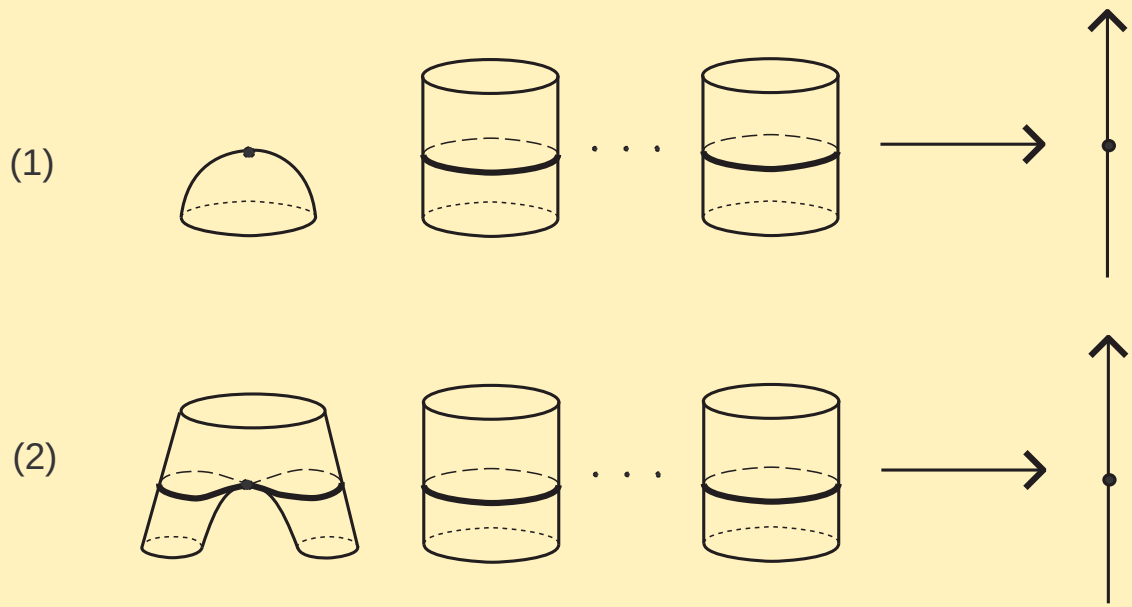


(2)



Surface case

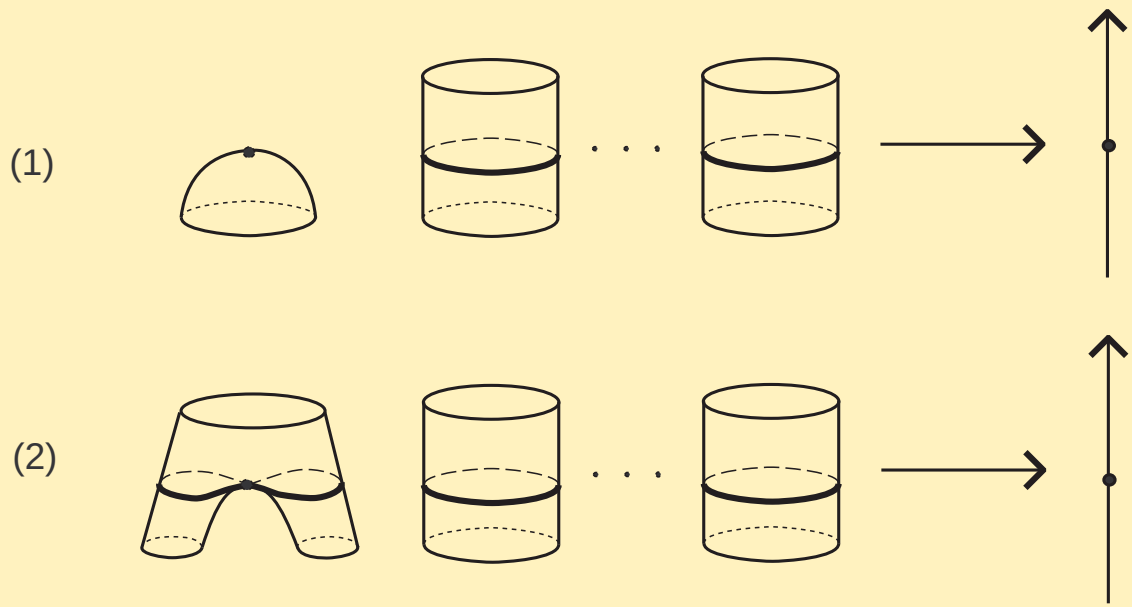
Let M^2 be a closed oriented surface, and $f : M^2 \to \mathbf{R}$ a Morse function. In this case, singular fibers are classified as follows:



