chapter 1 Singular Homology Theory

- $x, y \in \mathbb{R}^n$

  segment from x to y $= \{(1-t)x + ty \mid 0 \le t \le 1\}$

  $C \subseteq \mathbb{R}^n$: convex

  if $\forall x, y \in C$, segment from x to y $\subset C$

  $A \subset \mathbb{R}^n$

  convex hull of A

  $\Longleftrightarrow$ $\cap$ all convex sets in $\mathbb{R}^n$ containing A

- p-simplex s in $\mathbb{R}^n$

  $\Longleftrightarrow$ convex hull of a coll. of $(p+1)$ ~~pts~~ $\{x_0, \dots, x_p\}$ in $\mathbb{R}^n$ s.t. $x_1 - x_0, \dots, x_p - x_0$ form a lin. indep. set.

  Note : this is indep. of designation of which pt is x_0.

prop. 1.1

$\{x_0, \dots, x_p\} \subseteq \mathbb{R}^n$. TFAE

(a) $x_1 - x_0, \dots, x_p - x_0$: lin. indep.

(b) $\sum s_i x_i = \sum t_i x_i$ and $\sum s_i = \sum t_i$

$\Rightarrow$ $s_i = t_i$ for $i = 0, \dots, p$.

proof

(a) $\Rightarrow$ (b) If $\sum s_i x_i = \sum t_i x_i$ & $\sum t_i = \sum s_i$,

$$0 = \sum_{i=0}^{p} (s_i - t_i) x_i = \sum_{i=0}^{p} (s_i - t_i) x_i - \left[\sum_{i=0}^{p} (s_i - t_i)\right] x_0$$

$$= \sum_{i=1}^{p} (s_i - t_i)(x_i - x_0)$$

lin. indep. $\Rightarrow$ $s_i = t_i$, $i = 1, \dots, p$.

$s_0 = t_0$

(b) $\Rightarrow$ (a) If $\sum_{i=1}^{p} t_i(x_i - x_0) = 0$,

$$(0 \cdot x_0) + \sum_{i=1}^{p} t_i x_i = \left(\sum_{i=1}^{p} t_i\right) x_0 + \sum 0 \cdot x_i$$

By (b), $t_1, \cdots, t_n = 0$. ☐

- s: s-simplex in $\mathbb{R}^n$

Set of all pts of the form $t_0 x_0 + t_1 x_1 + \cdots + t_p x_p$, $\sum t_i = 1$, $t_i \geq 0$ $\forall i$

Prop 1.2

p-simplex s : convex hull of $\{x_0, \cdots, x_p\}$ $\Rightarrow$ $\forall$ pt of s has a distinct unique repr. in the form $\sum t_i x_i$, $t_i \geq 0$ $\forall i$ & $\sum t_i = 1$

- s : ordered simplex if vertices of s have been given a specific order

s ; ordered sx with $x_0, \cdots, x_p$.

Define $\sigma_p = \{(t_0, t_1, \cdots, t_p) \in \mathbb{R}^{p+1} \mid \sum t_i = 1, t_i \geq 0 \ \forall i\}$

$f : \sigma_p \rightarrow s$ given by

$$f(t_0, \cdots, t_p) = \sum t_i x_i$$

$\Rightarrow$ f ; homeomorphism

Note : σ_p : p-sx with vertices $x_0' = (1, 0 \cdots, 0)$

$x_1' = (0, 1, \cdots, 0), \cdots, x_p' = (0, \cdots 0, 1)$

σ_p is called the standard p-sx with natural ordering

- X: sp.

  A <u>singular p-sx</u> in X

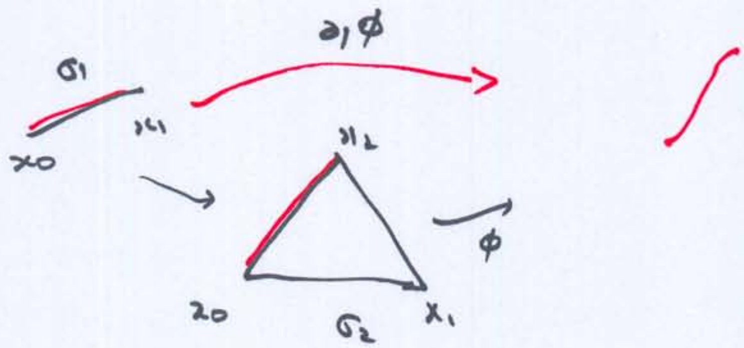
  $\Longleftrightarrow$ continuous fun. $\phi: \sigma_p \to X$

- ϕ: singular p-sx, i: integer with $0 \le i \le p$.

  define $\partial_i(\phi)$: <u>singular</u> $(p-1)$-sx in X, by

  $$\partial_i \phi(t_0, \cdots, t_{p-1}) = \phi(t_0, \cdots t_{i-1}, 0, t_i, \cdots, t_{p-1})$$

  $\partial_i \phi$: <u>i-th face</u> of ϕ



- $f: X \to Y$ continuous

  1. ϕ: singular p-sx in X

     define singular p-sx $f_\#(\phi)$ in Y by

     $$f_\#(\phi) = f \circ \phi$$

  2. $g: Y \to W$ continuous, $id: X \to X$ identity

     $\Rightarrow (g \circ f)_\# = g_\#(f_\#(\phi))$ & $(id)_\#(\phi) = \phi$.

- Abelian group is <u>free</u> if $\exists\ A \leq G$ s.t. $\forall\ g \in G$ has a unique repre.

$$g = \sum_{x \in A} n_x \cdot x$$

, where n_x : integer & equal to 0 for all but finitely many x in A.

A .. <u>basis</u> for G

$F(A) = \{ f : A \to \mathbb{Z} \mid f(x) \neq 0$ for only a finite numbers of elts of A.

Define an oper. on $F(A)$ by

$$(f+g)(x) = f(x) + g(x)$$

$\forall\ a \in A$, define $f_a \in F(A)$ by

$$f_a(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{otherwise} \end{cases}$$

$\rightsquigarrow \{ f_a \mid a \in A \}$; <u>basis for $F(A)$.</u>

<u>Note</u>

G : free abelian with basis A

H : ab. gp.

$\rightsquigarrow$ $\forall\ f : A \to H$ can be uniquely extended to a hom. $f : G \to H$.

- X : top. sp.

$S_n(X)$; free ab. → basis is the set of all singular n-sxes of X.

An elt of $S_n(X)$ is called a singular n-chain of X & has the form

$$\sum_{\phi} n_\phi \cdot \phi$$

, where n_ϕ : integer, equal to 0 for all but finite number of ϕ.

- There is unique extension to a hom.

$$\partial_i : S_n(X) \longrightarrow S_{n-1}(X) \quad \text{by}$$

$$\partial_i\left(\sum n_\phi \cdot \phi\right) = \sum n_\phi \cdot (\partial_i \phi)$$

Define the boundary operator by

$$\partial : S_n(X) \longrightarrow S_{n-1}(X)$$

by $\partial = \sum_{i=0}^{n} (-1)^i \partial_i$

prop. 1.3

The composition $\partial \circ \partial$ in

$$S_n(X) \xrightarrow{\partial} S_{n-1}(X) \xrightarrow{\partial} S_{n-2}(X)$$

is zero

- $c \in S_n(X)$: <u>n-cycle</u> if $\partial(c) = 0$

  $d \in S_n(X)$: <u>n-boundary</u> if $d = \partial(e)$ for some $e \in S_{n+1}(X)$.

  $Z_n(X) = \ker \partial$, $\partial : S_n(X) \to S_{n-1}(X)$.

  $B_n(X) = \operatorname{Im} \partial$, $\partial : S_{n+1}(X) \to S_n(X)$

<u>Note</u> $B_n(X) \leq Z_n(X)$ subgroup.

$$H_n(X) = Z_n(X) / B_n(X)$$

; n-th <u>singular</u> <u>homology</u> <u>group</u> of X

- <u>graded</u> (abelian) group G ; coll. of ab. groups
  - $\{G_i\}$ indexed by integers with componentwise oper.

  G, G': graded

  - hom $f : G \to G'$; <u>coll</u>. of $\{f_i\}$

    , where $f_i : G_i \to G'_{i+r}$ for some fixed r

    r is called the <u>degree of f</u>

  - $H \leq G$ subgp $\iff$ graded gp $\{H_i\}$

    , where H_i : subgp of G_i

  G/H .. graded gp $\{G_i/H_i\}$.

- 1. <u>Chain complex</u> : seq. of ab. gps & hom

$$\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}}$$

s.t. $\underline{\partial_{n-1} \circ \partial_n = 0}$.

$\Leftrightarrow$

2. $C = \{C_i\}$ together with hom $\partial : C \to C$ of degree -1 s.t. $\partial \circ \partial = 0$

3. C & C' : chain cxes with ∂, ∂'

$\leadsto$ <u>chain map</u> from C to C' is a hom. $\Phi : C \to C'$ of deg. 0 s.t.

$$\partial' \circ \Phi_n = \Phi_{n-1} \circ \partial \quad \forall n$$

4. Denote by $Z_*(C)$ & $B_*(C)$: kernel & image of ∂

The <u>homology of C</u> is the graded gp

$$H_*(C) = Z_*(C)/B_*(C)$$

<u>Note</u> : if Φ is a <u>chain map</u>,

$$\Phi(Z_*(C)) \subseteq Z_*(C'), \quad \Phi(B_n(C)) \subset B_n(C')$$

$\therefore$ Φ induces a <u>homomorphism</u> on homology gps

$$\Phi_* : H_*(C) \to H_*(C').$$

- 1. graded gp $S_*(X) = \{S_i(X)\}$ becomes a chain cx under ∂ s.t. <u>homology gp</u> of X is the homology of this <u>chain</u>.

2. $f: X \to Y$ continuous, ϕ: singular n-sx in X

$\leadsto$ $f_\#(\phi) = f \circ \phi$: " in Y

This extends uniquely to

$$f_\#: S_n(X) \to S_n(Y) \qquad \forall n.$$

3. $f_\#$: chain map

$$\begin{array}{ccc} S_n(X) & \xrightarrow{f_\#} & S_n(Y) \\ \downarrow \partial & & \downarrow \partial \\ S_{n-1}(X) & \xrightarrow{f_\#} & S_{n-1}(Y) \end{array}$$

4 $f_\#$ induces a hom. of deg zero

$$f_*: H_*(X) \to H_*(Y)$$

Note : for $g: Y \to W$ conti, $id: X \to X$,

$(g \circ f)_* = g_* \circ f_*$, id_* = identity.

Ex

$X = pt$

- $\forall p \geq 0$, $\exists!$ singular p-sx $\phi_p = \sigma_p \to X$

For $p > 0$, $\partial_i \phi_p = \phi_{p-1}$

$$\cdots \to S_2(pt) \to S_1(pt) \to S_0(pt) \to 0$$

$$\partial \phi_n = \sum_{i=0}^{n} (-1)^i \partial_i \phi_n = \sum_{i=0}^{n} (-1)^i \phi_{n-1}$$

$\therefore$ $\partial \phi_{2n-1} = 0$

$\partial \phi_{2n} = \phi_{2n-1}$

$Z_n(pt) = B_n(pt)$ for $n > 0$

$Z_0(pt) = S_0(pt) \cong \mathbb{Z}$, $B_0(pt) = 0$

$$H_n(pt) = \begin{cases} \mathbb{Z} & \text{if } n=0 \\ 0 & \text{if } n\neq 0 \end{cases}$$

- X = path-conn. $\iff$ $\forall x, y \in X$, $\exists$ conti. fun.

  $\varphi : [0,1] \to X$ s.t. $\varphi(0) = x$, $\varphi(1) = y$.

$$\begin{array}{c} S_1(X) \xrightarrow{\partial} S_0(X) \longrightarrow 0 \\ \qquad\qquad \parallel \\ \qquad\qquad Z_0(X) \end{array}$$

$\Rightarrow$ free ab. gp gen. by pts of X.

$y \in Z_0(X)$ has the form

$$y = \sum_{x \in X} n_x \cdot x$$

$\Rightarrow$ n_x : integer, all but finitely many equal to 0.

- 1. $S_1(X)$: free ab. gp gen. by the set of all paths in X.

vertices of σ_1 : v_0, v_1 ϕ : singular 1-sx

$$\longmapsto \quad \partial\phi = \phi(v_1) - \phi(v_0) \in Z_0(X)$$

Define $\alpha : S_0(X) \to \mathbb{Z}$ by $\alpha(\sum n_x \cdot x) = \sum n_x$.

$\Rightarrow$ <u>epimorphism</u>

$$B_0(X) \subset \ker\alpha.$$

2. Conversely, sp'se $n_1 x_1 + \cdots + n_q x_q \in Z_0(X)$ with $\sum n_i = 0$.

pick $x \in X$, & $\forall i$, 1-sx $\phi_i : \sigma_1 \to X$ with

$$\partial_0(\phi_i) = x_i, \quad \partial_1(\phi_i) = x_0$$

Taking singular 1-chain $\sum n_i \phi_i \in S_1(X)$,

$$\partial(\sum n_i \phi_i) = \sum n_i x_i - (\sum n_i) x = \sum n_i x_i$$

$\therefore$ $\ker \alpha \subset B_0(X)$.

Prop 1.8

X: non-empty path-conn. sp.

$\Rightarrow$ $H_0(X) \cong \mathbb{Z}$.

- A : set, $\alpha \in A$ G_α : ab. gp.

1. Define ab. gp $\sum_{\alpha \in A} G_\alpha$:

   elts ; $f: A \to \cup G_\alpha$ s.t.

   $f(\alpha) \in G_\alpha$ $\forall \alpha$, $f(\alpha) = 0$ for all but finitely many elts $\alpha \in A$; $(f+g)(\alpha) = f(\alpha) + g(\alpha)$.

2. Set $g_\alpha = f(\alpha) \in G_\alpha$,

   $$f = \{ g_\alpha \mid \alpha \in A \}$$

3. $\sum G_\alpha$: weak direct sum of G_α's.

4. If ($f(\alpha) = 0$ for all but finitely many α) is omitted, the resulting gp = strong direct sum or direct product $\prod_{\alpha \in A} G_\alpha$.

Note : G: ab. gp. $\{G_\alpha\}_{\alpha \in A}$ subgps of G s.t. $g \in G$ has a unique repr.

$$g = \sum_\alpha g_\alpha \quad \text{with} \quad g_\alpha \in G_\alpha.$$

& $g_\alpha = 0$ for all but finitely many α $\Rightarrow$ $G \cong \sum_\alpha G_\alpha$

- $\forall \alpha \in A$, chain cx C^α

$$\cdots \longrightarrow C_p^\alpha \longrightarrow C_{p-1}^\alpha \to \cdots$$

Define a chain cx $\sum_{\alpha \in A} C^\alpha$ by

$$(\Sigma C^\alpha)_p = \Sigma C_p^\alpha$$

&

$$\partial (c_\alpha : \alpha \in A) = (\partial^\alpha c_\alpha : \alpha \in A)$$

Lemma 1.5

$$\boxed{H_k(\Sigma C^\alpha) = \sum_\alpha H_k(C^\alpha)}$$

proof

- $Z_k(\Sigma C^\alpha) = \Sigma(Z_k C^\alpha)$

$B_k(\Sigma C^\alpha) = \Sigma(B_k C^\alpha)$

$$\therefore \quad H_k(\Sigma C^\alpha) = Z_k(\Sigma C^\alpha) / B_k(\Sigma C^\alpha)$$

$$= \Sigma(Z_k(C^\alpha)) / \Sigma B_k(C^\alpha)$$

$$\simeq \Sigma (Z_k(C^\alpha)/B_k(C^\alpha))$$

$$= \Sigma H_k(C^\alpha)$$

$\boxtimes$

- X : top. sp.

$x \sim y \iff \exists$ path in X from x to y.

$\sim$: equivalence relation

$\sim_P$ path components.

prop. 1.6

X: sp, $\{X_\alpha \mid \alpha \in A\}$: path components of X

$\Rightarrow \quad H_*(X) = \sum_{\alpha \in A} H_*(X_\alpha)$

- homological properties of a sp.
  ; completely det. by those of its path components.

  $\therefore$ restrict our attention to the study of path-conn. spaces.

  Note $H_0(X)$; free ab. gp whose basis is in 1-1 corr. with the path components of X.

Thm 1.7

$f: X \to Y$ homeomorphism

$\leadsto \quad f_* : H_p(X) \to H_p(Y)$ is an isomorphism $\forall p$.

Thm 1.8

X: convex subset of $\mathbb{R}^n$.

$\Rightarrow \quad H_p(X) = 0 \quad$ for $p > 0$.

proof

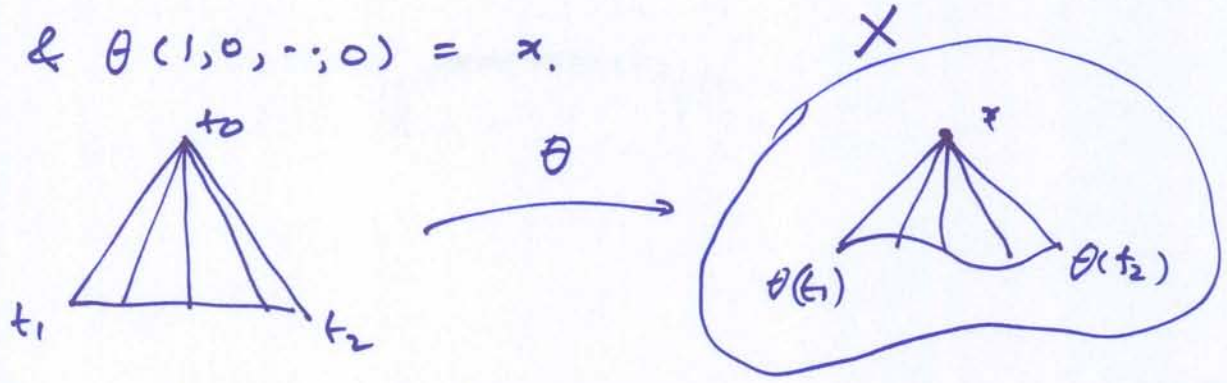
- $x \in X$, $\phi: \sigma_p \to X$ singular p-sx, $p \geq 0$.

  Define a singular $(p+1)$-sx $\theta: \sigma_{p+1} \to X$:

$$\theta(t_0, \cdots, t_{p+1}) = \begin{cases} (1-t_0)\cdot\left(\phi\left(\frac{t_1}{1-t_0}, \cdots, \frac{t_{p+1}}{1-t_0}\right)\right) + t_0 x, & t_0 < 1 \\ x, & t_0 = 1 \end{cases}$$

$\therefore\ \theta(0, t_1, \cdots, t_{p+1}) = \phi(t_1, \cdots, t_{p+1})$

$\&\ \theta(1, 0, \cdots, 0) = x.$



line segment from t_0 to face opposite to t_0

; linearly into corr. line segment in X.

this is possible since X is convex.

- θ : continuous except possibly at $(1, 0, \cdots, 0)$

To check continuity,

Show ;
$$\lim_{t_0 \to 1} \| \theta(t_0, \cdots, t_{p+1}) - x \| = 0$$

Now,

$$\begin{aligned}
&\lim_{t_0 \to 1} \| \theta(t_0, \cdots, t_{p+1}) - x \| \\
&= \lim_{t_0 \to 1} \| (1 - t_0) \left(\phi\left(\frac{t_1}{1-t_0}, \cdots, \frac{t_{p+1}}{1-t_0} \right) \right) - (1-t_0) x \| \\
&\le \lim_{t_0 \to 1} (1 - t_0) \left(\left\| \phi\left(\frac{t_1}{1-t_0}, \cdots, \frac{t_{p+1}}{1-t_0} \right) \right\| + \|x\| \right)
\end{aligned}$$

$\phi(\sigma_p)$: cpt. $\Rightarrow \left(\left\| \phi\left(\frac{t_1}{1-t_0}, \cdots, \frac{t_{p+1}}{1-t_0} \right) \right\| + \|x\| \right)$

is bdd.

$\therefore\ \theta$ is continuous.

- $\partial_0(\theta) = \phi$.

Since this may be applied to any singular k-sx, $k \geq 0$, $\exists!$ extension to a hom.

$T: S_k(X) \to S_{k+1}(X)$ s.t.

$\partial_0 \circ T =$ identity.

More generally, for ϕ, a singular k-sx

$d_i(T(\phi))(t_0, \dots, t_k)$

$= T(\phi)(t_0, \dots, t_{i-1}, 0, t_i, \dots, t_k)$

$= (1-t_0)\left(\phi\left(\frac{t_1}{1-t_0}, \dots, \frac{t_{i-1}}{1-t_0}, 0, \frac{t_i}{1-t_0}, \dots \frac{t_k}{1-t_0}\right)\right)$

$+ t_0 x$

$T(d_{i-1}(\phi))(t_0, \dots, t_k)$

$= (1-t_0)\left(d_{i-1}\phi\left(\frac{t_1}{1-t_0}, \dots, \frac{t_k}{1-t_0}\right)\right) + t_0 x)$

$= (1-t_0)\cdot\phi\left(\frac{t_1}{1-t_0}, \dots, \frac{t_{i-1}}{1-t_0}, 0, \frac{t_i}{1-t_0}, \dots, \frac{t_k}{1-t_0}\right) + t_0 x$

$\therefore$ For $1 \leq i \leq k+1$, $d_i T\phi = T(d_{i-1}\phi)$

- ϕ: any singular k-sx.

$\partial T\phi = d_0 T\phi + \sum_{i=1}^{k+1} (-1)^i d_i T(\phi)$

$= d_0 T\phi + \sum_{i=1}^{k+1} (-1)^i d_i T(\phi) - \left[\sum_{i=1}^{k+1} (-1)^i T d_{i-1}(\phi)\right.$

$\left. + \sum_{j=0}^{k} (-1)^j T d_j \phi\right]$

$= \phi - T\partial\phi$

$\therefore$ we've constructed a hom. $T : S_k(X) \to S_{k+1}(X)$

s.t. $\partial T + T\partial$: id. on $S_k(X)$, $k \geq 1$.

- $z \in Z_p(X)$

$\Rightarrow$ for $p > 0$, $(\partial T + T\partial)(z) = z$

z: cycle $\Rightarrow$ $\partial z = 0$

$\therefore$ $z = \partial(Tz) \in B_p(X)$

$\therefore$ $H_p(X) = 0$ $\forall p > 0$.

$\boxed{}$

- $C = \{C_i, \partial\}$, $C' = \{C_i', \partial'\}$: chain cxes

$T : C \to C'$: hom. of graded gps of deg 1.

1. Consider hom $\partial' T + T\partial : C \to C'$ of deg 0.

$\Rightarrow$ chain map

$(\because$ $\partial'(\partial' T + T\partial) = \partial'\partial' T + \partial' T\partial = \partial' T\partial$

$= \partial' T\partial + T\partial\partial = (\partial' T + T\partial)\partial$ $)$

2. $(\partial' T + T\partial)$ induces a hom of homology

$(\partial' T + T\partial)_* : H_p(C) \to H_p(C')$ $\forall p$

3. $z \in Z_p(C)$

$\rightsquigarrow$ $(\partial' T + T\partial)(z) = \partial' T(z) \in B_p(C')$

$\therefore$ $(\partial' T + T\partial)_*$: zero hom $\forall p$.

- chain maps $f, g: C \to C'$,

  f & g : <u>chain homotopic</u>

  $\Leftrightarrow \exists$ hom. $T: C \to C'$ of deg 1 with

  $$\partial' T + T\partial = f - g.$$

<u>prop 1.9</u>

$$\boxed{\begin{array}{l} f, g : C \to C' \quad \text{chain homotopic chain maps} \\ \Rightarrow f_* = g_* \quad \text{as homs from } H_*(C) \text{ to } H_*(C'). \end{array}}$$

<u>proof</u>

- $T: C \to C'$; <u>chain homotopy</u> bet. f & g

  $\leadsto \quad 0 = (\partial' T + T\partial)_* = (f - g)_* = f_* - g_*$ ▨

- $f, g: X \to Y$ maps $\Rightarrow$ induced chain maps

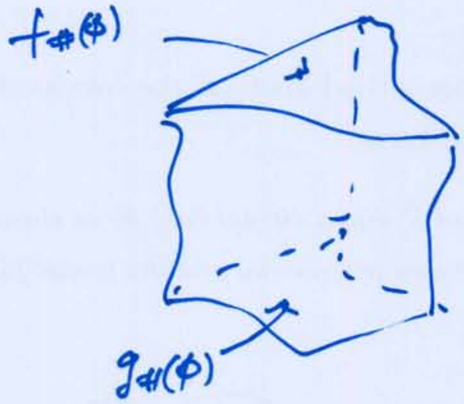
  $f_\#$ & $g_\#$: $S_*(X) \to S_*(Y)$ are <u>chain homotopic</u>.

  T: chain homotopy bet. $f_\#$ and $g_\#$

  $\hookrightarrow$ T : interpreted <u>geometrically</u> ;

- ϕ: singular n-sx in X

$T(\phi)$: continuous deformation of $f_\#(\phi)$ into $g_\#(\phi)$



$$\underline{\partial T(\phi) = f_\#(\phi) - g_\#(\phi) - T(\partial\phi)}$$

- $c = \sum m_i \phi_i$: n-cycle in X

$\rightsquigarrow$ $f_\#(c)$ & $g_\#(c)$: n-cycles in Y.

$$\Rightarrow \quad \partial T(c) = f_\#(c) - g_\#(c) - T(\underset{\overset{\shortparallel}{0}}{\partial c})$$

$\therefore$ $f_\#(c)$ & $g_\#(c)$: homologous cycles in Y.

- X, Y : sps

$f_0, f_1 : X \to Y$ are <u>homotopic</u> if $\exists\ F : X \times I \to Y$

with $F(x,0) = f_0(x)$, $F(x,1) = f_1(x)$

F ; homotopy bet. f_0 & f_1

$[X, Y]$: set of homotopy classes of maps

Thm 1.10

$$\boxed{\begin{array}{l} f_0, f_1 : X \to Y \quad \text{homotopic maps} \\ \Rightarrow f_{0*} = f_{1*} \;:\; H_*(X) \to H_*(Y) \end{array}}$$

proof

- It is sufficient to show ; chain maps $f_{0\#}, f_{1\#} : S_*(X) \to S_*(Y)$ are chain homotopic.

  $F : X \times I \to Y$ homotopy bet. f_0 & f_1

  Define $g_0, g_1 : X \to X \times I$ by $g_0(x) = (x, 0)$
  $g_1(x) = (x, 1)$

  $\rightsquigarrow$ $f_0 = F \circ g_0$, $f_1 = F \circ g_1$

- Spse $g_{0\#}$ & $g_{1\#}$ are chain homotopic.

  $\Rightarrow \exists$ hom $T : S_*(X) \to S_*(X \times I)$ of deg 1

  with $\partial T + T\partial = g_{0\#} - g_{1\#}$

  $\Rightarrow$ $F_\#(\partial T + T\partial) = F_\#(g_{0\#} - g_{1\#})$

  or $\partial(\underline{F_\# T}) + (\underline{F_\# T})\partial = f_{0\#} - f_{1\#}$

  $\underline{F_\# T}$: chain homotopic

  $\therefore$ It is suff. to show ; $g_{0\#}$ & $g_{1\#}$ are chain homotopic.

- σ_n : standard n-sx

  $\tau_n \in S_n(\sigma_n)$: elt repr. by id.

  If $\phi : \sigma_n \to X$ is any n-sx in X,

  $$\phi_\# : S_n(\sigma_n) \to S_n(X)$$
  $$\phi_\#(\tau_n) = \phi .$$

- Construct a chain homotopy T bet. $g_{0\#}$ & $g_{1\#}$ inductively on dim. of chain gp.

Sp'se $n>0$ and $\forall$ sp. X, $\forall\, i<n$, there is a hom. $T: S_i(X) \to S_{i+1}(X\times I)$ s.t.

$$\partial T + T\partial = g_{0\#} - g_{1\#}.$$

Assume further that this is <u>natural</u>:

$\forall\, h: X \to W$,

$$\begin{tikzcd} S_i(X) \arrow[r] \arrow[d, "h_\#"'] & S_{i+1}(X\times I) \arrow[d, "(h\times id)_\#"] \\ S_i(W) \arrow[r] & S_{i+1}(W\times I) \end{tikzcd} \quad \forall\, i<n.$$

- To define on <u>n-chains</u> on X,

it's sufficient to define T on singular n-sxes.

$\phi: \sigma_n \to X$ singular n-sx, $\phi_\#(\tau_n) = \phi$.