There are **no obstructions** to filling in the singular fiber neighborhoods. (The leftmost surfaces in (1) and (2) bound D^3.)

Surface case

Let M^2 be a closed oriented surface, and $f : M^2 \to \mathbf{R}$ a Morse function. In this case, singular fibers are classified as follows:



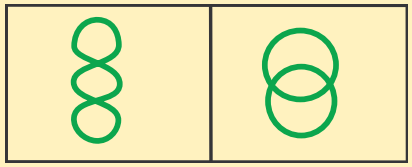
There are **no obstructions** to filling in the singular fiber neighborhoods. (The leftmost surfaces in (1) and (2) bound D^3.)

Corollary 2.1 *Every closed oriented surface is null-cobordant.*

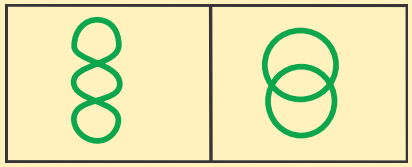
Let M^3 be a closed oriented 3-manifold, and $f : M^3 \to \mathbf{R}^2$ a stable map.

3- and 4-dimensional cases

Let M^3 be a closed oriented 3-manifold, and $f : M^3 \to \mathbf{R}^2$ a stable map. The possible obstructions lie near the following singular fibers.



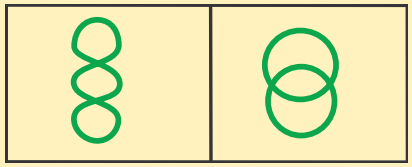
Let M^3 be a closed oriented 3-manifold, and $f : M^3 \to \mathbf{R}^2$ a stable map. The possible obstructions lie near the following singular fibers.



Costantino–Thurston, 2008:
There are **no obstructions** to filling in the neighborhoods of the above singular fibers of f.

3- and 4-dimensional cases

Let M^3 be a closed oriented 3-manifold, and $f : M^3 \to \mathbf{R}^2$ a stable map. The possible obstructions lie near the following singular fibers.

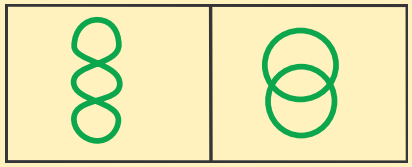


Costantino–Thurston, 2008:
There are **no obstructions** to filling in the neighborhoods of the above singular fibers of f.

Using this idea, they show that every closed oriented 3-manifold **efficiently** bounds a 4-manifold.

3- and 4-dimensional cases

Let M^3 be a closed oriented 3-manifold, and $f : M^3 \to \mathbf{R}^2$ a stable map. The possible obstructions lie near the following singular fibers.



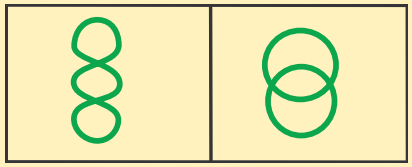
Costantino–Thurston, 2008:
There are **no obstructions** to filling in the neighborhoods of the above singular fibers of f.