$\therefore$ By defining $T\sigma_n: S_n(\sigma_n) \to S_{n+1}(\sigma_n\times I)$, naturality of construction $\rightsquigarrow$

$$T_X(\phi) = T_X(\phi_\#(\tau_n)) = (\phi\times id)_\#(T_{\sigma_n}(\tau_n))$$

$\therefore$ To define T_X, it's sufficient to define $T\sigma_n$ on $S_n(\sigma_n)$.

- d: singular n-sx in σ_n & consider

$$c = g_{0\#}(d) - g_{1\#}(d) - T\sigma_n(\partial d), \quad \partial d \in S_{n-1}(\sigma_n)$$

$\rightsquigarrow$

$$\begin{aligned} \partial c &= \partial g_{0\#}(d) - \partial g_{1\#}(d) - \partial T\sigma_n(\partial d) \\ &= g_{0\#}(\partial d) - g_{1\#}(\partial d) - [g_{0\#}(\partial d) - g_{1\#}(\partial d) - T\sigma_n \partial\partial d] \\ &= 0 \end{aligned}$$

$\therefore$ c ; cycle of dim n in $\underline{\sigma_n \times I}$.
convex

Thm 1.8 $\leadsto$ c is also bdry.

Let $b \in S_{n+1}(\sigma_n \times I)$ with $\partial b = c$.

$\leadsto$ define $T\sigma_n(d) = b$ &

$$\partial T(d) + T\partial(d) = g_{0\#}(d) - g_{1\#}(d)$$

- $\forall$ singular n-sx $\phi : \sigma_n \to X$, define

$$T_X(\phi) = (\phi \times id)_\# T\sigma_n(\tau_n).$$

$\Downarrow$

$\exists!$ extension to a hom.

$$T_X : S_n(X) \to S_{n+1}(X \times I)$$

This inductive construction indicates the proper definition for T on 0-chains.

Recall : σ_0 is a pt & consider c in $S_0(\sigma_0 \times I)$

by $c = g_{0\#}(\tau_0) - g_{1\#}(\tau_0)$

Take a singular 1-sx b in $\sigma_0 \times I$ with bdry $g_{0\#}(\tau_0) - g_{1\#}(\tau_1)$ and define $T\sigma_0(\tau_0) = b$.

This defines T on 0-chains.

- If ϕ is a singular n-sx in X,

$$g_{0\#}(\phi) = g_{0\#}\phi_\#(\tau_n) = (\phi \times id)_\# g_{0\#}(\tau_n)$$

$$g_{1\#}(\phi) = g_{1\#}\phi_\#(\tau_n) = (\phi \times id)_\# g_{1\#}(\tau_n)$$

Consider

$$\begin{aligned}
\partial T(\phi) + T\partial(\phi) &= \partial T \phi_{\#}(\tau_n) + T\partial \phi_{\#}(\tau_n) \\
&= \partial (\phi \times id)_{\#} T(\tau_n) + T\phi_{\#} \partial \tau_n \\
&= (\phi \times id)_{\#} \partial T(\tau_n) + (\phi \times id)_{\#} T\partial(\tau_n) \\
&= (\phi \times id)_{\#} (g_{0\#}(\tau_n) - g_{1\#}(\tau_n)) \\
&= g_{0\#}(\phi) - g_{1\#}(\phi).
\end{aligned}$$

The naturality follows similarly.

$\therefore$ T_X $\rightsquigarrow$ chain homotopy bet $g_{0\#}$ & $g_{1\#}$

$\therefore$ $f_{0\#} = f_{1\#}$ $\boxed{\square}$

- $f: X \to Y$, $g: Y \to X$: maps

  $f \circ g$ & $g \circ f$; homotopic to id. resp.

  $\rightsquigarrow$ f & g are homotopy inverses of each other.

  $f: X \to Y$ is a <u>homotopy equivalence</u> if f has a homotopy inverse.

  (X and Y are said to have the same homotopy type)

<u>prop 1.11</u>

$f: X \to Y$ is a homotopy equiv.

$\rightsquigarrow$ $f_*: H_n(X) \to H_n(Y)$ is an isomorphism $\forall n$.

- $i : A \to X$ inclusion map of a subsp. A of X.

1. $g : X \to A$ s.t. $g \circ i$ is the id. on A

   ... retraction of X onto A.

2. If furthermore, $i \circ g : X \to X$ is homotopic to id., g is a <u>deformation retraction</u> and A is a <u>deformation</u> <u>retract</u> of X.

<u>Note</u>: in this case the inclusion i is a <u>homotopy equiv.</u>

<u>Cor 1.12</u>

1. $i : A \to X$: inclusion of a retract A of X

   $\Rightarrow$ $i_* : H_*(A) \to H_*(X)$ is a <u>monomorphism</u> onto a direct summand.

2. If A is deformation retract of X, then i_* is an <u>isomorphism</u>

pf

- $g : X \to A$; retraction.

  $\rightsquigarrow$ $g_* \circ i_* = (g \circ i)_* = (\mathrm{id})_* = \mathrm{id}$.

  $\therefore$ i_* : <u>monomorphism</u>

Define

- $G_1 = \operatorname{im} i_*$, $G_2 = \ker g_*$.

  $\alpha \in G_1 \cap G_2$ $\Rightarrow$ $\alpha = i_*(\beta)$ for some $\beta \in H_*(A)$

  and $g_*(\alpha) = 0$

However $\quad 0 = g_*(\alpha) = g_* i_*(\beta) = \beta$

$\therefore \quad \alpha = i_*(\beta) = 0$

On the other hand, let $\gamma \in H_*(X)$

$\rightsquigarrow \quad \gamma = \underset{\underset{G_1}{\cap}}{i_* g_*(\gamma)} + \underset{\underset{G_2}{\cap}}{(\gamma - i_* g_*(\gamma))}$

$\therefore \quad H_*(X) \approx G_1 \oplus G_2$

∎

- 1. triple $C \xrightarrow{f} D \xrightarrow{g} E$ of ab. gps & homomorphisms, ; **exact** if im f = kernel g.

  2. A seq. of ab. gps and homs,

  $$\cdots \to G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \to \cdots \to G_n \xrightarrow{f_n} \cdots$$

  ; **exact** if each triple is exact.

  3 exact seq. $\quad 0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$

  is called **short exact**.

Note : in a short exact seq as above,

f ; **mono** & $C \approx$ subgp $C' \subseteq D$

g: epi. with ker C'.

$\therefore$ up to iso, seq $\rightsquigarrow$

$$0 \to C' \to D \to D/C' \to 0.$$

However $\quad 0 = g_*(\alpha) = g_* i_*(\beta) = \beta$

$\therefore \quad \alpha = i_*(\beta) = 0$

On the other hand, let $\gamma \in H_*(X)$

$\rightsquigarrow \quad \gamma = \underset{\underset{G_1}{\cap}}{i_* g_*(\gamma)} + \underset{\underset{G_2}{\cap}}{(\gamma - i_* g_*(\gamma))}$

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- 1. triple $C \xrightarrow{f} D \xrightarrow{g} E$ of ab. gps & homomorphisms,
  ; exact if im f = kernel g.
  2. A seq. of ab. gps and homs,
  $$\cdots \to G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \to \cdots \to G_n \xrightarrow{f_n} \cdots$$
  ; exact if each triple is exact.
  3. exact seq. $\quad 0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$
  is called short exact.

Note : in a short exact seq as above,

f ; mono & $C \approx$ subgp $C' \subseteq D$

g : epi. with ker C'.

$\therefore$ up to iso, seq $\rightsquigarrow$

$$0 \to C' \to D \to D/C' \to 0.$$

- Suppose $C = \{C_n\}$, $D = \{D_n\}$, $E = \{E_n\}$ chain cxes, $0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$: short exact seq., where f, g : chain maps of deg 0.

  $\forall p$, $\exists$ asso. triple of homology gps
  $$H_p(C) \xrightarrow{f_*} H_p(D) \xrightarrow{g_*} H_p(E)$$

-
$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \arrow[d] & \\
0 \arrow[r] & C_n \arrow[r, "f"] \arrow[d, "\partial"] & D_n \arrow[r, "g"] \arrow[d, "\partial"] & E_n \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r] \arrow[d] & D_{n-1} \arrow[r] \arrow[d] & E_{n-1} \arrow[r] \arrow[d] & 0 \\
 & & & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & d \arrow[r] \arrow[d] & z \\
c \arrow[r, "f"] & \partial d &
\end{tikzcd}$$

1) $z \in Z_n(E) \Rightarrow \partial z = 0$

g: epi. $\Rightarrow \exists\, d \in D_n$ with $g(d) = z$
$$g(\partial d) = \partial g(d) = \partial z = 0$$
$\partial d \in \ker g = \operatorname{Im} f \quad \therefore \exists\, c \in C_{n-1}$ with $f(c) = \partial d$

Note $\quad f(\partial c) = \partial f(c) = \partial \partial d = 0$

f: mono $\Rightarrow \partial c = 0 \quad \therefore c \in Z_{n-1}(C)$

2) $z \longmapsto c$ of $Z_n(E)$ to $Z_{n-1}(C)$ is not well-defined

However, asso. correspondence on homology gps is well-defined.

- 1) $z, z' \in \mathcal{Z}_n(E)$: homologous.

  $\rightsquigarrow$ $\exists\ e \in E_{n+1}$ with $\partial e = z - z'$

2) $d, d' \in D_n$ with $g(d) = z$, $g(d') = z'$

   $c, c' \in C_{n-1}$ with $f(c) = \partial d$, $f(c') = \partial d'$

   Show: c & c' are <u>homologous</u>.

3) $\exists\ a \in D_{n+1}$ with $g(a) = e$.

   $g(\partial a) = \partial g(a) = \partial e = z - z'$

   $\therefore\ (d - d') - \partial a \in \ker g = \operatorname{Im} f$.

4) $b \in C_n$ with $f(b) = (d - d') - \partial a$.

   $f(\partial b) = \partial f(b) = \partial(d - d' - \partial a) = \partial d - \partial d'$

   $= f(c) - f(c') = f(c - c')$

   f: mono. $\Rightarrow$ $c - c' = \partial b$

   $\therefore$ c & c': homologous cycles.

   $\therefore$ correspondence induced on homology gps is well-defined hom.

$\hookrightarrow$ denoted by $\Delta : H_n(E) \to H_{n-1}(C)$ & called the connecting homomorphism for $0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$.

Thm 1.13

$0 \to C \xrightarrow{f} D \xrightarrow{g} E \to 0$: short exact seq. of chain cxes and deg. 0 chain maps.

$\leadsto$ the long exact sequence

$$\cdots \to H_n(D) \xrightarrow{g_*} H_n(E) \xrightarrow{\Delta} H_{n-1}(C) \xrightarrow{f_*} H_{n-1}(D) \to \cdots$$

is <u>exact</u>.

- construction of connecting hom is natural.

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r] \arrow[d, "\alpha"] & D \arrow[r] \arrow[d, "\beta"] & E \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \arrow[r] & 0 \end{tikzcd}$$

$\leadsto$

$$\begin{tikzcd} \cdots \arrow[r] & H_n(D) \arrow[r] \arrow[d, "\beta_*"] & H_n(E) \arrow[r] \arrow[d, "\gamma_*"] & H_{n-1}(C) \arrow[r] \arrow[d, "\alpha_*"] & H_{n-1}(D) \arrow[r] \arrow[d, "\beta_*"] & \cdots \\ \cdots \arrow[r] & H_n(D') \arrow[r] & H_n(E') \arrow[r] & H_{n-1}(C') \arrow[r] & H_{n-1}(D') \arrow[r] & \cdots \end{tikzcd}$$

- 1) X : top. sp, $A \subseteq X$ subsp.

  $\mathcal{U}$: coll. of subsets of X ∋ covering of X

  int $\mathcal{U}$; 〃 interiors of elts of $\mathcal{U}$.

  $S_n^{\mathcal{U}}(X)$; subgp of $S_n(X)$ gen. by singular n-sxes $\phi : \sigma_n \to X$ ∋ $\phi(\sigma_n) \subset U (\in \mathcal{U})$ for some $U \in \mathcal{U}$.

2) $\forall i$, $\quad \mathrm{im}\, \partial_i \phi \subseteq \mathrm{im}\, \phi$

$\therefore \quad \partial : S_n^{\mathcal{U}}(X) \to S_{n-1}^{\mathcal{U}}(X)$ is the boundary.

3) $\forall$ covering $\mathcal{U}$ of X, $\exists$ chain cx $S_*^{\mathcal{U}}(X)$

and $\quad i : S_*^{\mathcal{U}}(X) \to S_*(X)$ is a chain map.

4) $\mathcal{V}$: covering of a sp. Y and $f : X \to Y$ is a map s.t. $\forall\, U \in \mathcal{U}$, $f(U) \subset V$ for some $V \in \mathcal{V}$

$\hookrightarrow \quad \exists$ chain map $\quad f_\# : S_*^{\mathcal{U}}(X) \to S_*^{\mathcal{V}}(Y)$

and $\quad f_\# \circ i_X = i_Y \circ f_\#$

Thm 1.14

$\mathcal{U}$: family of subsets of X s.t. $\mathrm{Int}\, \mathcal{U}$ is a covering of X.

$\Rightarrow \quad i_* : H_n(S_*^{\mathcal{U}}(X)) \to H_n(S_*(X))$

is an isomorphism $\forall n$.

- $\mathcal{U} = \{U, V\}$: covering of X

$\rightarrow \quad \mathrm{Int}\, U \cup \mathrm{Int}\, V = X$.

1) A' : set of all singular n-cxes in U

A'' : $\quad$ " $\quad$ in V

$\rightsquigarrow \quad S_n(U) = F(A')$, $\quad S_n(V) = F(A'')$

$S_n(U \cap V) = F(A' \cap A'')$, $\quad S_n^{\mathcal{U}}(X) = F(A' \cup A'')$

2) $\exists$ natural hom

$$h : F(A') \oplus F(A'') \to F(A' \cup A'') \quad \text{by}$$

$$h(a_i', a_j'') = a_i' + a_j''$$

: epimorphism

3) $$g = F(A' \cap A'') \to F(A') \oplus F(A'') \quad \text{by}$$

$$g(b_i) = (b_i, -b_i)$$

$\Rightarrow$ g: mono, $h \circ g = 0$

4) $\forall n$, $\exists$ a short exact seq.

$$0 \to S_n(U \cap V) \xrightarrow{g_\#} S_n(U) \oplus S_n(V) \xrightarrow{h_\#} S_n^{\mathcal{U}}(X) \to 0$$

Define a chain cx $S_*(U) \oplus S_*(V)$ by

$\left(S_*(U) \oplus S_*(V)\right)_n = S_n(U) \oplus S_n(V)$ &

letting the bdry operator be the usual

bdry on each component.