Using this idea, they show that every closed oriented 3-manifold **efficiently** bounds a 4-manifold.

How about the 4-dimensional case?

3- and 4-dimensional cases

Let M^3 be a closed oriented 3-manifold, and $f : M^3 \to \mathbf{R}^2$ a stable map. The possible obstructions lie near the following singular fibers.



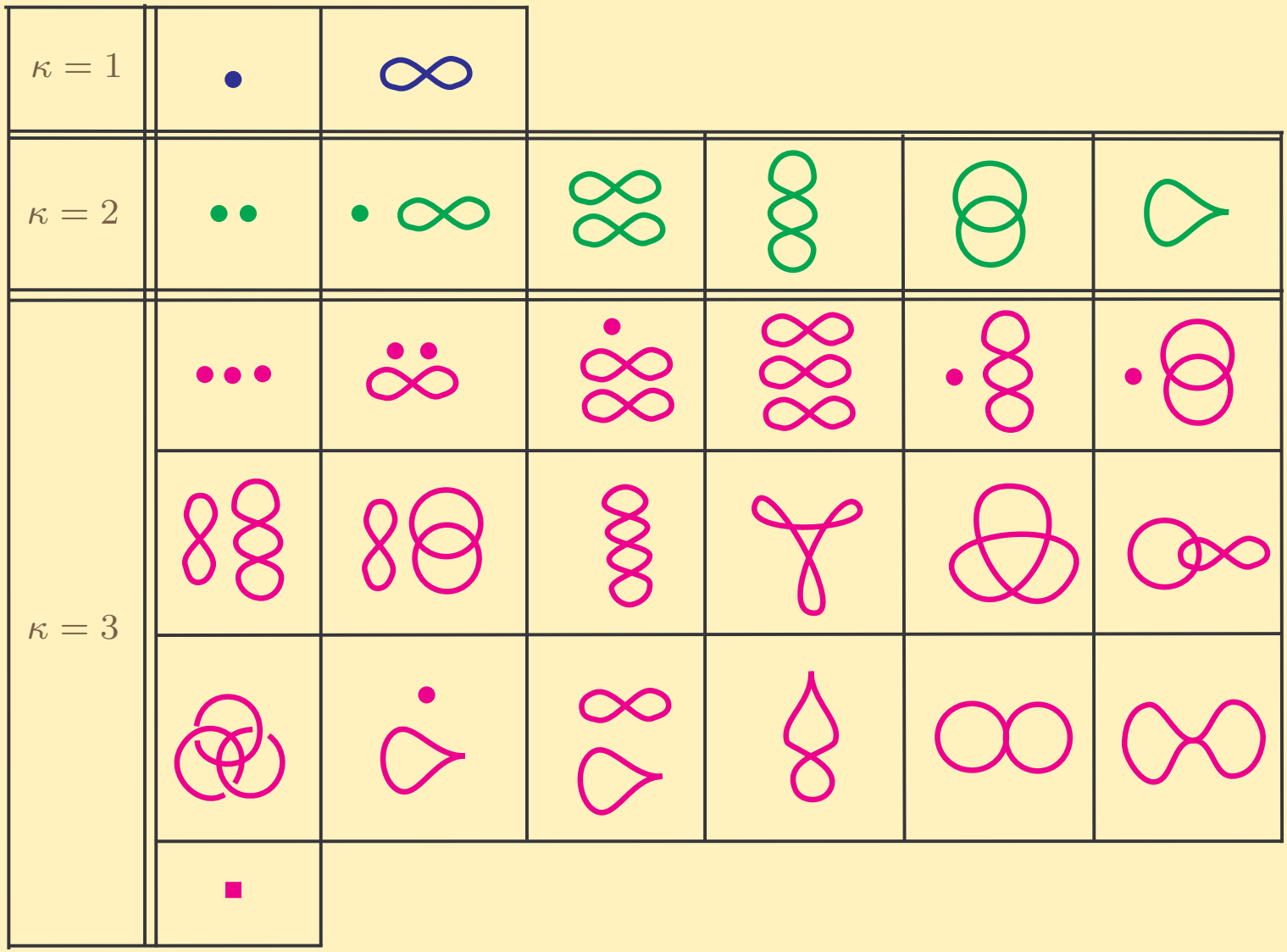
Costantino–Thurston, 2008:
There are **no obstructions** to filling in the neighborhoods of the above singular fibers of f.