$\therefore$ short exact seq. of chain cxes and

deg 0 chain maps.

•

1) By Thm 1.13, $\exists$ long seq of homology gps

$$\xrightarrow{\Delta} H_n(U \cap V) \xrightarrow{g_*} H_n(S_*(U) \oplus S_*(V)) \xrightarrow{h_*} H_n(S_*^{\mathcal{U}}(X)) \xrightarrow{\partial} H_{n-1}(U \cap V) \to \cdots$$

$$H_n(S_*(U) \oplus S_*(V)) \cong H_n(U) \oplus H_n(V), \qquad H_n(S_*^{\mathcal{U}}(X)) \cong H_n(X)$$

$\therefore$ Mayer-Vietoris seq :

$$\cdots \xrightarrow{\Delta} H_n(U \cap V) \xrightarrow{g_*} H_n(U) \oplus H_n(V) \xrightarrow{h_*} H_n(X) \xrightarrow{\Delta} H_{n-1}(U \cap V) \to \cdots$$

2)
$$\begin{tikzcd} & U \arrow[dr, "k"] & \\ U\cap V \arrow[ur, "i"] \arrow[dr, "j"'] & & U\cup V = X \\ & V \arrow[ur, "l"'] & \end{tikzcd}$$

i, j, k, l: inclusions.

⇝
$$g_*(x) = (i_*(x), -j_*(x)), \quad h_*(y,z) = k_*(y) + l_*(z).$$

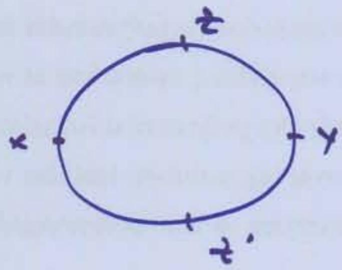
3) Mayer-Vietoris sequence is <u>natural</u>.

If X' is a space, $\mathrm{Int}\, U' \cup \mathrm{Int}\, V' = X'$,
$f: X \to X'$ map s.t. $f(U) \subseteq U'$, $f(V) \subseteq V'$

$\Rightarrow$
$$\begin{tikzcd} \cdots \arrow[r, "\Delta"] & H_n(U\cap V) \arrow[r, "g_*"] \arrow[d, "f_*"] & H_n(U)\oplus H_n(V) \arrow[r, "h_*"] \arrow[d, "f_*\oplus f_*"] & H_n(X) \arrow[r, "\Delta"] \arrow[d, "f_*"] & H_{n-1}(U\cap V) \arrow[r] \arrow[d, "f_*"] & \cdots \\ \cdots \arrow[r, "\Delta'"] & H_n(U'\cap V') \arrow[r, "g'_*"] & H_n(U')\oplus H_n(V') \arrow[r, "h'_*"] & H_n(X') \arrow[r, "\Delta'"] & H_{n-1}(U'\cap V') \arrow[r] & \cdots \end{tikzcd}$$

<u>Example</u>

$X = S^1$,
$U = S^1 \setminus \{z'\}$
$V = S^1 - \{z\}$



1. M-V seq;
$$\underbrace{H_1(U) \oplus H_1(V)}_{=\,0} \xrightarrow{h_*} H_1(S^1) \xrightarrow{\Delta} H_0(U\cap V) \xrightarrow{g_*} H_0(U)\oplus H_0(V)$$

2. Δ: mono, $H_1 S^1 \cong \mathrm{im}\,\Delta = \ker g_*$

An elt of $H_0(U\cap V)$: $ax + by$, a, b: integers

$g_*(ax+by) = (i_*(ax+by), -j_*(ax+by))$

U, V: path-conn, $i_*(ax+by) = 0 \Leftrightarrow a = -b$
j_* " "

$$\therefore \quad \ker g_* = \{ a(x-y) \mid a : \text{integer} \} \cong \mathbb{Z}.$$

$$\therefore \quad H_1(S^1) \cong \mathbb{Z}.$$

3. $\forall n > 1$, $\quad H_n(U) \oplus H_n(V) \xrightarrow{\text{lin}} H_n(S^1) \xrightarrow{\partial} H_{n-1}(U \cap V)$

$\qquad \qquad H_n(U) \oplus H_n(V) = 0, \quad H_{n-1}(U\cap V) = 0$

$$\therefore \quad H_n(S^1) = 0.$$

//

- $S^n = \{ (x_1, \cdots, x_{n+1}) \mid x_i \in \mathbb{R}, \ \sum x_i^2 = 1 \}$

$z = (0, 0 \cdots 0, 1)$

$z' = (0 \cdots 0, -1)$

$\rightsquigarrow \quad S^n - \{z\} \cong \mathbb{R}^n \quad) \quad S^n - \{z, z'\} \cong \mathbb{R}^n - \{0\}.$

$\quad S^n - z' \quad \text{"}$

S^{n-1} ; deformation retract of $\mathbb{R}^n - \{0\}$.

$U = S^n - z$, $\quad V = S^n - z'$, s.t. $U \cap V = S^n - \{z, z'\}$

$\Rightarrow \quad H_m(U) \oplus H_m(V) \to H_m(S^n) \to H_{m-1}(S^{n-1}) \to 0$

$m > 1$: $\quad H_m(U) \oplus H_m(V) = 0$

Thm 1.15 $\therefore$ $\quad H_*(S^n) = \begin{cases} \mathbb{Z} & \text{if } * = n, 0 \\ 0 & \text{otherwise.} \end{cases}$

Cor 1.16

For $n \neq m$, S^n and S^m do not have the same homotopy type.

Cor 1.17

$$\boxed{\text{There is no retraction of } D^n \text{ onto } S^{n-1}}$$

pf

- $n=1$; obvious since D^1 is connected and S^0 is not.
- $n>1$

  Suppose $f: D^n \to S^{n-1}$ s.t. $f \circ i = \mathrm{id}$, where $i: S^{n-1} \to D^n$ inclusion

$\rightsquigarrow$

$$\begin{tikzcd}
H_{n-1}(S^{n-1}) \arrow[rr, "id"] \arrow[dr, "i_*"'] & & H_{n-1}(S^{n-1}) \\
& H_{n-1}(D^n) \arrow[ur, "f_*"'] &
\end{tikzcd}$$

$$H_{n-1}(D^n) = 0$$

This is impossible. □

Cor 1.18 (Brower fixed point theorem)

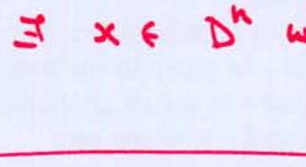
$$\boxed{\text{Given a map } f: D^n \to D^n,\ \exists\, x \in D^n \text{ with } f(x) = x.}$$

pf

Suppose $f: D^n \to D^n$ without fixed pts.

Define $g: D^n \to S^{n-1}$ as follows:

$\forall x \in D^n$, $\exists$ ray from $f(x)$ and passing through x



Define $g(x)$ = this ray $\cap S^{n-1}$.

$\rightsquigarrow$ g = <u>retraction</u>. □

- $n \geq 1$, $f: S^n \to S^n$ a map

1) α: generator of $H_n(S^n) \cong \mathbb{Z}$.

$\rightsquigarrow$ $f_*(\alpha) = m \cdot \alpha$ for some integer m.

m is the <u>degree</u> of f, denoted <u>$d(f)$</u>.

2) <u>basic properties</u> of <u>degree</u>.

(a) $d(\mathrm{id}) = 1$

(b) $f, g: S^n \to S^n$ $\rightsquigarrow$ $d(f \circ g) = d(f) \cdot d(g)$

(c) $d(\text{constant map}) = 0$

(d) f, g: homotopic $\rightsquigarrow$ $d(f) = d(g)$

(e) f: homotopy equiv. $\rightsquigarrow$ $d(f) = \pm 1$

- $\exists$ maps of any integral degree on S^n, $n > 0$.

3) (<u>Hopf</u>).

If $d(f) = d(g)$, then f and g are <u>homotopic</u>.

($\therefore$ degree is a <u>complete</u> invariant for studying homotopy classes of maps from S^n to S^n.)

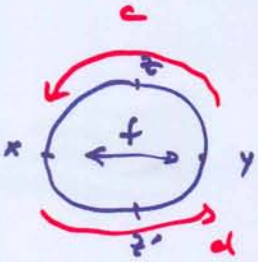
<u>Prop</u> 1.19.

$n > 0$, $f: S^n \to S^n$ by $f(x_1, \ldots, x_{n+1}) = (-x_1, x_2, \ldots, x_{n+1})$

$\Rightarrow$ $d(f) = -1$.

<u>proof</u>

- $n = 1$.



1). $U = S^1 \setminus \{z'\}$, $V = S^1 - \{z\}$ has:

$$f(U) \subseteq U, \quad f(V) \subseteq V.$$

2) naturality of M-V seq:

$$\begin{tikzcd} 0 \arrow[r] & H_1(S^1) \arrow[r] \arrow[d, "f_*"] & H_0(U \cap V) \arrow[d, "(f|)_*"] \\ 0 \arrow[r] & H_1(S^1) \arrow[r] & H_0(U \cap V) \end{tikzcd}$$

3) generator α of $H_1(S^1)$ was repr. by $c+d$, where $\partial c = x - y = -\partial d$.

$\Delta(\alpha)$ is repr. by $x-y$.

4)
$$\Delta f_*(\alpha) = (f|)_* \Delta(\alpha) = (f|)_*(x-y) = y - x = -\Delta(\alpha) = \Delta(-\alpha)$$

Δ is mono. $\rightsquigarrow$ $\underline{d(f) = -1}$.

- Suppose it's true in dim $n-1 \geq 1$ and $S^{n-1} \subseteq S^n$.

$U = S^n - \{z'\}$, $V = S^n - \{z\}$.

$i : S^{n-1} \to U \cap V$ is a h.e.

Since $n \geq 2$, connecting hom is an iso.

$\therefore$

$$\begin{tikzcd} H_n(S^n) \arrow[r, "\Delta", "\cong"'] \arrow[d] & H_{n-1}(U \cap V) \arrow[d, "(f|)_*"] & H_{n-1}(S^{n-1}) \arrow[l, "i_*", "\cong"'] \arrow[d, "f_*"] \\ H_n(S^n) \arrow[r, "\Delta", "\cong"'] & H_{n-1}(U \cap V) & H_{n-1}(S^{n-1}) \arrow[l, "i_*", "\cong"'] \end{tikzcd}$$

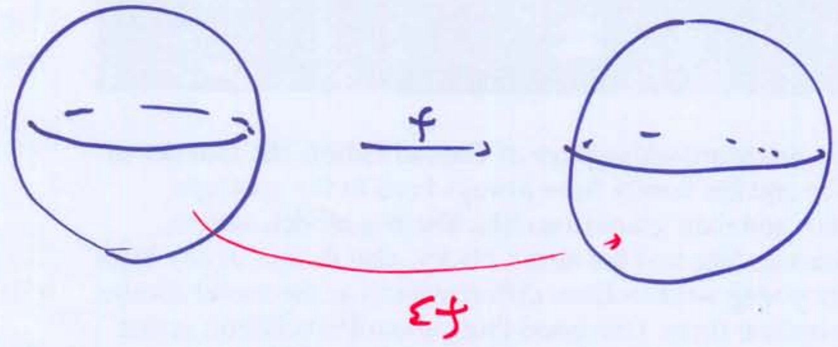
α: generator of $H_n(S^n)$

$$\rightsquigarrow \quad f_*(\alpha) = \Delta^{-1}(f|)_* \Delta(\alpha) = \Delta^{-1} i_* f_* \, i_*^{-1} \Delta(\alpha) = -\Delta^{-1} i_* \, i_*^{-1} \Delta(\alpha)$$
$$= -\alpha$$

$\boxtimes$

- $f: S^n \to S^n$, $n \geq 0$

  $\exists$ asso. map $g: S^{n+1} \to S^{n+1}$ called the **suspension of f** and denoted by Σf.



$$\Sigma f(x,t) = \begin{cases} (x,t) & , \; x = 0 \\ \left(\|x\| \cdot f\left(\frac{x}{\|x\|}\right), t\right) & x \neq 0 \end{cases}$$

Prop 1.20

$$\boxed{\begin{array}{l} f: S^n \to S^n, \; n \geq 1, \text{ map} \\ \Rightarrow \quad d(\Sigma f) = d(f) \end{array}}$$

- Note: if $f(x_1, \dots, x_{n+1}) = (-x_1, \dots, x_{n+1})$ and

  $g(x_1, \dots, x_{n+2}) = (-x_1, \dots, x_{n+2})$

  $\rightsquigarrow$ $g = \Sigma f$ and Prop. 1.19 is a special case of Prop. 1.20.

Cor 1.21

$f: S^n \to S^n$ is given by

$f(x_1, \cdots, x_{n+1}) = (x_1, \cdots, -x_i, \cdots, x_{n+1})$

$\Rightarrow$ $d(f) = -1$.

proof

$h: S^n \to S^n$; map that exchanges the 1-st coordinate and the i-th coord.

$\Rightarrow$ h: homeo ($h^{-1} = h$) $\therefore$ $\deg h = \pm 1$.

Let $g(x_1, \cdots, x_{n+1}) = (-x_1, \cdots, x_{n+1})$ $\ni$ $d(g) = -1$.

$\therefore$ $d(f) = d(h \circ g \circ h^{-1}) = d(h)^2 d(g) = -1$ ☒

Cor 1.22

The antipodal map $A: S^n \to S^n$ by $A(x_1, \cdots, x_{n+1}) = (-x_1, \cdots, -x_{n+1})$ has $d(A) = (-1)^{n+1}$

Exer.

Show that for $n > 0$ and m any integer, there exists a map $f: S^n \to S^n$ of degree m.

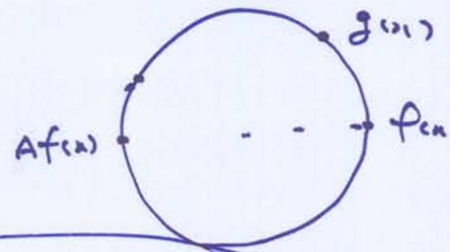
Prop 1.23

$f, g: S^n \to S^n$ maps with $f(x) \neq g(x)$ for all $x \in S^n$

$\leadsto$ g is homotopic to $-f$

pf

Define a function



$$F: S^n \times I \to S^n$$

by
$$F(x,t) = \frac{(1-t)\,Af(x) + t\,g(x)}{\|(1-t)\,Af(x) + t\,g(x)\|}$$
$\neq 0$

□

Cor 1.24

$$\boxed{\begin{array}{l} f: S^{2n} \to S^{2n} \;:\; \text{map} \\ \Rightarrow\ \exists\, x \in S^{2n} \text{ with } f(x) = x \text{ or} \\ \quad\ \exists\, y \in S^{2n} \text{ with } f(y) = -y. \end{array}}$$

pf

If $f(x) \neq x \ \forall x$, then by prop. f is homotopic to A.

If $f(x) \neq -x = A(x), \forall x$, $f \sim A \circ A = id$.

$$\therefore \quad \underset{\overset{\|}{(-1)^{2n+1}}}{d(A)} = d(f) = \underset{\overset{\|}{1}}{d(id)}$$

This is impossible ■

Cor 1.25

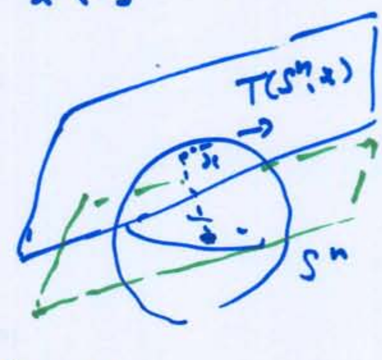
$$\boxed{\begin{array}{l} \text{There is no continuous map } f: S^{2n} \to S^{2n} \\ \quad \text{s.t. } f(x) \text{ and } x \text{ are orthogonal } \forall x. \end{array}}$$

- S^n ; manifold of dim. n.

  (i.e. locally homeomorphic to $\mathbb{R}^n$)

  $T(S^n, x)$: tangent space at $x \in S^n$

  $\hookrightarrow$ n-dim'l hyperplane in $\mathbb{R}^{n+1}$



  Translate this hyperplane to the origin where it becomes n-dim'l subsp orthogonal to x.

  A <u>vector field</u> on S^n ; continuous fun. to each $x \in S^n$ a vector in the corr. lin. subsp.

  A vector field ϕ is non-zero if $\phi(x) \neq 0$ $\forall x \in S^n$.

<u>Cor 1.26</u>

$$\boxed{\text{There exists no non-zero vector field on } S^{2n}}$$

<u>proof</u>

If ϕ is non-zero v.f. on S^{2n}, then $\psi(x) = \dfrac{\phi(x)}{\|\phi(x)\|}$ is a v.f. on S^{2n} of unit length.

$\therefore$ $\psi : S^{2n} \to S^{2n}$; map s.t. $\psi(x)$ is orthogonal to x, $\forall x$.

This is impossible by Cor 1.25. $\square$

- 1. Non-zero vector fields exist on odd dim'l spheres.
  2. coll. of vector fields $\phi_1, \cdots, \phi_k$ on S^n is lin. indep. $\iff$ $\forall x \in S^n$, $\phi_1(x), \cdots, \phi_k(x)$ are lin. indep.

- 1) A directed set Λ $\iff$ set ~~=~~ with partial order relation $\leq$ s.t. $\forall a, b \in \Lambda$, $\exists c \in \Lambda$ with $a \leq c$ and $b \leq c$
  2) A direct system of sets
  $\iff$ family of sets $\{X_a\}_{a\in\Lambda}$, Λ = directed set and functions $f_a^b : X_a \to X_b$, $a \leq b$
  satisfying :
  (i) f_a^a = identity on X_a, $\forall a \in \Lambda$
  (ii) if $a \leq b \leq c$, $f_a^c = f_b^c \circ f_a^b$
  3) particular case of interest
  : X_a : abelian groups,
  f_a^b : homomorphisms

- $\{X_a, f_a^b\}$; direct system of ab. gps and homs.
  Define a subgroup R of $\Sigma_a X_a$
  $$R = \{ \sum_{i=1}^n x_{a_i} \mid \exists c \in \Lambda, c \geq a_i : \sum_{i=1}^n f_{a_i}^c(x_{a_i}) = 0 \}$$
  $\leadsto$ the direct limit of $\{X_a, f_a^b\}$ is the group $\varinjlim X_a = \Sigma X_a / R$

Note: if $x_a \in X_a$, $x_b \in X_b$,
they are equal in the direct limit
if for some $c \in \Lambda$, $c \geq a, b$, $f_a^c(x_a) = f_b^c(x_b)$.

Lemma 1.27

X: sp, $\{X_a\}$: family of all cpt subsets of X partially ordered by incl.
$\Rightarrow$ $\{H_*(X_a)\}$: direct system where homomorphisms are induced by incl. maps.
& $\varinjlim_a H_*(X_a) = H_*(X)$

proof. Omitted.

Lemma 1.28

$A \subset S^n$ subset with $A \cong I^k$, $0 \leq k \leq n$

$$\Rightarrow \quad H_j(S^n \setminus A) = \begin{cases} \mathbb{Z}, & j = 0 \\ 0, & j > 0 \end{cases}$$

proof

- If $k = 0$, A is a pt and $S^n - A \cong \mathbb{R}^n$
  $\therefore$ the conclusion follows.
- Assume the result is true for $k < m$
  $\star$ $h : A \to I^m$; homeo.
  Split I^m into upper and lower halves by
  $$I^+ = \{(x_1, \cdots, x_m) \in I^m \mid x_1 \geq 0\}$$
  $$I^- = \{(x_1, \cdots, x_m) \in I^m \mid x_1 \leq 0\}$$
  s.t. $I^+ \cap I^- \cong I^{m-1}$

Let $A^+ = h^{-1}(I^+)$, $A^- = h^{-1}(I^-)$

$\rightsquigarrow$ $S^n - (A^+ \cap A^-)$

$= (S^n - A^+) \cup (S^n - A^-)$.

M-V seq $\Rightarrow$

$$H_{j+1}(S^n - (A^+ \cap S^-)) \to H_j(S^n - A) \to H_j(S^n - A^+) \oplus H_j(S^n - A^-) \to H_j(S^n - (A^+ \cap A^-)) = 0$$

0 ; inductive hyp. for $j > 0$

$\therefore$ $H_j(S^n - A) \xrightarrow{\cong} H_j(S^n - A^+) \oplus H_j(S^n - A^-)$

- Repeat this procedure by splitting A^+ into 2 pieces whose intersection is homeo. to I^{m-1}.

$\rightsquigarrow$ $\exists$ seq. of subsets of S^n

$A = A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$,

incl. $S^n - A \subseteq S_n - A_k$ induces a hom on homology, $\cap A_i \cong I^{m-1}$.

- $\varinjlim_{k} H_j(S^n - A_k) \cong H_j(S^n - \cap A_i) = 0$

by ind. hyp.

$\rightsquigarrow$ $H_j(S^n - A) = 0$

- For $j = 0$, M-V seq $\Rightarrow$ mono.

If $x, y \in S^n \setminus A$ with $x - y \neq 0$ in $H_0(S^n \setminus A)$, $x - y \neq 0$ in $H_0(S^n - \cap A_i)$. contradiction.

Cor 1.29

$B \subseteq S^n$; homeo. to S^k, $0 \le k \le n-1$.

$\Rightarrow$ $H_*(S^n - B)$; free ab. gp with 2 generators, one in dimension zero and one in dim. $n-k-1$.

proof

- $k=0$; S^k = two pts & $S^n - B$ has the homotopy type of S^{n-1}

  $\therefore$ the result is true for $k=0$

- Suppose the result is true for $k-1$

  and $B = B^+ \cup B^-$, where B^+, B^- ; closed hemispheres in S^k, $B^+ \cap B^- \cong S^{k-1}$.

  M-V seq. $\Rightarrow$

$$H_{j+1}(S^n - B^+) \oplus H_{j+1}(S^n - B^-) \to H_{j+1}(S^n - (B^+ \cap B^-))$$
$$\to H_j(S^n - B)$$
$$\to H_j(S^n - B^+) \oplus H_j(S^n - B^-)$$

For $j > 0$, both of end terms are zero by Lemma 1.28.

Inductive step implies the conclusion. $\boxtimes$

Thm 1.30 (Jordan - Brouwer Separation Theorem)

An $(n-1)$-sphere imbedded in S^n separates S^n into 2 components and it's the boundary of each component.

proof

- $B \subseteq S^n$ embedded copy of S^{n-1}

  By Cor 1.29, $H_*(S^n - B) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & * = 0 \\ 0, & * \neq 0 \end{cases}$

  $S^n - B$ has 2 path components

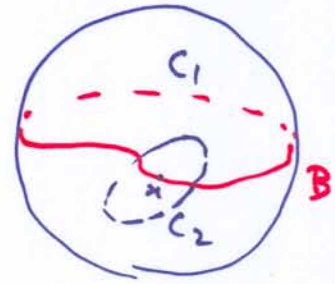
  B closed $\Rightarrow$ $S^n - B$: open $\therefore$ loc. path-conn

  $\therefore$ path components = components

- C_1, C_2 : components of $S^n - B$

  $C_1 \cup B$: closed $\Rightarrow$ $\partial C_1 \subset B$

  $\partial C_1 = \bar{C}_1 - C_1^{\circ}$



  Show : $B \subseteq \partial C_1$

  $x \in B$, U = nbd of x in S^n.

  $B \cong S^{n-1} \Rightarrow \exists$ subset $K \subset U \cap B$ with $x \in B$

  & $B - K \cong D^{n-1}$!

  Lemma 1.28 $\leadsto$ $H_*(S^n - (B-K)) \approx \mathbb{Z}$ with

  generator in dim. 0.

  $\therefore$ $S^n - (B-K)$ has one path component

  Let $p_1 \in C_1$, $p_2 \in C_2$. γ : path in $S^n - (B-K)$

  between p_1 & p_2.

  C_1, C_2 : distinct path components in $S^n - B$.

  $\Rightarrow$ γ must intersect K.

  $\therefore$ K contains pts of $\bar{C}_1$ and $\bar{C}_2$

  $\therefore$ <u>$x \in \partial C_1$</u>

Attaching Spaces with maps

- Purpose

  ; develop basic theory of CW complexes and their homology groups.

- Recall:

  1) relation on a set A : equivalence relation

     if (i) $a \sim a$

     (ii) $a \sim b \Rightarrow b \sim a$

     (iii) $a \sim b,\ b \sim c \Rightarrow a \sim c$, $\forall a, b, c \in A$.

  2) equivalence classes

  3) $A/\sim$; set of equiv. classes under $\sim$.

     $\pi : A \to A/\sim$: quotient fun.

     $a \mapsto [a]$

- $f : A \to B$ function of sets

  $\rightsquigarrow$ $\exists$ associated equiv. rel. on A

  $a_1 \sim a_2$ if and only if $f(a_1) = f(a_2)$

- $\sim$: equiv. rel. on a top. space X.

  1) the quotient sp. $X/\sim$ may be topologized by defining a subset $U \subseteq X/\sim$; open $\Longleftrightarrow$ $\pi^{-1}(U)$ is open in X.

  2) $\pi : X \to X/\sim$; continuous.

- X : top. sp.

1) define $\quad D = \{ (x,x) \mid x \in X \} \subseteq X \times X$

   diagonal in $X \times X$.

2) X : Hausdorff $\iff$ D : closed in $X \times X$.

3) $\sim$: equiv. rel. on X.

   Δ : diagonal in $(X/\sim) \times (X/\sim)$

   $\pi \times \pi : \ X \times X \longrightarrow (X/\sim) \times (X/\sim)$

   $\rightsquigarrow (\pi \times \pi)^{-1}(\Delta) = \{ (x,y) \mid x \sim y \}$

   : <u>graph of relation</u>

   $\sim$ on X is closed $\iff$ its graph is <u>closed</u>.

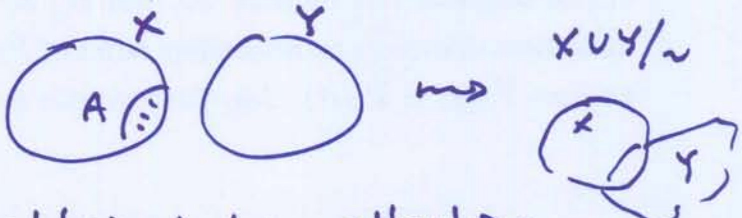
<u>prop 2.1</u>

$$\boxed{\begin{array}{l} \sim \text{ : closed relation on a cpt. Hausdorff sp.} \\ \Rightarrow X/\sim \text{ is Hausdorff} \end{array}}$$

- A, X, Y : spaces , $A \subseteq X$, $X \cap Y = \phi$

  $f : A \to Y$ continuous.

  $\sim$; equiv. rel on $X \cup Y$ s.t. $x \sim f(x) \quad \forall x \in A$



$X \cup Y/\sim$; space obtained by attaching X to Y via $f : A \to Y$.

Denote $X \cup Y/\sim$ by <u>$X \cup_f Y$</u>

Cor 2.2

$$\boxed{\begin{array}{l} X, Y : \text{ compact Hausdorff spaces} \\ A : \text{closed in } X, \quad f : A \to Y \text{ ; continuous} \\ \Rightarrow X \cup_f Y \text{ is a compact Hausdorff space.} \end{array}}$$

- $\exists$ homeomorphic copy of Y sitting in $X \cup_f Y$

  $i : Y \longrightarrow X \cup_f Y$ is an <u>embedding</u>.

1) $X = D^n$, $A = S^{n-1} = \partial D^n$.

   $D^n \cup_f Y$ is called the space by attaching n-cell to Y via f.

   (Denote $D^n \cup_f Y$ by Y_f)

<u>Example</u>

$X = D^2$, $A = S^1 = \partial D^2$, $\quad Y = S^1$ disjoint from X.

$f : A \to S^1 (= Y)$; map of deg 2 by

$$f(e^{i\theta}) = e^{2i\theta}$$

$\Rightarrow \quad X \cup_f Y = \mathbb{R}P(2)$ real projective plane

- homology groups of this space

1) U : open cell in the interior of D^2

D^2

   $p \in U$.

   $V = \mathbb{R}P(2) - \{p\}$.

   $U \cap V$ and V have homotopy type of S^1

   U : contractible

$$\begin{array}{ccccc} H_1(U\cap V) & \xrightarrow{\alpha} & H_1(U)\oplus H_1(V) & \xrightarrow{\beta} & H_1(\mathbb{R}P(2)) \\ \wr\wr & & \wr\wr & & \text{epi.} \\ \mathbb{Z} & & \mathbb{Z} & & \end{array}$$

α is a monomorphism onto $2\mathbb{Z}$.