Using this idea, they show that every closed oriented 3-manifold **efficiently** bounds a 4-manifold.

How about the 4-dimensional case?

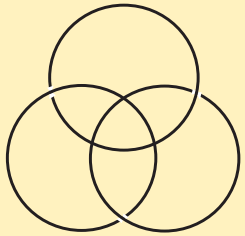
There do exist obstructions!

Singular fibers for dimension 4



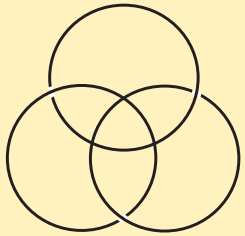
Obstructions

Theorem 2.2 (S., 2010) *There are* <u>*no obstructions*</u> *to filling in the singular fiber neighborhoods,* **except for the III^8-type below**. *Furthermore, around each singular fiber of type III^8, $\mathbf{C}P^2$ appears.*



Obstructions

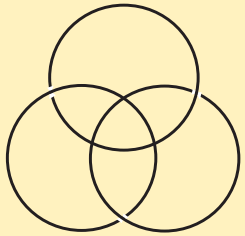
Theorem 2.2 (S., 2010) *There are* <u>*no obstructions*</u> *to filling in the singular fiber neighborhoods,* **except for the III^8-type below**. *Furthermore, around each singular fiber of type III^8, $\mathbf{C}P^2$ appears.*



Corollary 2.3 (Rohlin, 1952) *The oriented 4-dimensional cobordism group is infinite cyclic generated by the class of $\mathbf{C}P^2$.*

Obstructions

Theorem 2.2 (S., 2010) *There are no obstructions to filling in the singular fiber neighborhoods,* **except for the III^8-type below**. *Furthermore, around each singular fiber of type III^8, $\mathbf{C}P^2$ appears.*



Corollary 2.3 (Rohlin, 1952) *The oriented 4-dimensional cobordism group is infinite cyclic generated by the class of $\mathbf{C}P^2$.*

Corollary 2.4 (T. Yamamoto–S., 2006)
For a stable map $f : M^4 \to \mathbf{R}^3$, the **signature** *of M^4 is equal to the number of III^8-type singular fibers counted with signs.*

Complexity of stable maps, 1

Recall that for a stable map $f : M^4 \to \mathbf{R}^3$, its set of singular points $S(f)$ is a smooth closed surface embedded in M^4.

Furthermore, $f|_{S(f)} : S(f) \to \mathbf{R}^3$ is an immersion (with cusps and swallowtails).

Complexity of stable maps, 1

Recall that for a stable map $f : M^4 \to \mathbf{R}^3$, its set of singular points $S(f)$ is a smooth closed surface embedded in M^4.

Furthermore, $f|_{S(f)} : S(f) \to \mathbf{R}^3$ is an immersion (with cusps and swallowtails).

Corollary 2.5 *Let $f : M^4 \to \mathbf{R}^3$ be a stable map as above. Then, $f|_{S(f)}$ has at least $|\sigma(M^4)|$* **triple points**, *where $\sigma(M^4)$ stands for the signature of M^4.*

Complexity of stable maps, 1

Recall that for a stable map $f : M^4 \to \mathbf{R}^3$, its set of singular points $S(f)$ is a smooth closed surface embedded in M^4.

Furthermore, $f|_{S(f)} : S(f) \to \mathbf{R}^3$ is an immersion (with cusps and swallowtails).

Corollary 2.5 *Let $f : M^4 \to \mathbf{R}^3$ be a stable map as above. Then, $f|_{S(f)}$ has at least $|\sigma(M^4)|$* **triple points**, *where $\sigma(M^4)$ stands for the signature of M^4.*

The complexity of a stable map reflects the topology of M^4.

Complexity of stable maps, 1

Recall that for a stable map $f : M^4 \to \mathbf{R}^3$, its set of singular points $S(f)$ is a smooth closed surface embedded in M^4.

Furthermore, $f|_{S(f)} : S(f) \to \mathbf{R}^3$ is an immersion (with cusps and swallowtails).

Corollary 2.5 *Let $f : M^4 \to \mathbf{R}^3$ be a stable map as above. Then, $f|_{S(f)}$ has at least $|\sigma(M^4)|$* **triple points**, *where $\sigma(M^4)$ stands for the signature of M^4.*

The complexity of a stable map reflects the topology of M^4.

We also have the following related results.

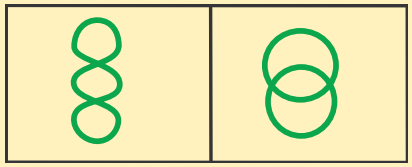
Complexity of stable maps, 2

Theorem 2.6 (Gromov, 2009) *$f : M^n \to \mathbf{R}^2$ stable map $\Longrightarrow \operatorname{rank} H_*(M^n) \leq 2N_2 + N_{\mathrm{cusp}} + 2N_{\mathrm{comp}}$, where N_2 is the number of double points of the plane curve $f|_{S(f)} : S(f) \to \mathbf{R}^2$, N_{cusp} is the number of cusps, and N_{comp} is the number of components of $S(f)$.*

Complexity of stable maps, 2

Theorem 2.6 (Gromov, 2009) *$f : M^n \to \mathbf{R}^2$ stable map $\Longrightarrow \operatorname{rank} H_*(M^n) \leq 2N_2 + N_{\mathrm{cusp}} + 2N_{\mathrm{comp}}$, where N_2 is the number of double points of the plane curve $f|_{S(f)} : S(f) \to \mathbf{R}^2$, N_{cusp} is the number of cusps, and N_{comp} is the number of components of $S(f)$.*

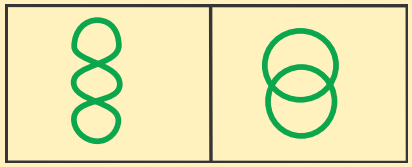
Theorem 2.7 (Costantino–Thurston, Gromov) *$f : M^3 \to \mathbf{R}^2$ stable $\Longrightarrow ||M^3||_\Delta \leq 10N_{\mathrm{sf}} \leq 10N_2$, where $||M^3||_\Delta$ is the simplicial volume, N_{sf} is the number of singular fibers as below, and N_2 is the number of double points of $f|_{S(f)}$.*



Complexity of stable maps, 2

Theorem 2.6 (Gromov, 2009) *$f : M^n \to \mathbf{R}^2$ stable map $\Longrightarrow \operatorname{rank} H_*(M^n) \leq 2N_2 + N_{\mathrm{cusp}} + 2N_{\mathrm{comp}}$, where N_2 is the number of double points of the plane curve $f|_{S(f)} : S(f) \to \mathbf{R}^2$, N_{cusp} is the number of cusps, and N_{comp} is the number of components of $S(f)$.*

Theorem 2.7 (Costantino–Thurston, Gromov) *$f : M^3 \to \mathbf{R}^2$ stable $\Longrightarrow ||M^3||_\Delta \leq 10N_{\mathrm{sf}} \leq 10N_2$, where $||M^3||_\Delta$ is the simplicial volume, N_{sf} is the number of singular fibers as below, and N_2 is the number of double points of $f|_{S(f)}$.*



Note: $||M^3||_\Delta \leq$ minimal number of 3-simplices of any triangulation, $||M^3||_\Delta = 0$ for graph manifolds M^3

§3. Invariants of Manifolds

Invariants

M^3: closed oriented connected 3-manifold

$\exists f : M^3 \to \mathbf{R}$ **Morse function**

Invariants

M^3: closed oriented connected 3-manifold
$\exists f : M^3 \to \mathbf{R}$ **Morse function**

Such a Morse function is **not unique**, but every pair of such functions can be connected by a "generic path" in the space of smooth functions.