$\therefore \quad H_1(\mathbb{R}P(2)) \cong \mathbb{Z}_2$.

2) connecting hom. $H_2(\mathbb{R}P(2)) \xrightarrow{\Delta} H_1(U\cap V)$

is a mono. whose image $= \ker\alpha$.

$\therefore \quad H_2(\mathbb{R}P(2)) = 0$

& $H_n(\mathbb{R}P(2)) = 0 \quad \forall\, n>2$

Prep 2.3

$f: S^{n-1} \to Y$ continuous where Y is Hausdorff.

$\Rightarrow \exists$ exact seq.

$$\cdots \to H_m(S^{n-1}) \xrightarrow{f_*} H_m(Y) \to H_m(Y_f) \xrightarrow{\Delta} H_{m-1}(S^{n-1}) \to \cdots$$

$$\to H_0(S^{n-1}) \to H_0(Y) \to H_0(Y_f).$$

- If n-cell has been attached to Y,

1) $H_n(Y) \xrightarrow{i_*} H_n(Y_f)$; mono. with cokernel either zero or infinite cyclic.

2) $H_{n-1}(Y) \xrightarrow{i_*} H_{n-1}(Y_f)$; epi. with ker either zero or cyclic.

- (X, A) pair of spaces, $Y = pt.$

1) $\exists !$ map $A \xrightarrow{f} Y$ for $A \neq \phi$.

$X \cup_f Y$ is denoted by X/A

2) X: cpt Hausdorff, A: closed in X

$\Rightarrow X/A$; cpt Hausdorff

prop 2.4

X, W: cpt Hausdorff spaces

$g: X \to W$; continuous, onto s.t. for some $w_0 \in W$,

$g^{-1}(w_0)$ is a closed set $A \subseteq X$.

for $w \neq w_0$, $g^{-1}(w)$: single pt of X

$\Rightarrow W$ is homeo. to X/A.

prop 2.5

X, Y, W ; cpt Hausdorff sps

A : closed in X

$f: A \to Y$; continuous, $g: X \cup Y \to W$ continuous onto

If $\forall w \in W$, $g^{-1}(w)$ is either a single pt of $X \setminus A$

or union of a single pt $y \in Y$ with $f^{-1}(y)$ in A

$\Rightarrow W \cong X \cup_f Y$.

proof

- $\pi : X \cup Y \to X \cup_f Y$; identification map

$$\begin{tikzcd}
X \cup Y \arrow[rr, "g"] \arrow[dr, "\pi"'] & & W \\
& X \cup_f Y \arrow[ur, "k"'] &
\end{tikzcd}$$

k is induced by g.

$\Rightarrow$ k : 1-1 and onto.

To see that k is continuous,

C : closed in W.

$\Rightarrow$ $k^{-1}(C)$ is closed iff $\pi^{-1}k^{-1}(C)$ is closed

But $\pi^{-1}k^{-1}(C) = g^{-1}(C)$: closed.

Since $X \cup_f Y$, W are cpt Hausdorff spaces, k is a homeomorphism $\square$

Example

$S^{n-1} = \partial D^n$, $h_1 : D^n - S^{n-1} \to \mathbb{R}^n$ homeo.

$z \in S^n$, $h_2 : S^n - \{z\} \to \mathbb{R}^n$ homeo. given by stereographic proj.

Define a function

$$g : D^n \to S^n \text{ by } g(x) = \begin{cases} z, & x \in S^{n-1} \\ h_2^{-1}h_1(x), & x \in D^n - S^{n-1}. \end{cases}$$

$\rightsquigarrow$ g satisfies hyp. of Prop. 2.4 with $A = S^{n-1}$

$\therefore$ $D^n / S^{n-1} \cong S^n$

(sp. given by attaching n-cell to a pt)

Example

$\mathbb{R}P^n = S^n/\sim \quad , \quad x \sim -x \quad \forall x.$

$\pi : S^n \to \mathbb{R}P(n)$ quotient map.

- what sp is produced by attaching an $(n+1)$-cell to $\mathbb{R}P(n)$ via π?

$S^n \subseteq S^{n+1}$ by identifying $(x_1, \ldots, x_{n+1}) \in S^n$ with $(x_1, \ldots, x_{n+1}, 0) \in S^{n+1}$.

$i : \mathbb{R}P(n) \to \mathbb{R}P(n+1)$ inclusion.

$S^{n+1} = E_+^{n+1} \cup E_-^{n+1}, \quad E_+^{n+1} \cap E_-^{n+1} = S^n.$

$\exists$ homeo. $g : D^{n+1} \to E_+^{n+1}$

Denote by $f_1 : D^{n+1} \to \mathbb{R}P^{n+1}$; composition

$$D^{n+1} \xrightarrow{g} E_+^{n+1} \subseteq S^{n+1} \xrightarrow{h} \mathbb{R}P(n+1)$$

quotient map.

$\therefore$ we have a map

$$D^{n+1} \cup \mathbb{R}P(n) \xrightarrow{f_1 \cup i} \mathbb{R}P(n+1)$$

; onto.

Note : $z \in \mathbb{R}P(n+1) \leadsto f_1^{-1}(z)$ is either a single pt of $D^{n+1} - S^n$ or $\{x, -x\}$ in S^n.

the ~~l~~ latter ; true $\iff z \in \mathbb{R}P(n)$.

$\therefore$ hyp. of prop. 2.5 are satisfied &

$\mathbb{R}P(n+1) \cong D^{n+1} \cup_\pi \mathbb{R}P(n)$

$\square$

- X, Y: top. sps, $x_0 \in X$, $y_0 \in Y$.

Define $X \vee Y$, the <u>wedge of X and Y</u>, to be

$$X \times \{y_0\} \cup \{x_0\} \times Y.$$

<u>Example</u>

n-cube $I^n \subseteq \mathbb{R}^n$ has $\partial I^n = \{(x_1, \cdots, x_n) \mid \text{some } x_i = 0 \text{ or } 1\}$

$\therefore I^m \times I^n = I^{m+n}$, $\partial(I^{m+n}) = (\partial I^m \times I^n) \cup (I^m \times \partial I^n)$

- $z_m \in S^m$, $z_n \in S^n$; base pts.

$\exists$ $f: (I^m, \partial I^m) \to (S^m, z_m)$; relative homeo.

& $g: (I^n, \partial I^n) \to (S^n, z_n)$;

$\Rightarrow$ $f \times g : I^m \times I^n \to S^m \times S^n$

On $\partial(I^m \times I^n)$,

$$\left\{\begin{array}{l} I^m \times I^n - \partial(I^{m+n}) = (I^m - \partial I^m) \times (I^n - \partial I^n) \\ (S^m - z_m) \times (S^n - z_n) = S^m \times S^n - (S^m \times \{z_n\} \cup \{z_m\} \times S^n) \\ \qquad = S^m \times S^n \setminus S^m \vee S^n \end{array}\right.$$

$\leadsto$

$S^m \times S^n$ is homeo. to sp. obtained by attaching an $(m+n)$-cell to $S^m \vee S^n$ via the map $\partial(I^{m+n}) \cong S^{m+n-1} \to S^m \vee S^n$

($S^m \times S^n$ is called a generalized torus)

Example

$\forall n$, $\mathbb{R}^{2n} \cong \mathbb{C}^n$ and denote $S^{2n-1} \subseteq \mathbb{C}^n$ by

$$S^{2n-1} = \{ (z_1, \dots, z_n) \mid \sum |z_i|^2 = 1 \}$$

Define an equiv. rel. on S^{2n-1} by

$(z_1, \dots, z_n) \sim (z_1', \dots, z_n') \iff \exists \lambda \in \mathbb{C}$ with $|\lambda| = 1$

s.t. $z_1' = \lambda z_1, \dots, z_n' = \lambda z_n$.

The space $S^{2n-1}/\sim$ is denoted $\mathbb{C}P(n-1)$,

$(n-1)$-dim'l complex projective space.

Exercise

$f : S^{2n-1} \to S^{2n-1}/\sim \; = \mathbb{C}P(n-1)$; identification map.

Show that the space formed by attaching a $2n$-cell to $\mathbb{C}P(n-1)$ via f is homeo. to $\mathbb{C}P(n)$.

- $n=1$ two any pts in S^1 are equiv. $\therefore$ $\mathbb{C}P(0)$ is a pt.

  $\mathbb{C}P(1)$ is formed by attaching D^2 to $\mathbb{C}P(0)$

  $\approx S^2$

- $S^3 \to S^3/\sim \; = \mathbb{C}P(1) = S^2$

  This map $h : S^3 \to S^2$ is called the Hopf map and is important in homotopy theory.

Example

- Identify $\mathbb{R}^4$ with division ring of ~~quater~~ quaternions by $(x_1, x_2, x_3, x_4) \to x_1 + i x_2 + j x_3 + k x_4$.

  This identifies $\mathbb{R}^{4n}$ with $\mathbb{H}^n$ &

  $$S^{4n-1} = \{ (\alpha_1, \dots, \alpha_n) \in \mathbb{H}^n \mid \Sigma |\alpha_i|^2 = 1 \}.$$

  On S^{4n-1}, set $(\alpha_1, \dots, \alpha_n) \sim (\alpha_1', \dots, \alpha_n') \iff \exists \gamma \in \mathbb{H}$ with $|\gamma| = 1$ s.t. $(\alpha_1', \dots, \alpha_n') = \gamma(\alpha_1, \dots, \alpha_n)$.

  Then $S^{4n-1}/\sim$ is $\mathbb{H}P(n-1)$, $(n-1)$-dim'l quaternionic projective space.

- $\mathbb{H}P(0) = pt$, $\mathbb{H}P(1) = S^4$,

  $\mathbb{H}P(n)$: space by attaching a $4n$-cell to $\mathbb{H}P(n-1)$ via identification map $S^{4n-1} \to \mathbb{H}P(n-1)$.

- The identification map $h : S^7 \to \mathbb{H}P(1) = S^4$ is called the Hopf map.

- Computation of homology groups.

  $S^m \times S^n$, $m, n \geq 2$

  1) $S^m \times S^n$ is given by attaching an $(m+n)$-cell to $S^m \vee S^n$.

Denote

$-*_m, -*_n$: antipodes of $*_m, *_n$.

Define $U = S^m \vee S^n - \{-*_n\}$, $V = S^m \vee S^n - \{*_m\}$

$\{U, V\}$: open covering of $S^m \vee S^n$

$U \simeq S^m$, $V \simeq S^n$, $U \cap V \sim pt.$

$\therefore$ M-V seq. $\Rightarrow$ $H_j(S^m) \oplus H_j(S^n) = H_j(S^m \vee S^n)$ for $j > 0$

$$\therefore H_*(S^m \vee S^n) = \begin{cases} \mathbb{Z}, & * = 0 \\ \mathbb{Z}, & * = m \text{ or } n \ (\text{or } \mathbb{Z} \oplus \mathbb{Z}, m = n) \\ 0 & \text{otherwise} \end{cases}$$

By prop. 2.23,

$\exists$ exact seq.

$$\cdots \to H_i(S^{m+n-1}) \xrightarrow{f_*} H_i(S^m \vee S^n) \to H_i(S^m \times S^n) \to H_{i-1}(S^{m+n-1}) \to \cdots$$

$m, n \geq 2$, $\Rightarrow$ $m+n-1 > m, n$.

$\therefore$ f_* : zero-map in positive dimensions.

On the other hand, if $i = m+n$,

conn. hom. $H_i(S^m \times S^n) \to H_{i-1}(S^{m+n-1})$: iso.

prop 2.6

$$\boxed{\begin{array}{l} H_*(S^m \times S^n),\ m, n \geq 0, \text{ is a free ab. group of} \\ \text{rank 4 having one basis elt of each dim} \\ 0, m, n, m+n \end{array}}$$

- $\alpha \in H_k(S^k) = \mathbb{Z}$ generator

1) $\beta \in H_k(X)$: <u>spherical</u> if $\exists\ f : S^k \to X$ s.t. $f_*(\alpha) = \beta$.

2) if $\beta \in H_2(S^1 \times S^1)$ is a generator, then β is <u>not</u> spherical.

(We'll prove this later)

Prop. 2.7

$$\boxed{H_i(\mathbb{C}P(n)) = \begin{cases} \mathbb{Z} & i = 0, 2, 4, \cdots, 2n \\ 0 & \text{otherwise} \end{cases}}$$

proof

S_i : group of singular i-chains in $\mathbb{C}P(n)$ $\Rightarrow S_i = \mathbb{Z}$, $i = 2j$.

$$\begin{array}{ccccccccc} S_{2j+1} & \to & S_{2j} & \xrightarrow{\partial} & S_{2j-1} & \xrightarrow{\partial} & S_{2j-2} & \to & \cdots \\ \parallel & & \parallel & & \parallel & & \parallel & & \\ 0 & & \mathbb{Z} & & 0 & & \mathbb{Z}. & & \end{array}$$

$$\therefore\ H_i(\mathbb{C}P(n)) = \begin{cases} \mathbb{Z} & , i = 0, 2, 4, \cdots, 2n \\ 0 & \end{cases}$$

Prop 2.8

$$\boxed{H_i(\mathbb{H}P(n)) = \begin{cases} \mathbb{Z} & i = 0, 4, 8 \cdots 4n \\ 0 & \text{otherwise.} \end{cases}}$$

- $C = \{C_n, \partial\}$ chain complex

1) $D = \{D_n, \partial\}$: <u>subcomplex</u> of C

($D_n \subseteq C_n \ \forall n$, ∂ for D is the restriction of ∂ for C)

Define the quotient chain cx

$$C/D = \{C_n/D_n, \partial'\}$$

, $\partial'\{c\} = \{\partial c\}$

2) $\exists$ natural short exact seq. of chain cxes and chain maps

$$0 \to D \xrightarrow{i} C \xrightarrow{\pi} C/D \to 0$$

, where i is the incl. & π : projection.

$\rightsquigarrow$ $\exists$ long exact seq. of homology gps

$$\cdots \to H_n(D) \to H_n(C) \to H_n(C/D) \xrightarrow{\Delta} H_{n-1}(D) \to \cdots$$

For clarity denote by $\{\ \}$ the equiv. rel. in C/D and by $\langle\ \rangle$ the equiv. rel. in homology.