Invariants

M^3: closed oriented connected 3-manifold
$\exists f : M^3 \to \mathbf{R}$ **Morse function**

Such a Morse function is **not unique**, but every pair of such functions can be connected by a "generic path" in the space of smooth functions.
$\Longrightarrow$ Kirby Calculus

Invariants

M^3: closed oriented connected 3-manifold
$\exists f : M^3 \to \mathbf{R}$ **Morse function**

Such a Morse function is **not unique**, but every pair of such functions can be connected by a "generic path" in the space of smooth functions.
$\Longrightarrow$ Kirby Calculus
$\Longrightarrow$ Lots and lots of **topological invariants** have been constructed and are still under investigation.

Invariants

M^3: closed oriented connected 3-manifold
$\exists f : M^3 \to \mathbf{R}$ **Morse function**

Such a Morse function is **not unique**, but every pair of such functions can be connected by a "generic path" in the space of smooth functions.
$\Longrightarrow$ Kirby Calculus
$\Longrightarrow$ Lots and lots of **topological invariants** have been constructed and are still under investigation.

How about using **stable maps** $M^3 \to \mathbf{R}^2$?

Reshetikhin–Turaev invariant

Reshetikhin–Turaev (1990) defined a quantum invariant for 3-manifolds, just after Witten's celebrated proposal to use (yet mathematically un-justified) path-integral in order to define invariants of 3-manifolds (associated to each Lie algebra).

Reshetikhin–Turaev invariant

Reshetikhin–Turaev (1990) defined a quantum invariant for 3-manifolds, just after Witten's celebrated proposal to use (yet mathematically un-justified) path-integral in order to define invariants of 3-manifolds (associated to each Lie algebra).

This can, in fact, be interpreted as an invariant derived from stable maps $M^3 \to \mathrm{R}^2$.

Stein factorization set

M^3: closed connected orientable 3-manifold
$f : M^3 \to \mathbf{R}^2$ stable map

Stein factorization set

M^3: closed connected orientable 3-manifold
$f : M^3 \to \mathbf{R}^2$ stable map
$\Longrightarrow W_f$ is a (certain) 2-dimensional polyhedron.

Stein factorization set

M^3: closed connected orientable 3-manifold
$f : M^3 \to \mathbf{R}^2$ stable map
$\Longrightarrow W_f$ is a (certain) 2-dimensional polyhedron.

Define $\mathcal{W}(M^3) = \{W_f \mid f : M^3 \to \mathbf{R}^2 \text{ stable}\}/\text{homeo}$.

Stein factorization set

M^3: closed connected orientable 3-manifold
$f : M^3 \to \mathbf{R}^2$ stable map
$\Longrightarrow W_f$ is a (certain) 2-dimensional polyhedron.

Define $\mathcal{W}(M^3) = \{W_f \mid f : M^3 \to \mathbf{R}^2 \text{ stable}\}/\text{homeo}$.

Theorem 3.1 (Motta–Porto–S., 1995)
For any finite set $M_1^3, M_2^3, \ldots, M_k^3$ of 3-manifolds, we have

$$\bigcap_{i=1}^{k} \mathcal{W}(M_i^3) \neq \emptyset,$$

but

$$\bigcap_{M^3} \mathcal{W}(M^3) = \emptyset.$$

A problem

Problem: $\mathcal{W}(M_0^3) = \mathcal{W}(M_1^3) \Longrightarrow M_0^3 \cong M_1^3$?

A problem

Problem: $\mathcal{W}(M_0^3) = \mathcal{W}(M_1^3) \Longrightarrow M_0^3 \cong M_1^3$?

Theorem 3.2 (Burlet–de Rham, 1974)
Yes, if $M_0^3 = S^3$ *or* $\sharp^k(S^1 \times S^2)$.

A problem

Problem: $\mathcal{W}(M_0^3) = \mathcal{W}(M_1^3) \Longrightarrow M_0^3 \cong M_1^3$?

Theorem 3.2 (Burlet–de Rham, 1974)
Yes, if $M_0^3 = S^3$ *or* $\sharp^k(S^1 \times S^2)$.

The answer to the above problem would be "YES", if for each 3-manifold M^3, we can construct a stable map $f : M^3 \to \mathbf{R}^2$ such that the regular fibers form a "trivial framed link" in M^3.

Problem. How about constructing invariants for **higher dimensional** manifolds?

Broken Lefschetz fibrations

Problem. How about constructing invariants for **higher dimensional** manifolds?

A possibility for dimension 4
Broken Lefschetz fibrations (Auroux–Donaldson–Katzarkov, 2005)
$=$ certain smooth maps $f : M^4 \to \Sigma^2$, with Σ^2 a closed oriented surface.

Broken Lefschetz fibrations

Problem. How about constructing invariants for **higher dimensional** manifolds?

A possibility for dimension 4
Broken Lefschetz fibrations (Auroux–Donaldson–Katzarkov, 2005)
$=$ certain smooth maps $f : M^4 \to \Sigma^2$, with Σ^2 a closed oriented surface.

Such a map is allowed to have unstable singularities (Lefschetz critical points), but can basically be treated in the category of stable maps (with a bit more effort).

Broken Lefschetz fibrations

Problem. How about constructing invariants for **higher dimensional** manifolds?

A possibility for dimension 4
Broken Lefschetz fibrations (Auroux–Donaldson–Katzarkov, 2005)
$=$ certain smooth maps $f : M^4 \to \Sigma^2$, with Σ^2 a closed oriented surface.

Such a map is allowed to have unstable singularities (Lefschetz critical points), but can basically be treated in the category of stable maps (with a bit more effort).

To such a map are associated **near-symplectic structures**.
$\Longrightarrow$ Gauge theoretic invariants can be defined...

Broken Lefschetz fibrations

Problem. How about constructing invariants for **higher dimensional** manifolds?

A possibility for dimension 4
Broken Lefschetz fibrations (Auroux–Donaldson–Katzarkov, 2005)
$=$ certain smooth maps $f : M^4 \to \Sigma^2$, with Σ^2 a closed oriented surface.

Such a map is allowed to have unstable singularities (Lefschetz critical points), but can basically be treated in the category of stable maps (with a bit more effort).

To such a map are associated **near-symplectic structures**.
$\Longrightarrow$ Gauge theoretic invariants can be defined...

Two such maps are connected by a generic path.
$\Longleftarrow$ "Moves" can be described (Porto–Furuya, 1986; Baykur–S., 2012)

Broken Lefschetz fibrations

Problem. How about constructing invariants for **higher dimensional** manifolds?

A possibility for dimension 4
Broken Lefschetz fibrations (Auroux–Donaldson–Katzarkov, 2005)
$=$ certain smooth maps $f : M^4 \to \Sigma^2$, with Σ^2 a closed oriented surface.

Such a map is allowed to have unstable singularities (Lefschetz critical points), but can basically be treated in the category of stable maps (with a bit more effort).

To such a map are associated **near-symplectic structures**.
$\Longrightarrow$ Gauge theoretic invariants can be defined...

Two such maps are connected by a generic path.
$\Longleftarrow$ "Moves" can be described (Porto–Furuya, 1986; Baykur–S., 2012)
Still under investigation!

Thank you!