- To see how Δ is defined,

$\{c\}$ in $Z_n(C/D)$ $\rightsquigarrow$ $c \in C_n$, $\partial c \in D_{n-1}$.

$\partial c \in Z_{n-1}(D)$ $\therefore$ represents a class in $H_{n-1}(D)$

$\therefore$ $\Delta(\langle\{c\}\rangle) = \langle \partial c \rangle$.

• (X, A) , X = sp with $A \subseteq X$.

The singular chain complex of X mod A

is defined by $S_*(X,A) = S_*(X)/S_*(A)$

1) The homology of this chain cx, the relative singular homology of X mod A ;

$$H_*(X,A) = H_n(S_*(X))/H_n(S_*(A))$$

2)

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{\pi_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(X) \to \cdots$$

prop 2.9

(X, A) : pair $\ni$ A is a deformation retract of X

$\Rightarrow$ $H_*(X, A) = 0$

• (X, A, B) : triplet spaces, $B \subseteq A \subseteq X$

1) $\leadsto$ $0 \to S_*(A,B) \to S_*(X,B) \to S_*(X,A) \to 0$

& $\cdots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \xrightarrow{\partial} H_{n-1}(A,B) \to \cdots$

2) (X,A), (Y,B) , $f: (X,A) \to (Y,B)$ (continuous, $f(A) \subseteq B$)

$\exists$ associated a hom.

$$f_*: S_*(X,A) \to S_*(Y,B)$$

: chain map.

3) $f, g: (X,A) \to (Y,B)$: homotopic

$\Leftrightarrow$ $\exists$ $F: (X \times I, A \times I) \to (Y,B)$ s.t.

$F(x,0) = f(x)$, $F(x,1) = g(x)$.

Thm 2.10

$$\boxed{\begin{array}{l} f, g : (X, A) \to (Y, B) \quad \text{homotopic as maps of pairs} \\ \Rightarrow f_* = g_* : H_*(X, A) \to H_*(Y, B) \end{array}}$$

proof

- As before, $i_0, i_1 : (X, A) \to (X \times I, A \times I)$ by $i_0(x) = (x, 0)$
  $i_1(x) = (x, 1)$
  It is sufficient to show that $i_{0\#}$ & $i_{1\#}$ are chain homotopic.
- Use the same technique as the absolute case, construct a natural hom

$$T : S_n(X) \to S_{n+1}(X \times I)$$

$\to$

$$\partial T + T \partial = i_{0\#} - i_{1\#}$$

and

$$T(S_n(A)) \subseteq S_{n+1}(A \times I)$$

$\therefore$ $\exists$ induced chain homotopy

$$T : S_n(X, A) \to S_{n+1}(X \times I, A \times I)$$

Example

$X = [0, 1]$, $A = \{0, 1\}$, $Y = S^1$, $B = \{1\}$.

$g, f : X \to Y$ by $f(x) = e^{2\pi i x}$, $g(x) = 1$

$\Rightarrow$ f and g : maps of pairs $(X, A) \to (Y, B)$

f & g ; absolutely homotopic as maps from X to Y

but not homotopic as maps of pairs.

Exercise (five lemma)

$$\begin{tikzcd}
C_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & C_2 \arrow[r, "d_2"] \arrow[d, "f_2"] & C_3 \arrow[r, "d_3"] \arrow[d, "f_3"] & C_4 \arrow[r, "d_4"] \arrow[d, "f_4"] & C_5 \arrow[d, "f_5"] \\
D_1 \arrow[r, "\beta_1"] & D_2 \arrow[r, "\beta_2"] & D_3 \arrow[r, "\beta_3"] & D_4 \arrow[r, "\beta_4"] & D_5
\end{tikzcd}$$

diagram of ab. groups & homs.

→ rows are exact.

(1) f_2, f_4 : epimorphisms, f_5 : mono.

$\Longrightarrow$ f_3 : epimorphism.

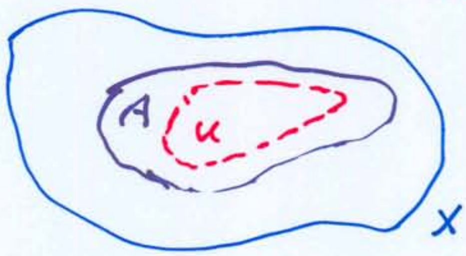
(2) f_2, f_4 : mono, f_1 : epimor.

$\Longrightarrow$ f_3 : mono.

Thm 2.11

$$\boxed{\begin{array}{l} (X,A) : \text{ pair of spaces, } U \subseteq A \text{ with } \bar{U} \subset \operatorname{int} A. \\ \Rightarrow \text{ the incl. } \quad i : (X-U, A-U) \to (X,A) \\ \text{induces an isomorphism on rel. homology gps} \\ \qquad i_* : H_*(X-U, A-U) \to H_*(X,A) \end{array}}$$

proof



- $\mathcal{U} = \{ X-U, \operatorname{Int} A \}$

  : covering of X

  $\mathcal{U}' = \{ A-U, \operatorname{Int} A \}$

  covering of A.

Thm 1.14 ⤳ $i : S_*^{\mathcal{U}}(X) \to S_*(X)$, $i' : S_*^{\mathcal{U}'}(A) \to S_*(A)$

both induce isomorphisms on homology.

$S_*^{\mathcal{U}'}(A) \subseteq S_*^{\mathcal{U}}(X)$

$\leadsto \quad j : S_*^{\mathcal{U}}(X)/S_*^{\mathcal{U}'}(A) \to S_*(X)/S_*(A) = S_*(X,A)$

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(S_*^{\mathcal{U}'}(A)) \arrow[r] \arrow[d, "i_*'"] & H_n(S_*^{\mathcal{U}}(X)) \arrow[r] \arrow[d, "i_*"] & H_n(S_*^{\mathcal{U}}(X)/S_*^{\mathcal{U}'}(A)) \arrow[r] \arrow[d, "j_*"] & H_{n-1}(S_*^{\mathcal{U}'}(A)) \arrow[r] \arrow[d, "i_*'"] & \cdot \\
& H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[r] & H_{n-1}(A) &
\end{tikzcd}$$

Five lemma $\Rightarrow$ j_* : <u>isomorphism</u>.

- $S_*^{\mathcal{U}}(X) = S_*(X-U) + S_*(\mathrm{Int}\,A)$

  $S_*^{\mathcal{U}'}(A) = S_*(A-U) + S_*(\mathrm{Int}\,A)$

  $$\Rightarrow \quad S_*^{\mathcal{U}}(X)/S_*^{\mathcal{U}'}(A) = S_*(X-U)/S_*(A-U) = S_*(X-U, A-U)$$

  $\therefore \quad H_*(X-U, A-U) \to H_*(X,A)$; isomorphism $\boxtimes$

- short exact seq. of ab. gps and homs,

  $$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \quad \text{is split exact}$$

  if $f(A)$ is a <u>direct summand</u> of B.

<u>Exer.</u>

$0 \to A \xrightarrow{f} B \xrightarrow{g} C$; short exact.

$\Rightarrow$ TFAE :

(1) the seq. is split exact

(2) $\exists$ hom $\bar{f} : B \to A$ with $\bar{f} \circ f = id.$

(3) $\exists$ hom $\bar{g} : C \to B$ with $g \circ \bar{g} = id.$

- X : space , y : single pt.

1) $\alpha : X \to y$ map.

$\exists$ induced hom. on homology

$$\alpha_* : H_*(X) \to H_*(y)$$

Denote $\ker \alpha_*$ by $\tilde{H}_*(X)$

; reduced homology of X

2) $H_i(y) = 0$ for $i \neq 0$

$$\Rightarrow \tilde{H}_i(X) = H_i(X) \quad \text{for } i \neq 0$$

3) If $X \neq \phi$,

α_* ; epimorphism s.t. $\tilde{H}_0(X)$ is free ab. with one ~~fewer~~ basis elt than $H_0(X)$.

4) $f : X \to Y$; map

$$\Rightarrow f_* : \tilde{H}_*(X) \to \tilde{H}_*(Y)$$

Prop. 2.12

$$\boxed{x_0 \in X \Rightarrow H_*(X, x_0) \cong \tilde{H}_*(X)}$$

pf.

- $\forall i$, hom $H_i(x_0) \to H_i(X)$ is a monomorphism.

$$\therefore \quad 0 \to H_i(x_0) \xrightarrow{i_*} H_i(X) \xrightarrow{j_*} H_i(X, x_0) \to 0$$

$\alpha : X \to x_0$ induces $\alpha_* : H_i(X) \to H_i(x_0)$

$\to$ splits the seq.

$$\therefore \quad H_*(X, x_0) \cong \tilde{H}_*(X).$$

- $A \subset X$: strong deformation retract of X

$\Longleftrightarrow$ $\exists$ $F: X \times I \to X$ s.t.

(i) $F(x,0) = x$, $\forall x \in X$

(ii) $F(x,1) \in A$ "

(iii) $F(a,t) = a$ $\forall a \in A, t \in I$

Prop 2.17

(X,A) : pair, X cpt Hausdorff, A closed in X

A ; strong deformation retract of X

$\pi : X \to X/A$ identification map

$y = \pi(A)$ in X/A

$\Rightarrow$ $\{y\}$: s. d. r. of X/A.

pf

1) $F: X \times I \to X$; map s.t. A: s. d. r. of X.

Show ; $\exists$ map $\tilde{F} : (X/A) \times I \to X/A$ $\ni$

$\tilde{F}(\tilde{x},0) = \tilde{x}$, $\tilde{F}(\tilde{x},1) = y$ $\forall \tilde{x} \in X/A$

$\tilde{F}(y,t) = y$, $\forall t \in I$.

$$\begin{CD} X \times I @>F>> X \\ @V{\pi \times id}VV @VV{\pi}V \\ (X/A) \times I @>>> X/A \end{CD}$$

2) Define $\tilde{F} = \pi \circ F \circ (\pi \times id)^{-1}$

$\hookrightarrow$ well-defined & continuous

$\boxtimes$

Thm 2.14

(X,A) pair with X cpt Hausdorff, A closed in X

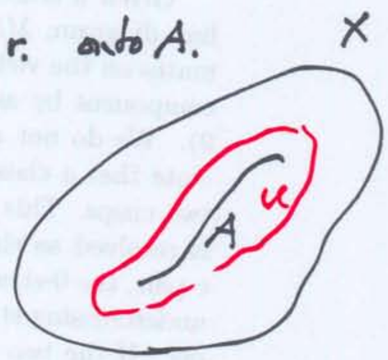
A: s.d.r. of some closed nbd of A in X

$\pi: (X,A) \to (X/A, y)$ identification map

$\Rightarrow \pi_*: H_*(X,A) \to H_*(X/A, y)$ isomorphism.

proof

1) U: cpt nbd of A in X $\to$ s.d.r. onto A.



Apply prop 2.13 to (U,A)

exact seq in $(X/A, \pi(U), y)$

$$\cdots \to H_n(\pi(U), y) \to H_n(X/A, y) \to H_n(X/A, \pi U) \to H_{n-1}(\pi U, y) \to \cdots$$

$$H_*(\pi U, y) = 0$$

$\therefore$ incl map induces an iso

$$H_*(X/A, y) \to H_*(X/A, \pi U)$$

2) X = cpt H.S $\Rightarrow$ normal.

$\mathrm{Int}\, U \supset A \Rightarrow \exists V$: open s.t. $A \subseteq V$, $\bar{V} \subset \mathrm{Int}\, U$.

V is excised from (X,U) to induce an iso.

$$H_*(X-V, U-V) \cong H_*(X,U)$$

A; s.d.r. of $U \Rightarrow H_*(X,A) \cong H_*(X,U)$

$$\therefore H_*(X,A) \cong H_*(X-V, U-V).$$

3) Similarly, $\pi(V)$ is excised from $(X/A, \pi U)$

to give an iso.

$$H_*(X/A, y) \cong H_*(X/A, \pi U) \cong H_*(X/A - \pi(U), \pi U - \pi V)$$

4) V: nbd of A → collapsed

∴ $\pi|$, gives a homeo. of pairs

$$\pi : (X-V, U-V) \to (X/A - \pi U, \pi U - \pi V)$$

and so an iso. of their homology gps

Combine all of these :

$$H_*(X, A) \cong H_*(X/A, y)$$

Cor 2.15

(X, A) : ~~cpt~~ compact Hausdorff pair ∋ A : s.d.r of some cpt nbd of A in X

$\Rightarrow$ $H_*(X, A) \cong \tilde{H}_*(X/A)$

- $f : (X, A) \to (Y, B)$ map of pairs s.t. f maps $X-A$ 1-1 & onto $Y-B$. then f is a <u>relative</u> <u>homeo.</u>

Thm 2.16

$f : (X, A) \to (Y, B)$ relative homeo. of cpt Hausdorff pairs

∋ A : s.d.r. of some cpt nbd in X

B " " Y

$\Rightarrow$ $f_* : H_*(X, A) \to H_*(Y, B)$ is an iso.

proof.

1) Consider diagram of spaces and maps

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow \pi & & \downarrow \pi' \\ X/A & \xrightarrow{f'} & Y/B \end{array}$$

Define $f' = \pi' \circ f \circ \pi^{-1}$.

$\Rightarrow$ f' ; well-defined and continuous

f : rel. homeo $\Rightarrow$ f' : 1-1 & onto.

X/A, Y/B ; compact Hausdorff spaces

$\therefore$ f' ; homeo.

2) Denote $x_0 = \pi(A)$, $y_0 = \pi'(B)$

$$\begin{array}{ccc} H_*(X,A) & \xrightarrow{f_*} & H_*(Y,B) \\ \downarrow \pi_* & & \downarrow \pi'_* \\ H_*(X/A, x_0) & \xrightarrow{f'_*} & H_*(Y/B, y_0) \end{array}$$

By thm 2.14, π_*, π'_* ; isomorphisms

f' : homeo $\Rightarrow$ f'_* ; iso.

$\therefore$ f_* is an iso. $\boxtimes$

Example

(1) $\exists$ rel. homeo. $f : (D^n, S^{n-1}) \to (S^n, *)$, $* \in S^n$.

$\therefore \quad f_* : H_*(D^n, S^{n-1}) \to H_*(S^n, *) \cong \tilde{H}_*(S^n)$ is an iso.

(2) Hypothesis of theorem is necessary.

- (X,A), (Y,B) : compact Hausdorff pairs.

  It's possible to define a map of pairs $f : (X,A) \to (Y,B)$ a rel. homeo.

  However, it can't induce an iso. on homology because $H_2(X,A) = 0$, $H_2(Y,B) \cong \mathbb{Z}$

  the result fails because A is not a s.d.r. of some cpt nbd of A in X.

- $\because$ $\underline{H_2(X,A) = 0}$

$$\cdots \to H_2(A) \to H_2(X) \to H_2(X,A) \to H_1(A) \to \cdots$$

X: contractible $\quad \therefore \; H_2(X) = 0$

On the other hand, if $\sum n_i \phi_i$ is a 1-chain in A, the sum is finite.

A is not loc. conn. $\Rightarrow$ union of images of these singular sxes can't bridge the gap in $\sin 1/x$ curve.

$\therefore$ chain is supported by some contractible subset of A

$\therefore$ if it is a cycle, it's also a bdry.

$\therefore$ $H_1(A) = 0$

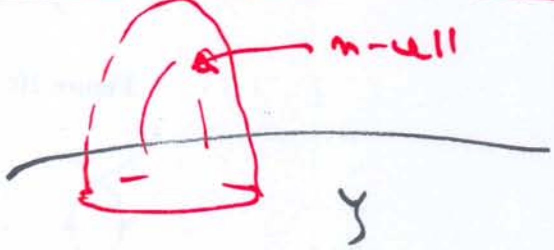
$\square$

Lemma 2.17

$f : S^{n-1} \to Y$ map, Y cpt Hausdorff space

Y_f ; space by attaching an n-cell to Y via f

$\Rightarrow$ Y ; s.d.r. of some cpt nbd of Y in Y_f



Note

h ; composition $D^n \xrightarrow{\text{incl}} D^n \cup Y \xrightarrow{\pi} Y_f$

$\Rightarrow$ h gives a map of pairs $h = (D^n, S^{n-1}) \to (Y_f, Y)$

$\exists$ rel. homeo.

$\therefore$ $h_* : H_*(D^n, S^{n-1}) \to H_*(Y_f, Y)$; iso.

$\therefore$ $H_*(Y_f, Y)$; free ab. gp on one basis elt of dim. n.

- $D_1^n, \dots, D_k^n$; finite number of disj. n-cells with bdries $S_1^{n-1}, \dots, S_k^{n-1}$.

  $\forall i,\ f_i : S_i^{n-1} \to Y$ map

  Define $\sim$; equiv. rel. on $D_1^n \cup \dots \cup D_k^n \cup Y$

  $\ni\ x_i \sim f_i(x_i)$, $x_i \in S_i^{n-1}$.

  $\Rightarrow$ $D_1^n \cup \dots \cup D_k^n \cup Y / \sim$: denoted $Y_{f_1 \dots f_k}$

  : space by attaching n-cells to Y via $f_1, \dots, f_k$.

- <u>finite</u> <u>CW-complex</u>

  1) $\Leftrightarrow$ cpt H.S X and a seq. $X^0 \subseteq X^1 \subseteq \dots \subseteq X^n = X$ of closed subspaces s.t.

  (i) X^0 : finite set of points

  (ii) X^k : homeomorphic to a space by attaching a finite number of k-cells to X^{k-1}

  2) $X^k - X^{k-1}$: finite disj. union of open k-cells, denoted $E_1^k, \dots, E_{r_k}^k$

  3) <u>cells</u> of X have the following:

  (a) $\{E_i^k \mid k = 0, 1, \dots, n;\ i = 1, \dots, r_k\}$; partition of X into disj. sets

  (b) $\forall k, i,\ \ \bar{E}_i^k - E_i^k$ is <u>contained</u> in the union of all cells of lower dim.

  (c) $X^k = \bigcup_{k' \leq k} E_j^{k'}$

(d) $\forall i, k$, $\exists$ rel. homeo.

$$h : (D^k, S^{k-1}) \to (\bar{E}_i^k, \bar{E}_i^k - E_i^k)$$

4) X^k .. k-skeleton of X

If $X^n = X$, $X^{n-1} \neq X$, then X ; n-dimensional.

Example

$z \in S^2$
ϕ
space by attaching a 2-cell to z.

$\Rightarrow$ S^2 ; cell structure $\ni$ $\exists$ one 0-cell & one 2-cell.

prop 2.18

X, Y : finite CW cxes
$\Longrightarrow$ $X \times Y$; finite CW cx.

Example

(1) S^1 

$\Big\}$ one 0-cell : z
one 1-cell : α

$S^1 \times S^1$ 

$\#(\text{0-cell}) = 1$

$\#(\text{1-cell}) = 2$

$\#(\text{2-cell}) = 1$

(2) $\mathbb{R}P(0) = pt$, $\mathbb{R}P(k)$: by attaching k-cell to $\mathbb{R}P(k-1)$

$\therefore$ $\mathbb{R}P(n)$; n-dim'l finite CW cx with one cell in each dim $0, \cdots, n$.

(3) $\mathbb{CP}(n)$: finite CW-cx of dim $2n$ with one cell in each even dim $0, 2, 4, \cdots, 2n$

- X: finite CW cx with cells $\{E_i^k\}$

1) $A \subset X$; <u>subcomplex</u> of X

$\iff$ if $A \cap E_i^k \neq \phi$, $\bar{E}_i^k \subseteq A$.

2) If A is a subcx of X,

A ; closed in X & inherits a natural CW cx structure.

<u>Thm 2.19</u>

A : subcomplex of a finite CW cx X

$\Rightarrow$ A ; <u>strong</u> <u>deformation</u> <u>retract</u> of some compact nbd of A in X.

<u>Prop 2.20</u>

X ; finite CW-cx , X^k ; k-skeleton of X

$\Rightarrow$ $H_j(X^k, X^{k-1}) = 0$ for $j \neq k$

& $H_k(X^k, X^{k-1})$; free ab. group with one basis elt for each k-cell of X

Pf

1) X^{k-1} ; subcx of X^k

$\Rightarrow$ s.d.r. of a opt nbd in X^k

X: finite CW-cx

$\Rightarrow$ $\exists$ rel. homeo

$$\phi : (D_1^k \cup \cdots \cup D_r^k,\ S_1^{k-1} \cup \cdots \cup S_r^{k-1}) \to (X^k, X^{k-1})$$

2) Thm 2.16

$$\Rightarrow H_k(X^k, X^{k-1}) \cong H_k(D_1^k \cup \cdots \cup D_r^k,\ S_1^{k-1} \cup \cdots \cup S_r^{k-1})$$

$\boxed{}$

- $\forall$ finite CW complex X, define

$$C_k(X) = H_k(X^k, X^{k-1})$$

$\Rightarrow$ $C_*(X) = \sum C_k(X)$; graded group $\ni$ non-zero in only finitely many dims.

& free ab., finitely generated $\forall$ dim.

<u>Connecting</u> <u>hom</u> of (X^k, X^{k-1}, X^{k-2}) defines an operator $\partial : C_k(X) \to C_{k-1}(X)$

$\Rightarrow$ $\partial \circ \partial = 0$

and $\{C_*(X), \partial\}$ is a <u>chain</u> <u>complex</u>

<u>Thm 2.21</u>

$$\boxed{\begin{array}{l} X : \text{finite CW complex} \\ \Rightarrow H_k(C_*(X)) \cong H_k(X),\ \forall k. \end{array}}$$

- $f: X \to Y$ map between finite CW cxes

1) is <u>cellular</u> if $f(X^k) \subseteq Y^k$, $\forall k$.

2) $f: X \to Y$: cellular

$\leadsto$ f defines a map of pairs

$$f: (X^k, X^{k-1}) \to (Y^k, Y^{k-1}) \quad \forall k$$

$\leftarrow$ chain map $f_*: C_*(X) \to C_*(Y)$

3) hom. induced by f_* on $H_*(C_*(X))$

corr. to hom. induced by f on $H_*(X)$.

- Computation of homology of <u>$\mathbb{R}P(n)$</u>

S^n ; finite CW cx. $\to$ k-skeleton is S^k

$$S^0 \subseteq S^1 \subseteq \cdots \subseteq S^n$$

s.t. $\exists$ 2 cells in each dim, denoted by E^k_+, E^k_-

Similarly $\mathbb{R}P(n)$; finite CW cx $\to$

$\mathbb{R}P(k)$; k-skeleton

$$\mathbb{R}P(0) \subseteq \mathbb{R}P(1) \subseteq \cdots \subseteq \mathbb{R}P(n)$$

& $\exists$ 1 cell in each dim.

We need to know : <u>$\partial: C_k(\mathbb{R}P(n)) \to C_{k-1}(\mathbb{R}P(n-1))$</u>

- antipodal map $A: S^n \to S^n$ cellular

Denoted by F^k ; composition

$$(D^k, S^{k-1}) \xrightarrow{\cong} (E^k_+, S^{k-1}) \xrightarrow{\text{incl}} (S^k, S^{k-1})$$

i_k ; gen. of $H_k(D^k, S^{k-1})$ $\Rightarrow$ $F^k_*(i_k) = e_k$: basis elt in $H_k(S^k, S^{k-1}) = C_k(S^n)$.

e_k ; basis elt corresponding to E^k_+.

$$\begin{tikzcd}
(D^k, S^{k-1}) \arrow[r, "\cong"] & (E^k_+, S^{k-1}) \arrow[r, "incl"] \arrow[d, "A"] & (S^k, S^{k-1}) \arrow[d, "A"] \\
 & (E^k_-, S^{k-1}) \arrow[r, "incl"] & (S^k, S^{k-1})
\end{tikzcd}$$

$\Rightarrow$ $A_*(e_k)$; basis elt corr. to E^k_-

$\therefore$ $C_k(S^n)$; free ab. gp with basis $\{e_k, A_*(e_k)\}$.

• 1)

$$\begin{tikzcd}
H_k(S^k, S^{k-1}) \arrow[rr, "\partial"] \arrow[dr, "\partial'"] \arrow[ddd, "A_*"'] & & H_{k-1}(S^{k-1}, S^{k-2}) \arrow[ddd, "A_*"] \\
 & H_{k-1}(S^{k-1}) \arrow[ur, "i_*"'] \arrow[d, "A_*"] & \\
 & H_{k-1}(S^{k-1}) \arrow[dr, "i_*"] & \\
H_k(S^k, S^{k-1}) \arrow[ur, "\partial'"] \arrow[rr, "\partial"] & & H_{k-1}(S^{k-1}, S^{k-2})
\end{tikzcd}$$

; commute

multiplication by $(-1)^k$.

2) $e_k \in H_k(S^k, S^{k-1}) = C_k(S^n)$

$\rightsquigarrow$ $\partial A_*(e_k) = i_* \partial' A_*(e_k) = i_* A_* \partial'(e_k)$

$= (-1)^k i_* \partial'(e_k) = (-1)^k \partial(e_k)$

$\therefore$ $e_k + (-1)^{k+1} A_*(e_k)$; <u>cycle</u> in $C_k(S^n)$

- In fact, set of cycles in $C_k(S^n)$; infinite cyclic gp
gen. by $\underline{e_k + (-1)^{k+1} A_k(e_k)}$.

$H_k(S^n) = 0$, $0<k<n$

$$\partial(e_{k+1}) = \pm(e_k + (-1)^{k+1} A_*(e_k))$$

Also holds for $k=0$

- $\pi : (S^k, S^{k-1}) \to (\mathbb{R}P(k), \mathbb{R}P(k-1))$ rel. homeo. on closure of each k-cell

generator e_k' is chosen $\ni$ $e_k' = \pi_*(e_k)$.

$\Rightarrow$ $\pi_* A_*(e_k) = \pi_*(e_k) = e_k'$.

$\therefore$ ∂ in $C_*(\mathbb{R}P(n))$ is given by

$$\begin{aligned} \partial(e_{k+1}') &= \partial \pi_*(e_{k+1}) = \pi_* \partial(e_{k+1}) \\ &= \pi_*(e_k + (-1)^{k+1} A_* e_k) \\ &= e_k' + (-1)^{k+1} e_k' \\ &= \begin{cases} 2e_k', & k \text{ : odd} \\ 0, & k \text{ : even} \end{cases} \end{aligned}$$

<u>prop 2.22</u>

$$\boxed{H_i(\mathbb{R}P(n)) = \begin{cases} \mathbb{Z}, & i=0 \\ \mathbb{Z}_2 & i \text{ : } \cancel{\text{odd}}\ \text{even}, \ 0<i<n \\ \mathbb{Z} & i \text{ : odd}, \ i=n \\ 0 & \text{otherwise} \end{cases}}$$

- <u>rank</u> of finitely generated ab. gp A is

1) given by

$$\text{rank } A = \text{lub} \{ n \mid \exists \text{ free ab. gp } B \subseteq A \text{ with basis having exactly } n \text{ elts} \}.$$

2) A, B : isomorphic ab. gps

$\Longrightarrow$ rank A = rank B.

3) H : subgp of a f.g. ab. gp G

$\Longrightarrow$ <u>rank G/H = rank G − rank H</u>

<u>**prop 2.23**</u>

$\boxed{\begin{array}{l} (X,A) \text{ ; finite CW-cx.} \\ \Rightarrow H_*(X,A) \text{ ; f.g. ab. gp.} \end{array}}$

Pf.

1) $H_*(X,A) \cong \tilde{H}_*(X/A)$.

& $H_*(X/A) = \tilde{H}_*(X/A) \oplus \mathbb{Z}$

it suffices to show ; $H_*(X/A)$ is f.g.

2) cells of X/A corr. to cells of X ∋ not in A with one 0-cell corr. to A.

$\therefore$ dim $(X/A) \leq$ dim X

3) thm 2.21 $\leadsto$ $H_k(X/A) \cong$ quotient of f.g. ab. gp by a subgp

& non-zero for only finitely many values of k

$\therefore$ f.g.

- For a sp. X,

1) i-th Betti number $b_i(X)$ is rank$(H_i(X))$.

By prop. 2.23, if X is finite CW cx, $b_i(X)$ is finite for all i, non-zero for only finitely many values i

2) Euler characteristic of X is given by

$$\chi(X) = \sum_i (-1)^i b_i(X).$$

Prop 2.24

X : finite CW cx. with α_i cells in dim i

$\Rightarrow \quad \sum (-1)^i \alpha_i = \chi(X)$

Thm 2.25 (Cellular approximation Theorem)

X, Y ; finite CW complexes, A : subcx of X.

$f : X \to Y$ map $\ni$ cellular on A

$\Rightarrow$ f is homotopic to a cellular map via a homotopy that does not change the restriction of f to A.